Design of Mechatronic Systems
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Lecture 27
Lagrange formulation examples

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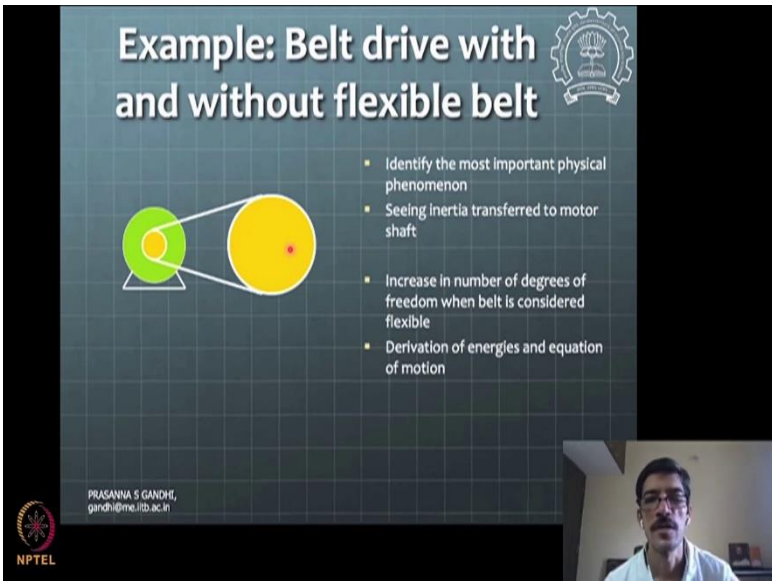

So, then we would consider now a next example of a belt drive with and without flexible belt. So, this is again a system wherein like you know that okay this is θ_m here and this is θ_l here. So, this part of this derivation, without flexible belt, we have seen in the previous class or we have seen previously one can see from there kinetic energy already be had expressed there.

So, you again, here also I would like you to kind of derive this in now. These are Lagrange formulations kind of a setting. So, potential energy in this case is 0, without flexible belt. So, without belt your potential energy is 0 and under that case if you apply your equation with the equivalent inertia. See, now, when you use generalized coordinate θ_m here.

The motion of the motor side θ_m, that is my generalized coordinate, if I defined it in that way, then I express my kinetic energy in terms of generalized coordinate and then I get like know this J equivalent naturally as a part of kinetic energy. So, half J equivalent which is equal to J_m plus J_l by N square that times theta m dot square, that will give me the kinetic energy expression and this is now J equivalent.

So, this J equivalent will naturally come and then when you apply Lagrange formulation. For this case, you will find the equation of motion coming up in terms of this J equivalent and things like that. And B equivalent, for example, here we will consider as an external force or we have to consider from the similar kind of energy principle that we saw in the last class, last time and previously what we have seen for the damping term we introduce that as a external force.

So, now, the question is how do you deal with this system when there is a belt which is flexible, which is to be considered flexible? So, what we do is, instead of like this kind of rigid connection, we say that okay there is a flexibility here. This part we have discussed a little bit in the last previous part also. So, then there is a flexibility or there is a spring here, then what are force that is getting applied to the belt by the motor will result in the tensioning of this or extension of this spring.

And that is how the θ for the load will no more be connected with θ for motor directly, it is not θ_m upon N will give you θ_l that is not possible now, so these become like an independent degrees of freedom. Now, if they are independent degrees of freedom, say θ_m is given θ_l is given as a independent degrees of freedom. Then, the question is can we find what is a extension in the spring, that will be, we will be able to find extension in the spring and once we have that extension in the spring we can express that as data, as an extension in the spring and half k x^2 will be our potential energy.

So, this is an additional part that will have, we will have previous kinetic energies and these additional potential energy of a belt that is we considered now and then we can use the Lagrange information and derive the equations. So, this part you can do as part of your work here, we can move on from here now.

So, you carry out this derivation and see what is the equation of motion that you get and one can interpret and once you get the equation of motion and you can interpret them in terms of Newtonian formulation, we are these, you are looking at the appropriate terms here later to become coming in the final equation. That exercise you can do. So, this becomes like a underactuated system. It has only one actuation for the motor but two degrees of freedom motor and the load, so we will see how these kind of systems can be dealt for the control.

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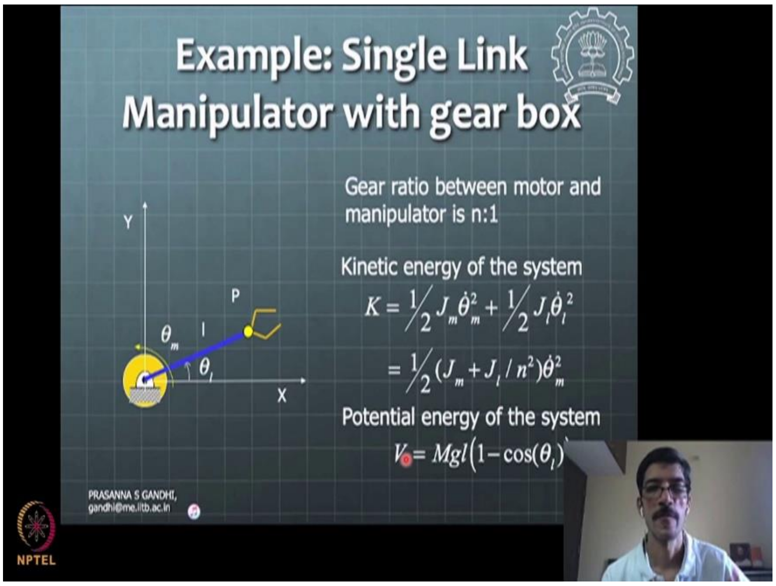

