Design of Mechatronic System
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Lecture: 26
Lagrange Formulation Fundamentals

So today's class we are going to look at mathematical modeling continuing with our discussion. And specifically will see this Lagrangian formulation of dynamics of problems. So, let us begin with the slides.

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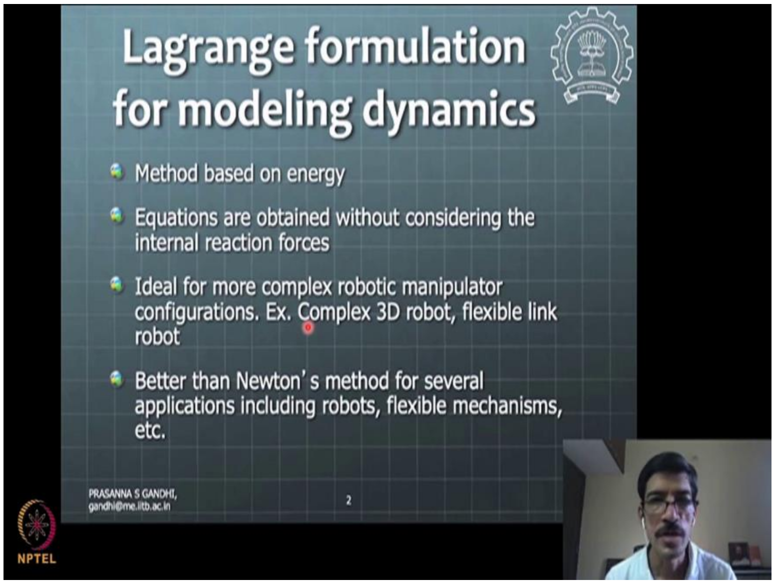

So, this Lagrangian formulation is a method based on energy. So, you do not need to really worry about the free body diagrams that you draw in the case of Newtonian formulation. So, this is purely based on energy and we will see how it is effective in many different cases. So, internal reaction forces are not to be considered during this formulation.

That is what it is important to see that and then this becomes like a very good tool for many different complex mechanical systems. Typically, like complexity robot dynamics with multiple links connected together. In fact, it can be applied to the flexible robotic systems as well, although we will not have bit in the scope of this class, but flexible to what also that can be modeled by using Lagrange formulation.

Then it is better in a then Newton's method of in the sense of a couple of things. So, as I said earlier, like this equations are based purely on energy principles and the equations that you finally get are having some kind of a mathematical structure which also will be there if the Newton's equation obtain by Newton's method both are equivalent to each other.

There is a proof that exists, mathematical proof that exists and both methods are equivalent. But the kind of way the Lagrange formulation gets derived, you get some kind of a structure and that structure becomes an important aspect for the control purposes. So we will see how and how and why it becomes important as we proceed.

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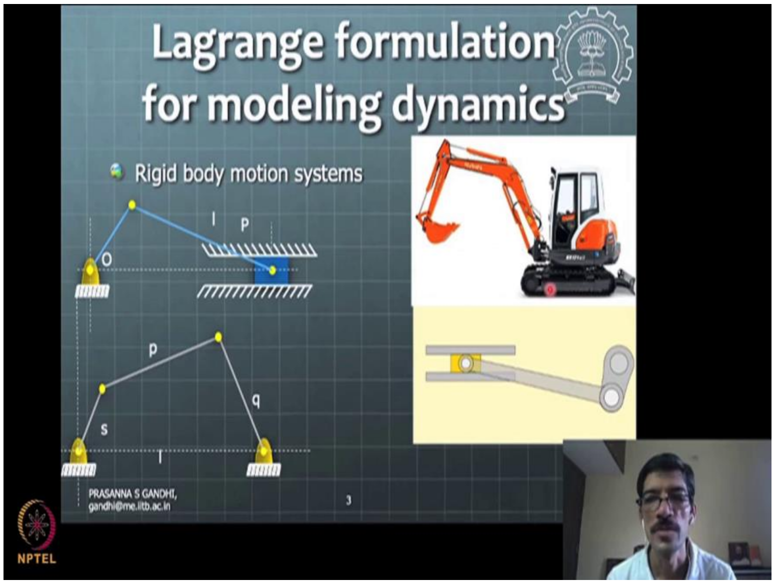

So, first thing, that will deal with the grand information of the systems which are mainly the rigid body motion systems. So, what are these rigid body motion systems mainly having like all these linkages here are rigid in the nature so that we can write kinetic energy and you do not have systems which are having infinite degrees of freedom, we have systems where the degrees of freedom are finite.

