Design of Mechatronic Systems
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Lecture 28
Dynamics: 2-R Manipulator

So, let us get started with the second part of the, this lecture. This is focused on generalization of some of the concepts that we started off developing with 2-R manipulator kind of a case. So, general mechanical system if we consider, how its Lagrange formulation gets developed and what are the properties corresponding to that. So, we will just go back and notice once again the expression for the kinetic energy that we got.

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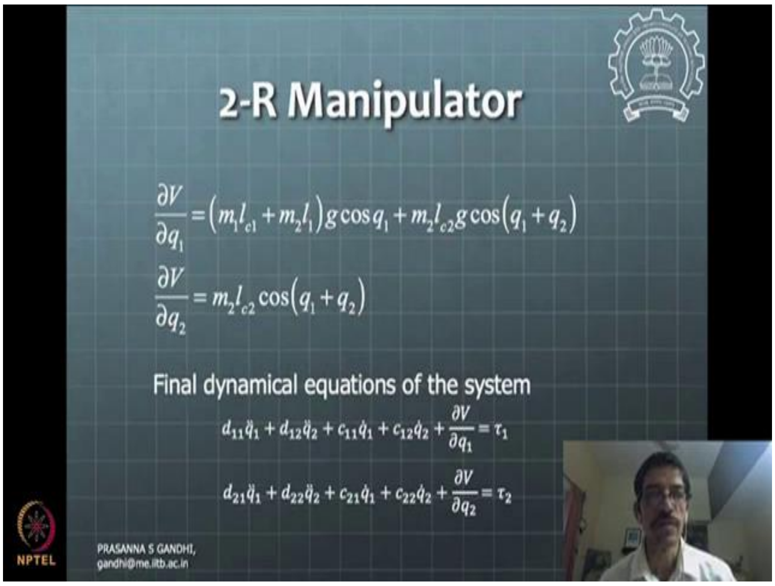

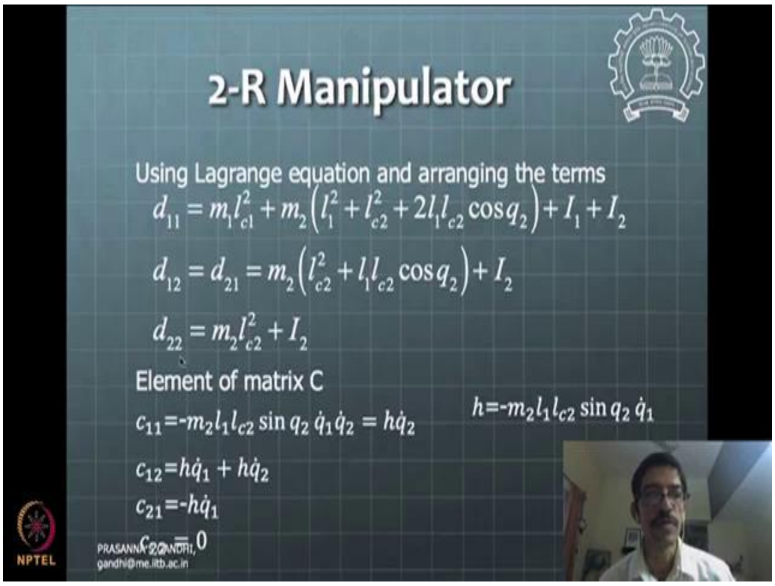

So, let us go into the slides. So, you see this kinetic energy expression that was there for 2-R manipulator was expressed in this, terms of this d_{11}, d_{12}, d_{22}.

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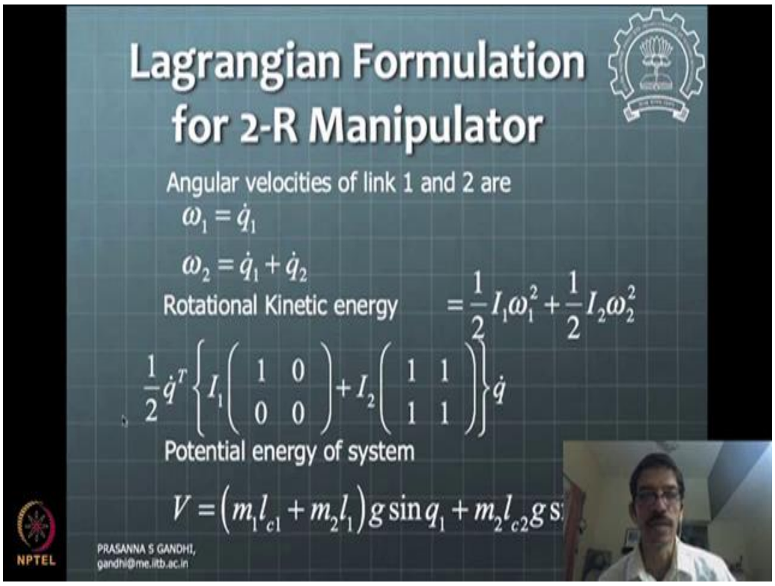