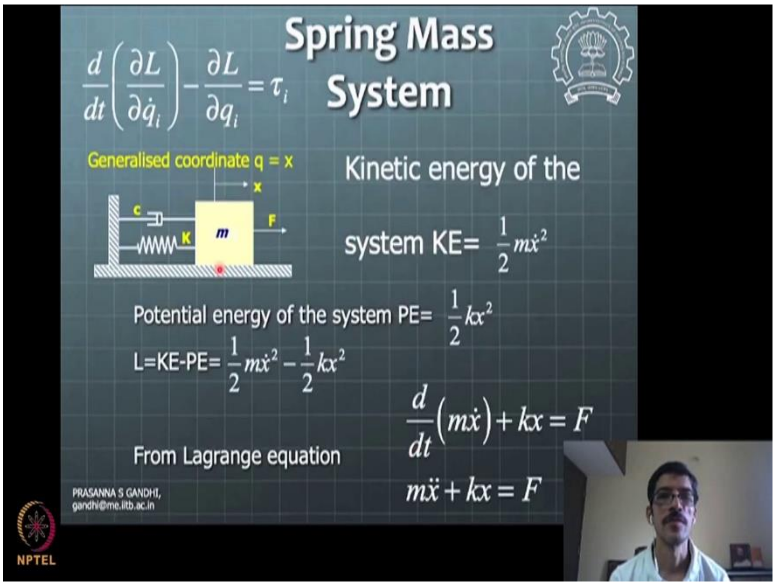

Now, we can further extend it for the other case, which is again another interesting problem. So, you have a gear attached to the motor shaft and to that output of the gear you have a link attached, this would be typically, this would be configuration for robotic systems. You have a motor and gear and then further you have some links coming up there, you may have multiple links for the robot of course, but we are considering now this is the most simplest kind of form for robotic manipulator.

So, the kinetic energy for the system can be written in terms of inertia of the motor side. So, this is $J_m \dot{\theta}_m{}^2$ and then you have an inertia of load coming up here, which is now hinged here. So, the inertia of this link that will be coming up as transferred to the hub, that will come up as a kinetic energy.

One can write this expression of kinetic energy in terms of this low inertia or one can say okay I will see that say this as a rigid body and for this rigid bodies the center of mass is here, for the center of mass, mass into velocity square half of that will be the transition kinetic energy and when I consider rotational kinetic energy and then whatever that total I get in terms of θ_l now alone, you have to express the translational kinetic energy of this rigid link also in terms of θ_l.

And then we can get this equivalent or this J about this point. So, one can directly consider that or one can see it in terms of your kinetic, translation kinetic energy and rotational kinetic energy separately. And then once you get this kinetic energy expression you use the connection between θ_m and θ_l. So, gearbox ratio will be the connection to express it in only in one generalized coordinate θ_m here.

So, this is typically an important step in the Lagrangian formation to see that the kinetic energy is expressed only in generalized coordinates. So, in this case it has only one degree of freedom system, you need to have only one generalized coordinate to be there in the kinetic energy, whatever you choose it can be θ_l also, that θ_l also can be generalized coordinate is also a possibility here. So, we have chosen these θ_m to be express here.

Now, if you see the potential energy, the potential energy of a system is Mgl 1 minus cosine θ_l. So, this can again be expressed this is, there is this l is a little different from this l because if you take a center of mass here the potential energy for that this should be the center of mass here. So, you have made that correction.

So, this potential energy is in terms of θ_l. But, we need to convert it in terms of θ_m again, so as to have consistency in the generalized coordinate. So, you define generalized coordinate as θ_m and then everything should be in consistency with that. Thats okay, all your kinetic energy and potential energy should be expressed in terms of this generalized coordinate. So, that is the main kind of thing that maybe you may miss.

We need to make sure that, okay all the time, the generalized coordinate energy expression should be in terms of generalized coordinate. So, you do this Lagrangian formulation and then you come back. I think you first do this formulation yourself and see that okay, what is the experience you define the Lagrangian to be kinetic minor potential energy, and then apply this equation what we saw here.

This equation if you apply you should be able to get the equation of motion for this case as well. So, do this exercise this is important, once you do you will understand how things work or some places you might get a miss out some things that will get corrected, and next time when you do will not make those kinds of mistakes.

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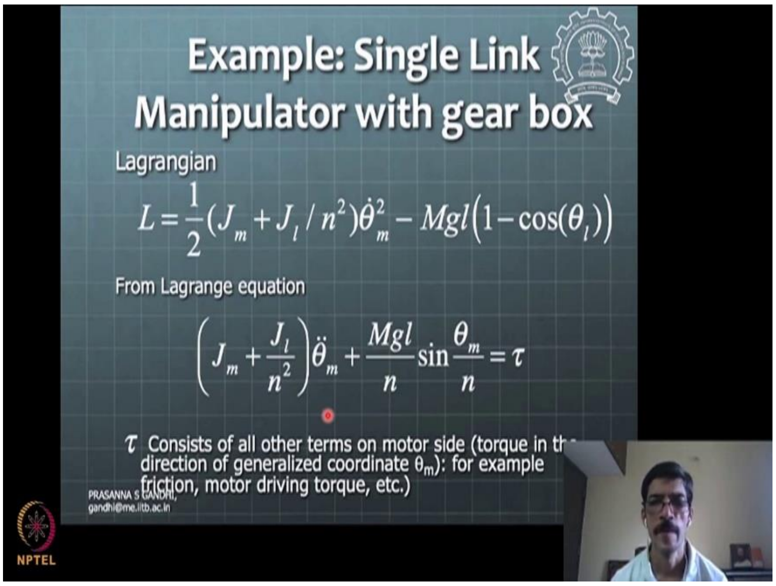

So, with that you will get this Lagrangian formulation like θ_l replaced by θ_m upon N, you get this kind of equation of motion. And for this τ is like your torque applied in the direction of generalized coordinate, θ_m. So, this is like a directly a motor torque here. So, this motor torque are, there are other torque that might be coming in.

So, say for example, if you want to integrate this now with the motor equation, how will you do that, think about that. So, it will be fiction torque and then there will be other torque that are coming. So, we have already incorporated, accounted for the motor inertia here, J_m is already considered here. So, considering all these, one can see this torque can be a consisting of missing torques that are not considered, so this inertia torque is considered for example, damping is not considered here.

So, those are the kind of things that one can think about to incorporate into these external forces in the direction of a motor, motor generalized coordinate, in the direction of the θ_m. So, those kinds of forces will come incorporate that and then you may be able to get complete equation and this can be integrated with for example with over motor equations. The motor equations we have derived in the previous classes.

So, that can be integrated with this equation to get appropriately the entire dynamics. So, I leave this exercise for you to do. You can see that in the absence of this torque. There is no torque that is applied by the motor. So, this term will be 0 here, when this term is 0 you can see clearly that this becomes and also there is no friction. This torque consists of all this friction and other forces towards, all the forces are 0.