For example, there are four rigid bodies involved with ground as including the ground and only one degree of freedom in this case similarly in four bar case then in this there are multiple rigid links that are involved and then there are actuators also, there is some kind of a mechanism that is get forms because of the placement of these actuators in different places. So, this will have different kind of evolution of dynamics of the system.

So, there can be many different so, there are these three actuators you can see. So, this is typically a three degree of freedom arm and if you can consider if you want to consider you have this rotary degree of freedom also, which is the fourth one. So, considering this of course, this button to be stationary. So, one can consider like, depending upon the needs of the problem of control. What are the degrees of freedom that are to be included for modelling.

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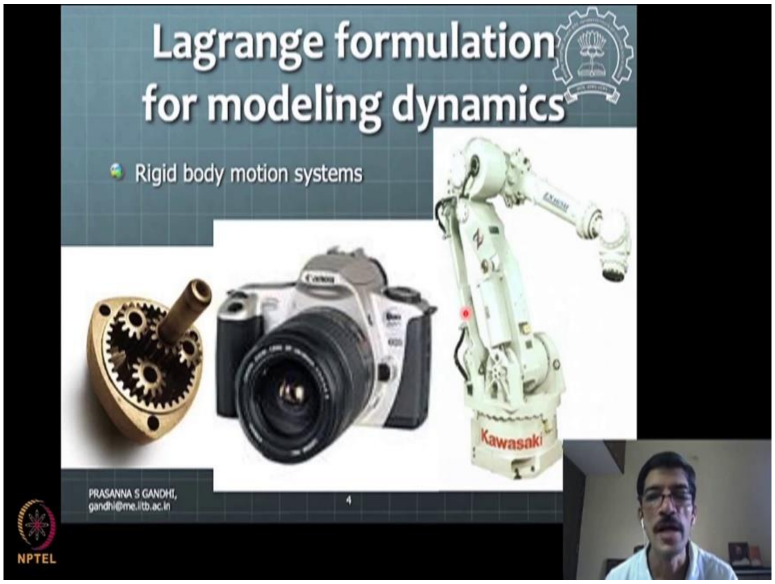

There are some other examples of these rigid body systems, robotic systems; you have gear based systems, in camera you will have like some kind of elements for gears and motors for motion of the lens, this like again a robotic system, Kawasaki robot for different industrial applications, things like that. So, all these systems, they can be very nicely modeled by using Lagrangian dynamics.

And based on the equations that you obtain into some generalized form, a control can be further proposed for, especially the systems which are fully actuated systems where all these robot joints, every joint has actuation, no joint is left unactuated, then you will get nice control that can be developed later.

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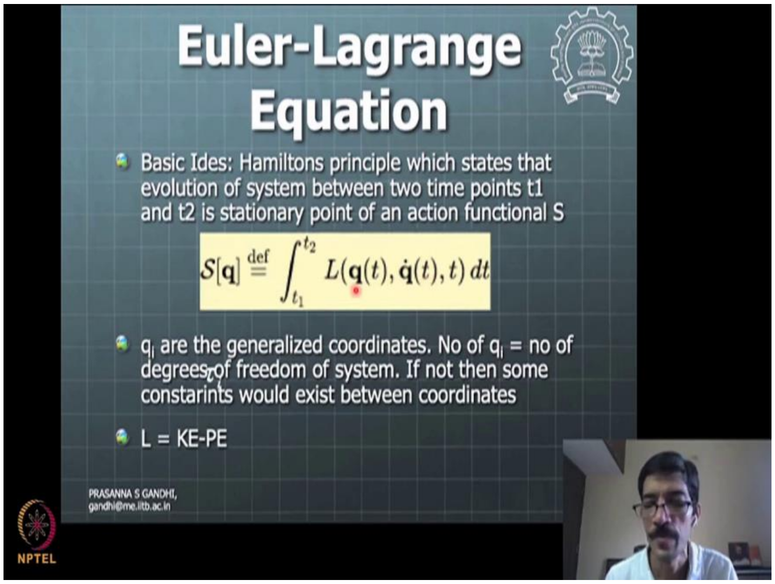

So, the basic principle in which the lagrange formulation is based on his Hamiltonian's principle. Now, what is this Hamilton's principle? It states that the evolution of a system between the two time points, t_1 and t_2, is a stationary point of an action functional. Now, this action functional is defined in this kind of a way. So, you have an integral of a Lagrangian function here which is a function the states of a system q and derivatives of the… so states are basically q and $\dot{q}$.

q are generalized coordinates and their derivatives also will be coming in the form of states and then this, it can be a general function of explicit function of time as well. So, this Lagrangian is defined in a specific way which is kinetic energy minus potential energy. So, this is not a general function of generalized quantities it is a very specific function called the Lagrangian of the system.