Then this gives you the equation of a pendulum only. So, if you do not have even the gear box then it will be simply $J_m \ddot{\theta}_m$ plus Mgl upon n sine theta m upon n here which is nowadays n is 1, there is no gearbox, so these are one quantities and then this is J_m plus J_l it will become, sorry it will not be, this cannot be 0, but this will be 1 here, a gearbox is not there means like you have a direct transmission.

So, this n is equal to 1, then this falls into, reduces to equation for pendulum, without gearbox, with the gearbox you can see this n coming up here and things like that. Like that one can have this physical interpretation of final equation formed and say that this makes sense.

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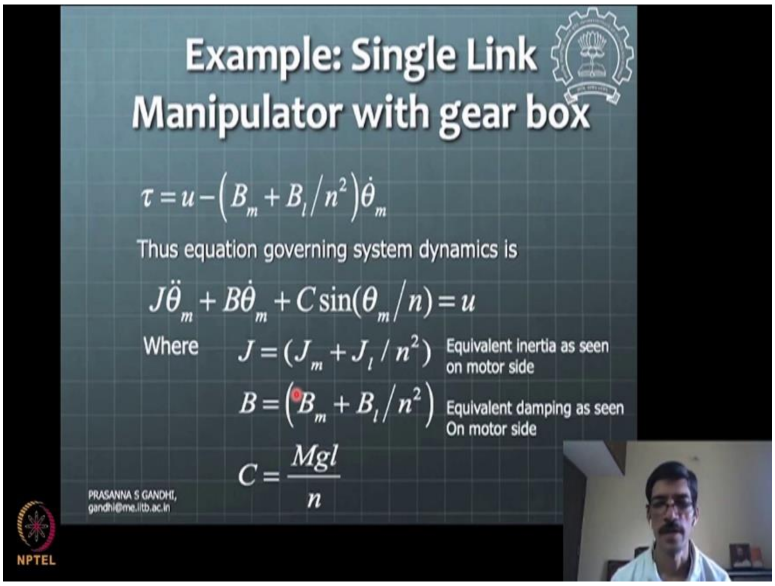

So, now this can be further developed by incorporating these other terms also into this equation, this damping term especially and you can additionally have, in the coulomb and other kind of friction models that can be incorporated, so that I will leave you to look at.

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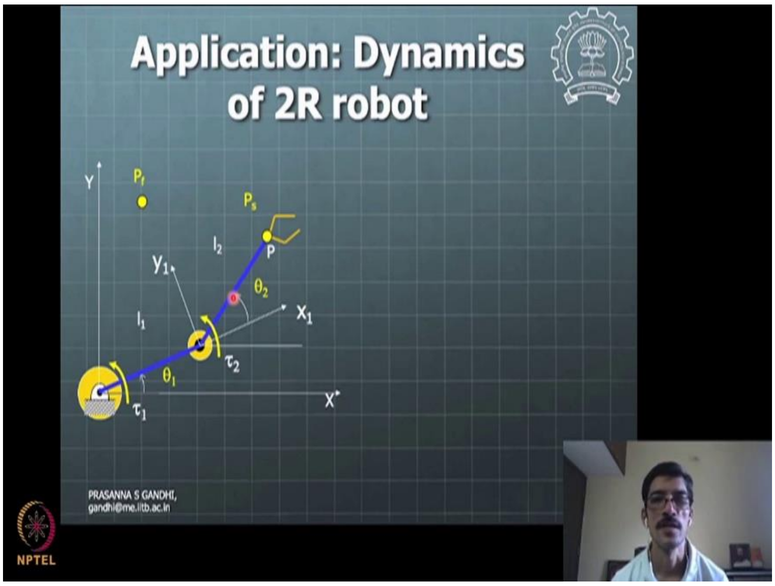

Then we can see this other application which is dynamics of these 2R manipulator. So, that is, so I would suggest here again you can take this kind of configuration, pause here and think of your fundamentals from kinematics and dynamics, and determine what are the energies here, kinetic energy and potential energy. So, kinetic energy for that you need a center of mass of these linkages and at that point you need a transmission and rotational kinetic energy.

So, kinetic energy from the center of mass velocity and rotational kinetic energy from the rigid body ω, angular velocity of that particular rigid body. Notice here that this θ_2 is measured in the coordinate frame which is moving here. So, while writing energies you need to confer that and write proper expressions because this is not, this θ_2 is not defined with respect to the stationary x axis.

So, stationary, with respect to stationary x axis is that the angle will be θ_1 plus θ_2 for this link. This is done because θ_2 is deformation or is not deforming, is angle of rotation of the motor that is existing at this point. So, if we have a robot, which is like now having these multiple motors attached to its different- different joints, then we typically need final dynamics in terms of the joint motions, rather than like, otherwise, we will have to find out the joint motion from the other coordinates.

If we defined θ_2 differently, I will have to find out what is the joint motion? So, at this point, let us you just pause and derive the equations for velocity first. So, we just need velocity, we do not need an acceleration analysis here, we just need a velocity analysis here and get the expressions for total velocity and further kinetic energy in terms of those velocities.

So, velocities of the center of masses of these two linkages, is what we need to get from our kinematics. So, how do you go about that? You can write the coordinates of x, x and y coordinates of the CG point here. Given all the geometry parameters, and same thing you can do for the CG point here. For second link, what are the coordinates x and y for the CG point for the second link.

And based on those coordinates, once you get the coordinates, you can say x_{CG2} and y_{CG2}, then you differentiate that to get $\dot{x}_{CG2}$ and $\dot{y}_{CG2}$ and your total velocity of CG will be square root of x^2 plus y^2, $\dot{x}_{CG2}{}^2$ plus $\dot{y}_{CG2}{}^2$. So, that is how one can get total velocity. And once you get those velocities, or velocity squared, you can use that in the expression of kinetic energy for writing kinetic energy of this. So, please do this derivation yourself.

And then you can proceed to look at the further developments that they are discussing. So, I will pause here and discuss this further in the next part of the lecture. So now we will continue with our topic of discussion of Lagrange formulation with this to 2R manipulator and then we will continue further about some of the properties, very interesting properties that these mechanical systems yield with the Lagrange formulation.

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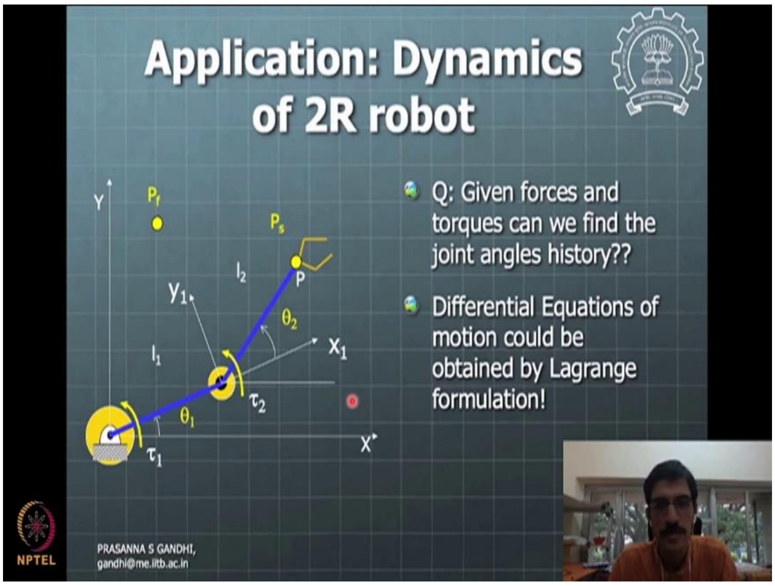

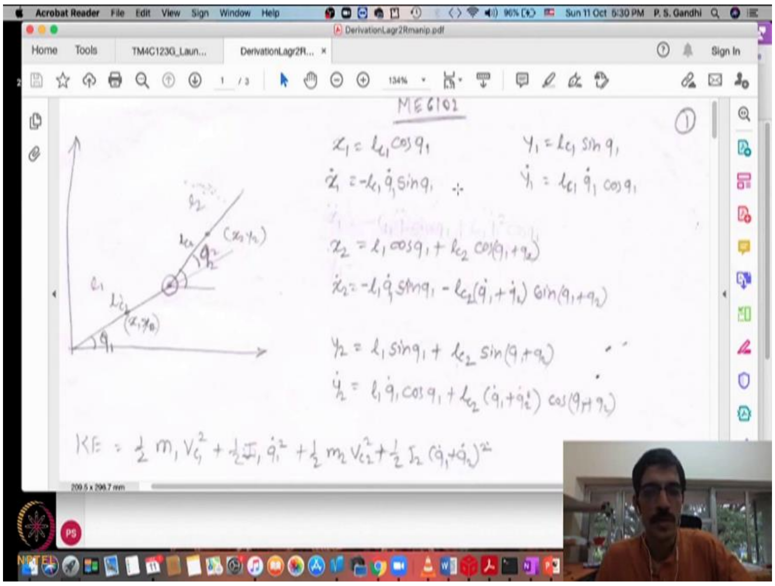
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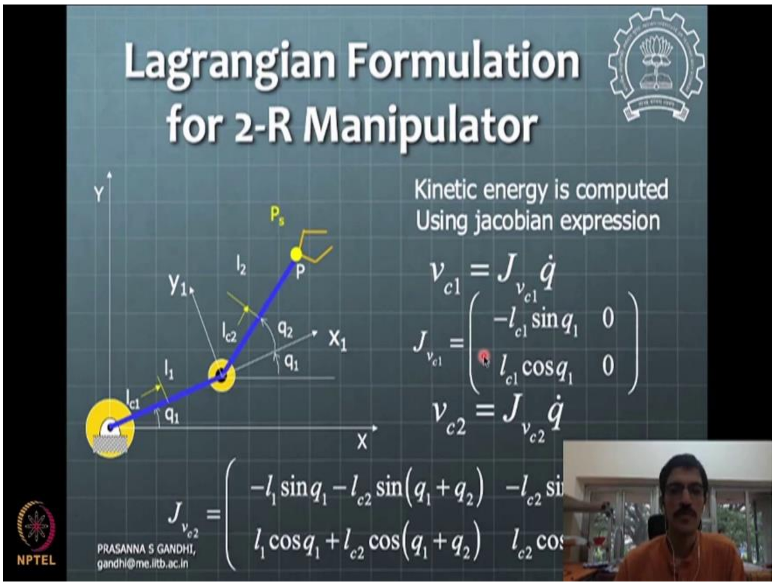
$x_1 = l_{c1}\cos q_1$
$y_1 = l_{c1}\sin q_1$
$\dot{x}_1 = -l_{c1}\dot{q}_1\sin q_1$
$\dot{y}_1 = l_{c1}\dot{q}_1\cos q_1$
$x_2 = l_1\cos q_1 + l_{c2}\cos(q_1+q_2)$
$\dot{x}_2 = -l_1\dot{q}_1\sin q_1 - l_{c2}(\dot{q}_1+\dot{q}_2)\sin(q_1+q_2)$
$y_2 = l_1\sin q_1 + l_{c2}\sin(q_1+q_2)$
$\dot{y}_2 = l_1\dot{q}_1\cos q_1 + l_{c2}(\dot{q}_1+\dot{q}_2)\cos(q_1+q_2)$
$KE = \frac{1}{2}m_1 v_{c1}^2 + \frac{1}{2}I_1\dot{q}_1^2 + \frac{1}{2}m_2 v_{c2}^2 + \frac{1}{2}I_2(\dot{q}_1+\dot{q}_2)^2$