So, this is a stationary point of this action functional which is based on this kinetic energy and potential energy. So, in some gross sense one can see that the kinetic energy. So, for a system say let us say example of pendulum, when the pendulum motion happens typically along the path of the pendulum what is happening to the energy when you think about that. You can see that the energy remains constant for the pendulum, kinetic energy getting converted into potential energy and vice versa at different points.

For example, if the pendulum goes to towards the end, the potential energy is maximum, kinetic energy is 0. So, it becomes stationary they are in for a moment it is velocity is 0 and when it comes to the center and the potential energy is completely converted into kinetic energy and it has a maximum energy in the center, again it goes to other side.

That is how the pendulum can see that the total energy in some senses is conserved along the path and the same kind of idea is there in this form of integral form that is therefore the functional. So, if you take this point t_1 and t_2 are two states of the system say somewhere along the path of pendulum one can say corresponding to q_1 and q_2 as a generalized coordinates or q in the generalized coordinate case of pendulum will be θ of the angle of the pendulum.

So, between the two kind of θ as you will have this functional defined and now what we are interested is in finding the path between these two points of a system along which this functional as a stationary value, what it means as you put up say suppose instead of the circular path that we take some other path for θ not really.

So, the circular path remains the same. But in time see this q is defined as a function of time here. So, we take whatever function of time, some function of time for θ and that defines one path between the two θ points of our pendulum. But that may not be the final path, I mean the pendulum has some kind of a sinusoidal variation of θ. So, first let us assume for the discussion purpose a small angle θ kind of a variation.

So, if some path is linear that is defined and that is not a path where these functions will have a stationary point in the sense like the this function is minimum. So, what do you mean by the stationary point is like knowing when you take a derivative of this, the derivative will give you with respect to q with respect to the curve q In terms of time that the derivative will be equal to 0, first derivative will be equal to 0 which will give us a stationary point.

So, that is a kind of idea that goes into your. So, you have many different paths defined for this evolution of q and you choose a path in some way or mathematical, we will see what is mathematical way of doing that by using the variation principles, you choose a path along which this functional is stationary and to come to that point we will need to take like the partial derivative or δ of a S or variation of S with respect to q that will be 0.

Variation of S with respect to q will be 0 that will give us equation which is the Lagrange equation, we will not get into the derivation part of this equation. But we need to have this like a fundamental concept understood so as to see the applications. So, for example, in this case this L has only q, $\dot{q}$. and t, and the L is defined as kinetic energy minus potential energy.

So, if these q_s are, these are generalized coordinates and they are equal to the number of degrees of freedom. So, if you choose qs in such a way that they are not matching the degrees of freedom of the system, then we will be in trouble. So, we will have to kind of express the kinetic energy and potential energy in terms of generalized coordinates that is the most important aspect or point that one needs to remember.

So, as we proceed like we will talk a little more about this point. So, what we are going to see is not a derivation of this equation, but application of the Lagrange formulation. So, let us see what is the Lagrange equation that we get after like doing this variation principles applied to this functional to minimize for a general system of n degree of freedom kind of system?

How this functional minimization or variation of δS variation of S which is δS with respect to q_s will be equal to 0, what it gives as a many kind of mathematically derive it in terms of L that gives us this final equation.

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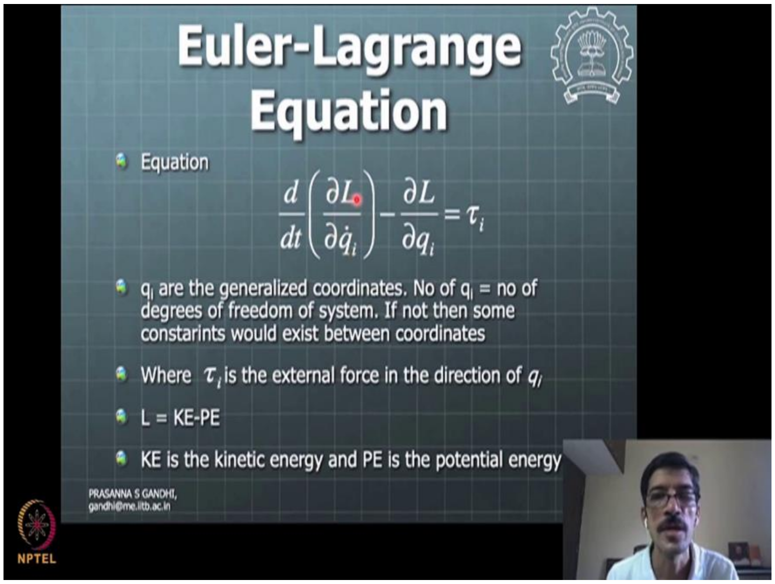