Lagrangian Formulation for 2-R Manipulator
Kinetic energy is computed Using jacobian expression
$v_{c1} = J_{v_{c1}}\dot{q}$
$J_{v_{c1}} = \begin{pmatrix} -l_{c1}\sin q_1 & 0 \\ l_{c1}\cos q_1 & 0 \end{pmatrix}$
$v_{c2} = J_{v_{c2}}\dot{q}$
$J_{v_{c2}} = \begin{pmatrix} -l_1\sin q_1 - l_{c2}\sin(q_1+q_2) & -l_{c2}\text{si} \\ l_1\cos q_1 + l_{c2}\cos(q_1+q_2) & l_{c2}\text{cos} \end{pmatrix}$
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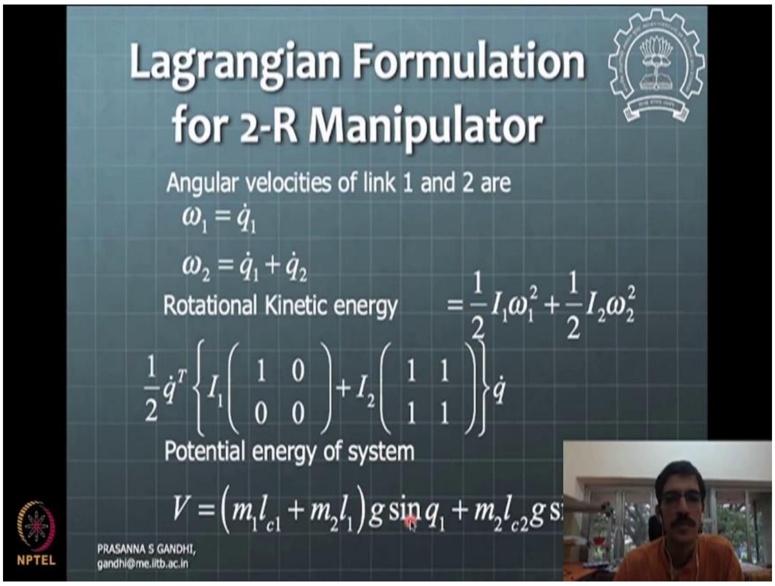
Lagrangian Formulation for 2-R Manipulator
Angular velocities of link 1 and 2 are
$\omega_1 = \dot{q}_1$
$\omega_2 = \dot{q}_1 + \dot{q}_2$
Rotational Kinetic energy $= \frac{1}{2}I_1\omega_1^2 + \frac{1}{2}I_2\omega_2^2$
$\frac{1}{2}\dot{q}^T\left\{I_1\begin{pmatrix}1 & 0\\0 & 0\end{pmatrix} + I_2\begin{pmatrix}1 & 1\\1 & 1\end{pmatrix}\right\}\dot{q}$
Potential energy of system
$V = (m_1 l_{c1} + m_2 l_1)g\sin q_1 + m_2 l_{c2} g\,\text{s}$
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So, let us see the screen, let me get the pointer now right. So, for this formulation I had asked you to do in the as a part of your own activity, I hope you would have done that. Please see that you have done yourself before getting into seeing the further thing, because that will kind of give you the real feel of what is happening. And also, any mistakes you have kind of made along the path they will get cut off rectified and that kind of opened up some concepts.

So now, for such a manipulator, you can use your kinematics fundamentals and get to these equations for the velocity. For example, the velocity of the CG, V_{c1} is given here in terms of the coordinate q_1 here. So, this is a new concept called Jacobian, which expresses velocities of interest. This velocity is *c1* in terms of the velocity is of generalized coordinates.

So, this generalized coordinate q here. So, q_1 and q_2 are generalized coordinate. So, q_1 and q_2 dots will be formed by this vector and this matrix when which multiply by that you will see this velocity getting represented in this kind of form. If you just work out based on using our kinematics and dynamics fundamentals, then also you can see the same thing, let me get to that.

So, if you see here, these are the equations that you will start writing and you can see this q_1 given here and q_2 here. Based on that you can write the coordinates of the CG of the links x_1, y_1 and x_2 y_2 for link 1 and 2. And you are seeing the here there are the coordinates written here and x_2 y_2 are written over here, x_2 and y_2. And then further differentiation of that will give you $\dot{x}_2$ $\dot{y}_2$and $\dot{x}_1$ $\dot{y}_1$.

And you just take out this $\dot{q}_1$ out of this, and $\dot{q}_2$ is not there in the exponent case. So, that those terms will be 0 and you can form this Jacobian matrix as we have seen in the last part. So, this part. So, that is how you get this the velocities in terms of a Jacobian matrix. And this Jacobian has some value.

We will see what value it has, especially for the robotic systems it has significance when the Jacobian becomes singular matrix then you will have a problem in operating that robot. So, maybe that much is now in further discussion of Jacobian, but this is an interesting way of concept and write the equations of velocities of linkages, especially the CG here for the sake of writing kinetic energies.

So, that sounds you then develop the kinetic energy and in the Jacobian form and further you get a rotational form added to it and then this traditional form becomes based on the angular velocities of linkages, link 1 and link 2. You add that in here they are kind of added in some kind of form which can be like the matrix multiplication representation here. So, this rotational kinetic energy, I mean you can see that this is same as what this expression gives you. So, it is just written into a, by considering this q_1 and q_2 is in this kind of a form.

So, this may be useful for some cases when you kind of start developing things for more complicated systems. Then potential energy of the system is straightforward, you can write based on the relative CG locations and mass multiplied by the length and by the gravity. And you get this potential energy of the system.

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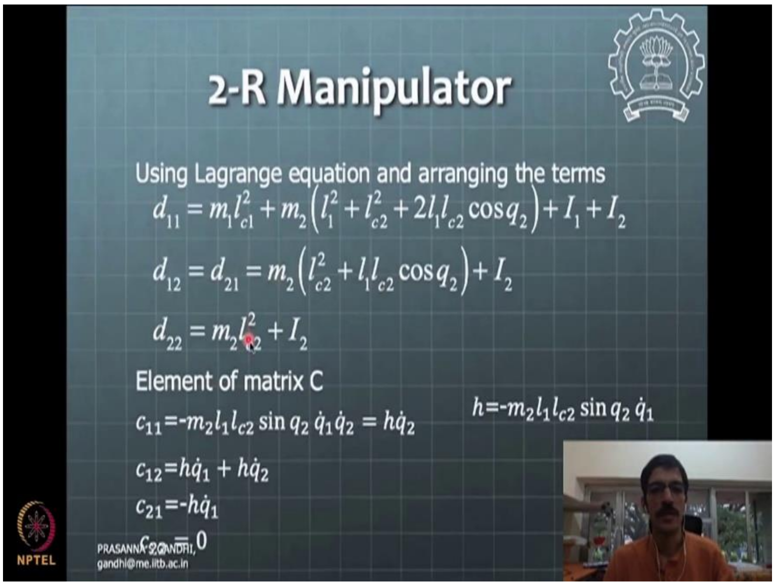
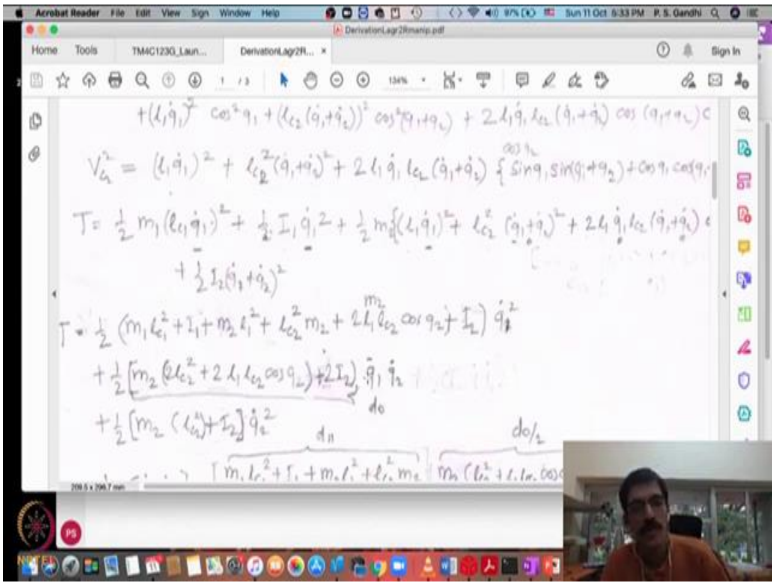




$$= \frac{1}{2}[\dot q_1\ \dot q_2]\begin{bmatrix} d_{11} & d_0 \\ 0 & d_{22}\end{bmatrix}\begin{bmatrix}\dot q_1 \\ \dot q_2\end{bmatrix}$$

(2)

$$T = \frac{1}{2} d_{11}\dot q_1^2 + \frac{1}{2} d_0 \dot q_1 \dot q_2 + \frac{1}{2} d_{22}\dot q_2^2$$

Always (!) quadratic form w.r.t. $\dot q$s?

$$V = m_1 g\, l_{c_1}\sin q_1 + m_2 g\,(l_1 \sin q_1 + l_{c_2}\sin(q_1+q_2))$$

$$= (m_1 l_{c_1} + m_2 l_1) g \sin q_1 + m_2 g\, l_{c_2}\sin(q_1+q_2)$$

$$L = T - V = \frac{1}{2} d_{11}\dot q_1^2 + \frac{1}{2} d_0 \dot q_1\dot q_2 + \frac{1}{2} d_{22}\dot q_2^2 - [(m_1 l_{c_1} + m_2 l_1) g\sin q_1 + m_2 g\, l_{c_2} \sin(q_1+q_2)]$$

$$\frac{\partial L}{\partial \dot q_1} = d_{11}\dot q_1 + \frac{1}{2} d_0 \dot q_2$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot q_1}\right) = d_{11}\ddot q_1 + \dot d_{11}\dot q_1 + \frac{1}{2} d_0 \ddot q_2 + \frac{1}{2}\dot d_0 \dot q_2$$

$$= d_{11}\ddot q_1 + 2 m_2 l_1 l_{c_2}(-\sin q_2)\dot q_2 \dot q_1 + \frac{1}{2} d_0 \ddot q_2 + \frac{1}{2}(2 m_2 l_1 l_{c_2}(-\sin q_2)\dot q_2)\dot q_2$$

$$= d_{11}\ddot q_1 + \frac{1}{2} d_0 \ddot q_2 - 2 m_2 l_1 l_{c_2} \sin q_2\, \dot q_1 \dot q_2 - m_2 l_1 l_{c_2}\sin q_2 \dots$$

$$\underbrace{\qquad}_{d_0}$$

$$+ \frac{1}{2}[m_2(l_{c_2}^2) + I_2]\dot q_2^2$$

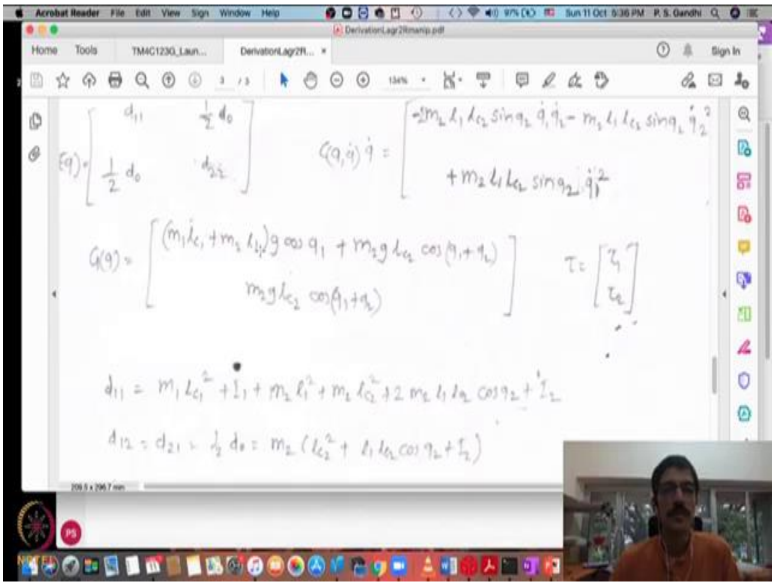
$$= \frac{1}{2}[\dot q_1\ \dot q_2]\begin{bmatrix} \overbrace{m_1 l_{c_1}^2 + I_1 + m_2 l_1^2 + l_{c_2}^2 m_2 + 2 m_2 l_1 l_{c_2}\cos q_2 + I_2}^{d_{11}} & \overbrace{m_2(l_{c_2}^2 + l_1 l_{c_2}\cos q_2) + I_2}^{d_0/2} \\ m_2(l_{c_2}^2 + l_1 l_{c_2}\cos q_2) + I_2 & \underbrace{m_2 l_{c_2}^2 + I_2}_{d_{22}} \end{bmatrix}\begin{bmatrix}\dot q_1 \\ \dot q_2\end{bmatrix}$$

$= \frac{1}{2} d_{11}\dot q_1^2 + \frac{1}{2} d \dots$

can also be expressed as

$$= \frac{1}{2}[\dot q_1\ \dot q_2]\begin{bmatrix} d_{11} & d_0 \\ 0 & d_{22}\end{bmatrix}\begin{bmatrix}\dot q_1 \\ \dot q_2\end{bmatrix}$$

Always (!)