So, just see carefully this equation states that this L here has a partial derivative with respect to $\dot{q}$This is so, L will in general be dependent upon q, $\dot{q}$ and time explicitly. But here when we take for example, there it is a partial derivative with respect to $\dot{q}$ and here you have a partial derivative with respect to q and then this is a full derivative with respect to time this is not a partial derivative.

So, this is important to note here. So, as we apply this, things will become really more clear now. And q as I said our generalized coordinate, so generalized coordinates will be equal to the number of degrees of freedom of the system. And if they are not, then there will be some constraints that would exist between the coordinates.

So, for example, if we talk of pendulum system, my favorite system the θ is like a generalized coordinate we can take but x and y coordinate, if you express kinetic energy, potential energy in terms of x and y. Now x and y we know are not independent of each other for pendulum kind of a system because it is moving along a circular path.

So, that path gives some kind of a constraint between x and y. So, then this particular equation may not be applicable there is something else that we need to do in terms of if you want to use the coordinates x and y, separately, we can make a choice to express everything in terms of x or everything in terms of y considering that it is moving along the circular path.

That is okay, no problem but if you want to use x and y separately. Then that is not a good that is not possible with this kind of discussion. That we are doing in the class now. There is some other formation called constraint Lagrange formulation that will have to be employed for the case where we have x and y which are dependent upon each other. So, we have to have additional constraint that equation that is put in then some constraint Lagrangian needs to be defined and then we need to apply the lagrange information for a different setting.

So, now, we are looking at only a case where there is no constraint in the system and you have all the degrees of freedom having independent of each other as a differential degree of freedom says and your kinetic energy, potential energy are expressible in the form of generalized coordinates. These are some of the important considerations to apply for Lagrange formation. So, the L is defined as the difference between kinetic energy and potential energy.

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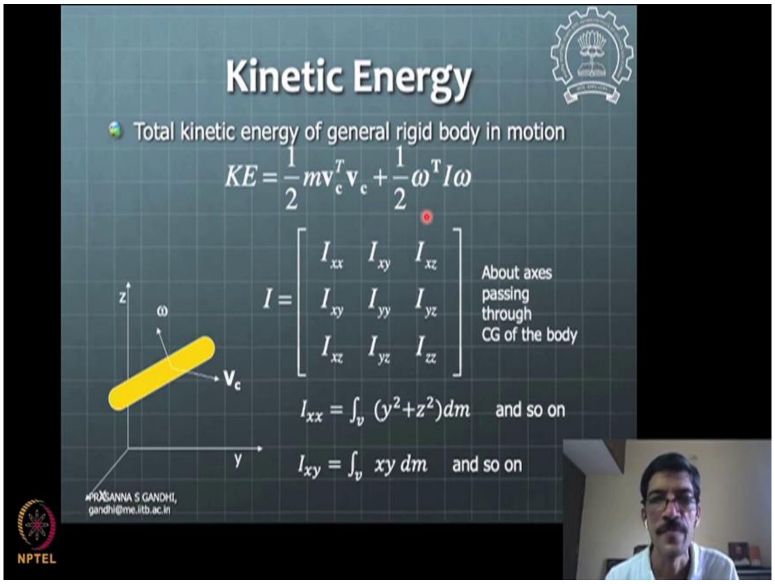

So, kinetic energy in general for rigid body system can be defined as you know from the concepts of kinematics and dynamics that we have seen is half *m v*, this *v* is of the center of mass of a DC point is a center of mass of rigid body. So, you just see the rigid body in motion it may be part of some mechanism or part of robotic system or part of whatever system, does not matter.