And then crank in the Lagrange equations. You do it by yourself, I will show you some part of the derivation here, you can pause and look at it, if at all, if you have to cross check your own data and stuff, you can do it in this fashion here. So, look, here we have a written expression for kinetic energy. Now, not in the Jacobian form, but let me complete expressions as you get and then they are simplified here V_{c1} squared V_{c2} squared there is some simplification that happens there.

And then further you go and like develop this complete form of kinetic energy here and start now collecting some terms to kind of expresses kinetic energy in the form something times $\dot{q}_1{}^2$ only, then something times like $\dot{q}_1\ \dot{q}_2$ terms here. So, these are we are kind of generating some kind of a quadratic form for the kinetic energy expression here.

So, this has also a value when that is to kind of see that we will, we will see what is, we will discuss what is this value that we had by using this kind of a form. So, one can write in kinetic energy directly, not directly, I mean by simplification, one can write the kinetic energy in terms of this vector here. And then this matrix multiplication and again, this another vector here. The vector is of the velocities, joint velocities.

So, this is the velocity, this is a total kinetic energy of the system expressed in the form of a generalized coordinate velocities and in general, these internal terms are functions of q_1, q_2 and so on and so forth, they are not functions of $\dot{q}_1$ and $\dot{q}_2$, and that is to the point to be noted. Once you get this kind of a kinetic energy expression then you further start developing the Lagrange formulation you would write the expression for potential energy here.

And then express the Lagrange and by kinetic energy by potential energy and further you start differentiating and this differentiation gives you different-different terms in the expression. So, while differentiating, if you are expressing in terms of like this d_{11} coefficients here, we need to make sure that these coefficients are further functions of their maybe for further functions of q.

So, wherever this full differentiation may come like this term need to be considered accordingly. So, say for example, this d_{11} term has this cosine q_2 part inside it. So, when you are differentiating with respect to time then your additional part of the terms will be coming there. So, this is like you can see here, $\dot{d}_{11}$ terms is coming here and you can confer for that accordingly.

So, like that you keep on doing this formulation and economic simplification of algebraic simplification of the terms considering all different derivatives that exist in Lagrange formulation and then for q_1 part you get this particular equation for q_1 alone, variable q_1, then you get another equation for q_2 part by considering the differentiation or generalized coordinate q_2 for Lagrange equation and then you get this second part of the equation, then you assemble these equations in certain kind of a form.

And why we do that, we will see later but the form is you consider this matrix Dq to be having this kind of a form then $Cq\,\dot{q}$ to be having some form here and then these Gq terms are gravity terms coming because of the gravity forces in the potential energy and you have a torque factor here and with this expression, you had a completely Lagrangian equation will get developed. So, this will get developed in the matrix form.

So, we will further see what are the properties of these matrices little later, but you are expressing this particular in this form so for some reason, okay and we will see that reason in a while. So, this is a formulation of 2-R manipulator that we have seen here and the same thing is put up here. Now, with this formulation, we further move on to see some of the interesting properties of these Lagrange formulation.

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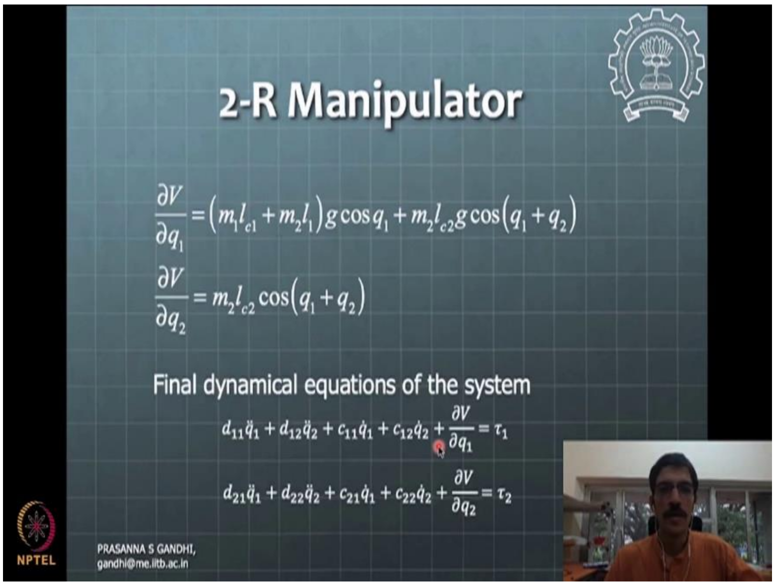

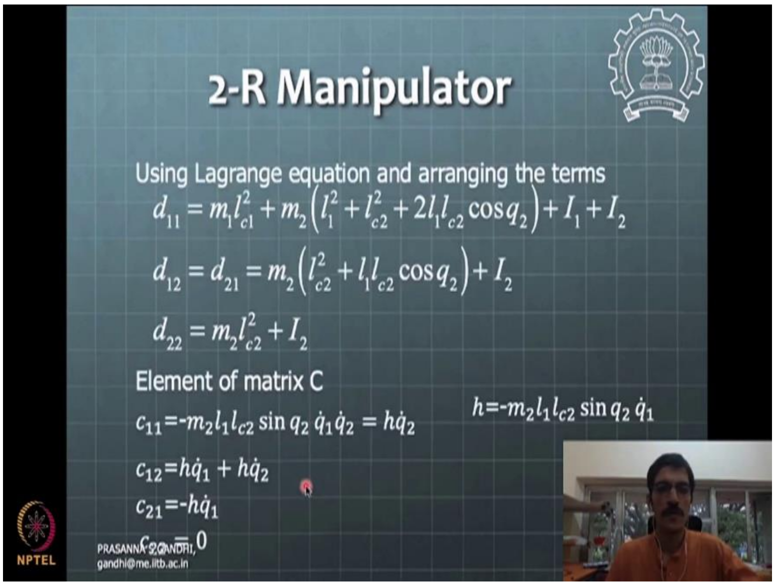

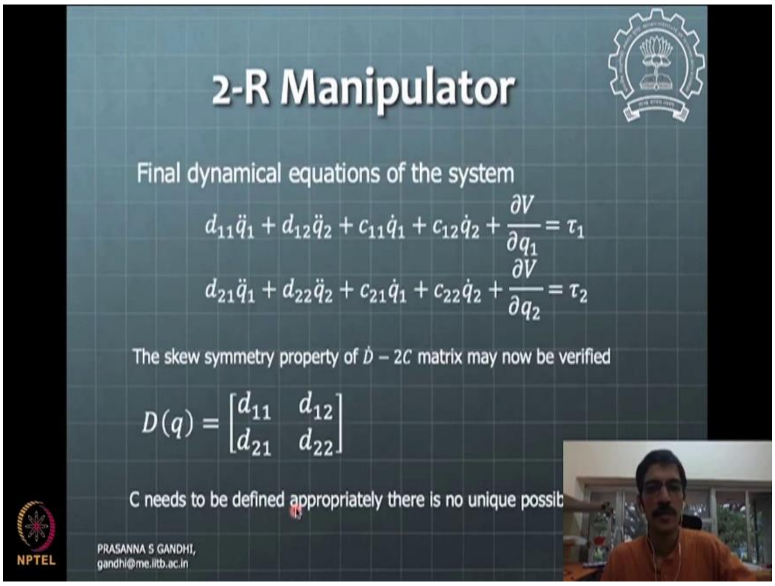

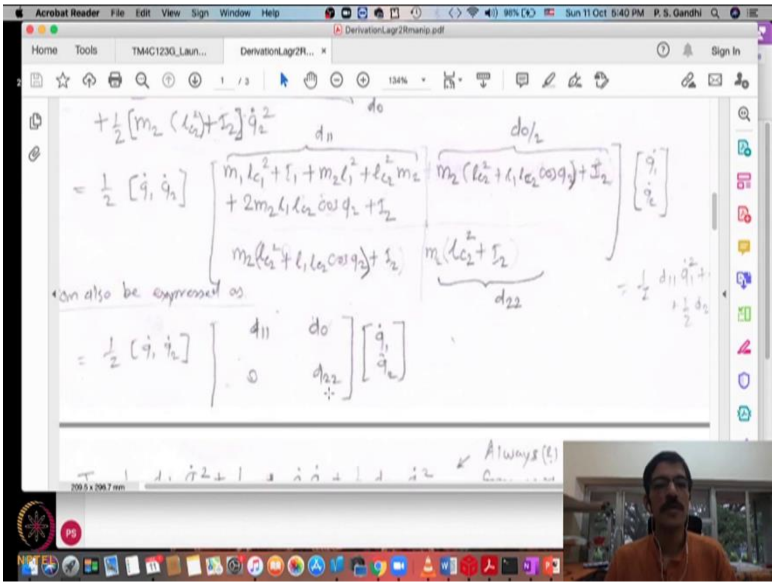

So, the equations are expressed in this form and as you have seen this d_{11}, d_{12} terms are defined as here also and in the other long hand derivation also. And these are the C terms mentioned here and with this you have these complete equations of dynamics of the manipulator derived. Now, if you notice here, this d_{11}, d_{12}, d_{21} and d_{22} form some kind of a matrix multiplying $\ddot{q}_1$ $\ddot{q}_2$ in the matrix equation form.

So, this is a *D matrix* here and then this *C matrix* we define out of these terms here, which are not having any double derivatives terms. And not, we are not considering damping in the system, no loss of the energy in the system and then there are these gravity terms that are coming here. So, with this there is this, there is a very interesting property called skew symmetry property of $\dot{D}$ minus *2C* matrix, which will come, which you can verify that or validate that property.

But for *D matrix* it says like know you have this property coming up here that this *D matrix* is a symmetric matrix and it is positive definite matrix. So, this is the invertible *D matrix* that we come here. So, we will look at this little more detail as a generalized thing, but you need to notice that this *D matrix* is coming out of the energy expression.

So, without even Lagrange formulation, one can write this *D matrix*, as once you write the kinetic energy of the system in the form that was expressed in the equations here. You can see this form of equation of kinetic energy was written in this form. So, d_{11} this was a d_0 term here, which can be split into two parts, one part can be written here other part can be written here which will be both same parts.

So, this d_0 by 2 you can come. So, this form of the expression, these terms directly appear in the expression of the Lagrange formulated equations of motion. So, that is what you is important thing to notice here and one can understand very easily why they would come once you have this energy expressed in this particular form. So, maybe we will stop here for now.

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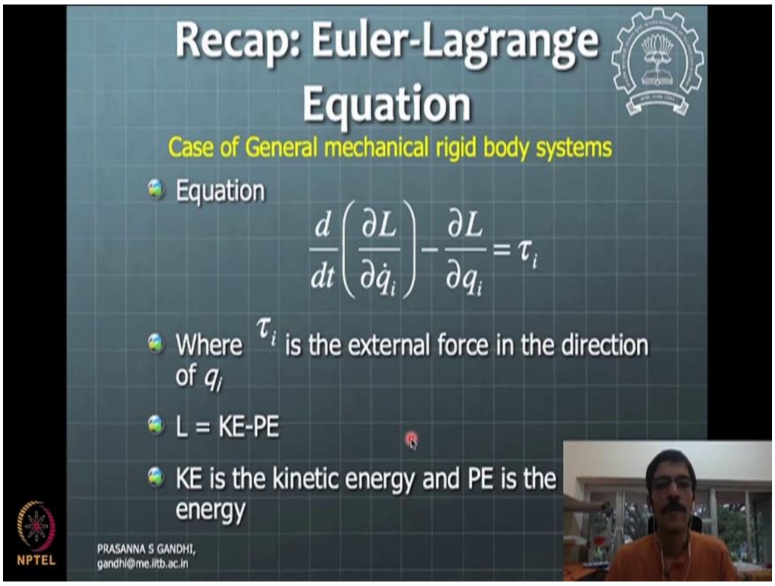

And we will continue the discussion of this generalized form of this rigid body equation in the next part of this lecture. So, you go ahead and make sure that you understand this application to 2-R manipulator, so that this generalization can be understood in a relatively easy way, so that is what I would suggest you to do. And we will stop for now here.