We just see a center of mass and a body the velocity of we ask the question velocity of center of mass of that rigid body and the kinetic energy is defined as mass times *v* transpose into *v* for the center of mass. These *v* we see is the velocity of the center of mass. So, it is like a half mv^2 but generalize now into this form for the rigid body or like in general kind of a motion in 3-D space.

And then it is this is not a complete kinetic energy. So, just a translation part of the kinetic energy and then you have this other rotational part of the kinetic energy I mean this we have seen in the kind of fundamentals. If you want you can just like just revise this kinetic energy expression revise some of the fundamentals of kinematics and dynamics and you can either...

This is expression, a general expression for kinetic energy when you have a rigid body motion having in the 3-D space and this omega transpose I omega where I is inertia matrix for the rigid body defined along the, define about like know the axis of the coordinate system passing through the center of gravity or center of mass of the body.

So, this is again another important point, we are not taking inertia, this inertia matrix about any other point in the system. So, you are taking it about the access passing through the center of mass of the body. These are the two terms that are coming in the kinetic energy expression and this inertias, these are mass moment of inertias are defined in this kind of a equation. So, this is like expression for general rigid body kinetic energy motion and then we can use that into Lagrangian formulation.

So, we will see now the application with some of the system. See potential energy one can, based on the some kind of reference one can define potential energy in terms of these different generalized coordinates. We will see how this definition of kinetic energy can be then finally expressing in terms of some generalized coordinates and support and then we can do some generalization in the later part of this lecture or maybe the next lecture.

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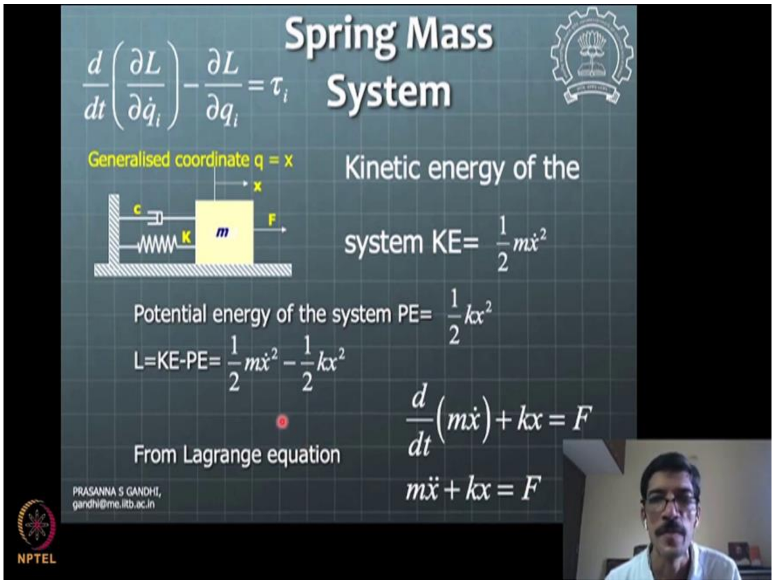

Now, we can see some simple examples for the formulation. So, let us take our famous spring mass damper system here. So, kinetic energy for the system is one can write very simply as half m $\dot{x}^2$ there is only one mass that is in motion we are considering the spring and damper to have like massless properties as compared to the end. So, this is all like this one has to think what are the considerations, what we are considering as degrees of freedom?

What are we or what part of the system we are modeling what is the dynamics of significant importance these are all user has to have that thing in their mind. What mechatronics engineer has to have a very clear understanding, look this is an important part of my system that is what I want to model and other parts I have to kind of ignore in the model, otherwise like you start thinking the spring also has a mass or like a damper as well as.

Where is the kinetic energy potential energy or like because of this masses, the kinetic energy because of these masses, which we are not considered here. So, then like one can consider of course, if we want to for formulation, but then whether it is really worth considering or not is a question. Because the terms that you are getting in the final equations are completely ignorable as compared to this term which is coming because of these big mass.

So, that those kind of calls is what user has to take and then he or she has to go ahead and define these energies. Now look here we are not really considering this damping in the system in any of these energies. Potential energy as from the first principles that energy stored in the spring is half times *k* times *x squared*. So, see the damper energy is not coming in these two equations.

So, where it will be, so late and then also this force is not coming into formulation of the Lagrange equation. So, these all terms like know these terms which are not coming finally in the energy expression they have to be get some are considered. In this generalized force along the direction of the coordinates.

So, this μ_i is a generalized force along the direction of q_i is a statement very important because we need to make sure that these forces along this direction if it is not then we need to find out what is a force along this direction by resolving the forces in appropriate kind of way. So, say for example, if I am applying these force to be in these direction, that is what is given, then I need to see what is the component of this force which is coming along the generalize coordinate which is x defined as in this case.

So, those are the kind of integrities that one has to be bothered about while applying this Lagrangian formulation to simple systems that we will consider. So, then we define this Lagrangian as kinetic energy minus potential energy and now you take a pause here and derive yourself what is the equation of motion coming when you apply this formula?

So, it is kind of going to be pretty straightforward because $\dot{x}$ is coming here only and then when you say δLby $\delta\dot{x}$ that time you get two $\dot{x}$ here and then you differentiate and everything like that.

So, you do that whole process and then after doing you come back. So, after coming back, you will see that this equation is of the sort, $m\,\ddot{x}$ plus $k\,x$ is equal to now this force F is in the direction of generalized coordinates x.

In the direction of x any other forces that exist would be bundled into this F here. And in addition to external force F that is applied on a system they see $\dot{x}$ damping also can be considered. So, this F can have value which is this F minus $c\,\dot{x}$. So, that is a generalized force that can exist on a system. Now, I have ignored the damping and I have just eaten this equation only with this F here.

So, one can consider also the damping in terms of energy part, which is called relay damping function. But we will introduce that later into the system. Most of the cases it is easy to see that the damping can be resolved in the direction of generalized coordinate and incorporated in the equation later as a external force, either way is fine.

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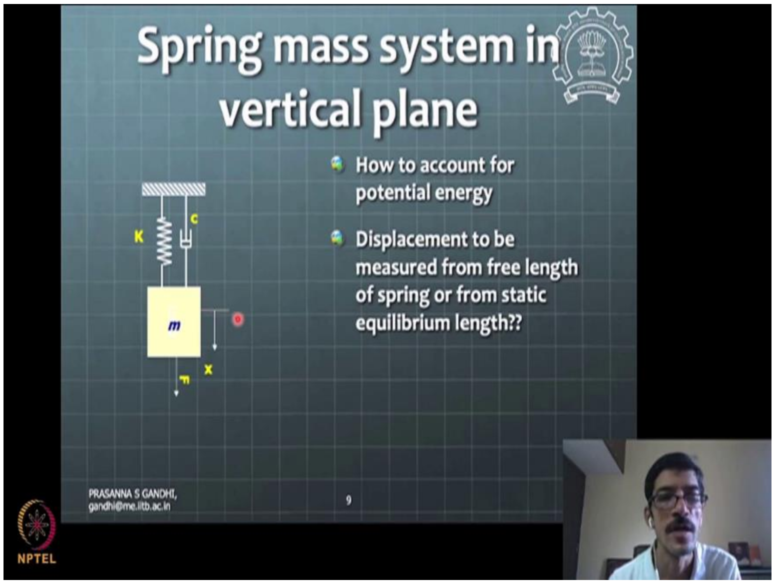

So, then we see some same system, but now if it is considered in a vertical plane, then how do you confer for potential energy in this case. So, it depends upon how you define your generalize coordinate to measure from. So, there are two possibilities, one is like you see that your generalized coordinate x is defined when the spring is having 0 extension.

That is one kind of a case and other kind of a case will be where the spring is having extension in such a way that the initial deformation of a spring balances the force mg. So, for the Lagrange formation we can consider both of the cases but in the energies that are to be considered then have to be different. So, in first case when you consider the spring to be having free lengthier from that is my reference point for *x* to begin.

Then you consider like a normal way all the forces of let us say now and then be additional potential energy because of the lowering of this mass that has to be accounted for and then you did you write like equations of motion in that kind of report that is a most. What you say general way to consider, that is the best way to consider the derivation in you start off with the spring having a 0 deformation.

That is how you should define there should be, stored energy in the elements to begin with that type that will give you formulation without any ambiguities and then other formulation where everyone can see that, if I deform the system allow me to kind of get into the equilibrium position with spring force balancing the mass then I defined that as my 0 and from there I start meeting *x*.

Then that from your Newton's formulation is also that then the additional change of the gravity or gravity force will not come into picture in the formulation because that initial balance of the spring force is already balancing the mass. So, then once that balance happens beyond that whatever displacement is happening exactly same as in the case of the horizontal version of the system.

So, I would suggest that you apply Lagrangian formulation for the case where you have free length of the system and you see the equations for yourself. And then if you have any doubts we can discuss more later. So, you do that, and then you proceed further recording.