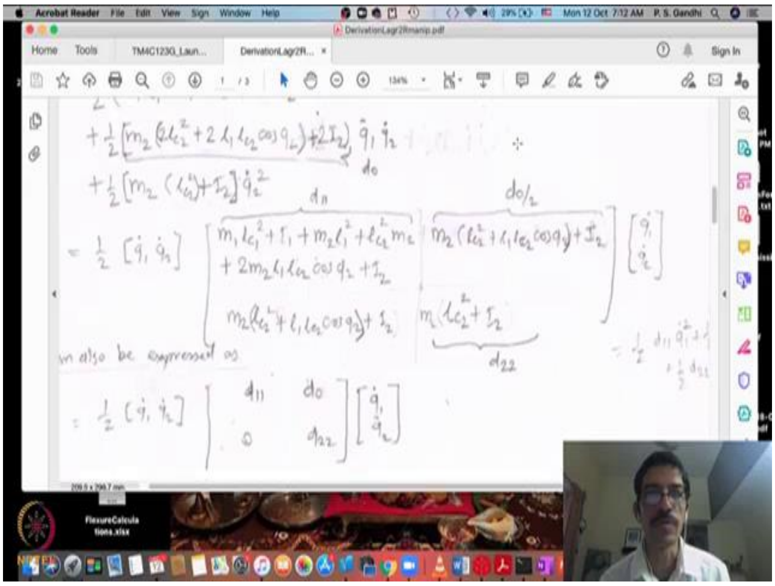

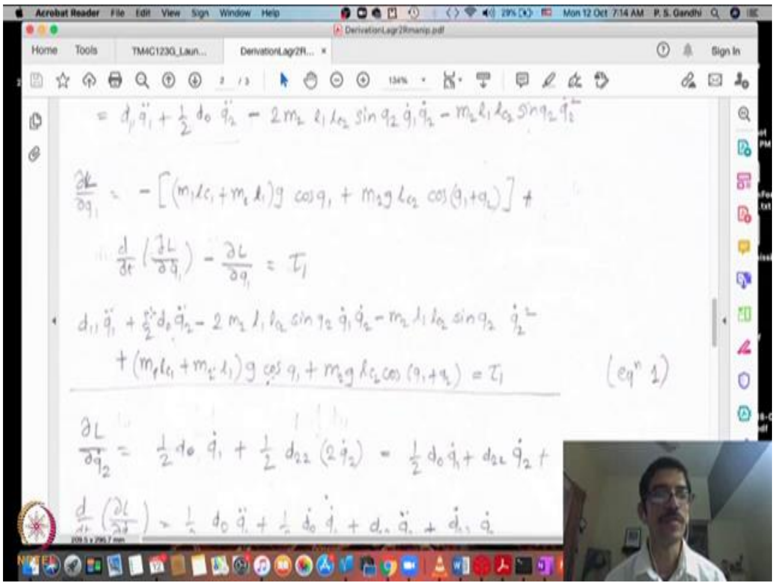

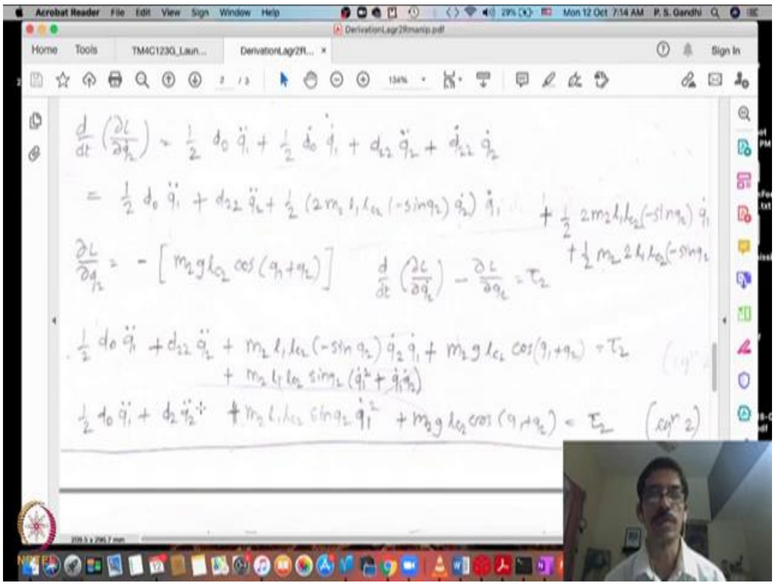

So, this is part of this kinetic energy expression, total kinetic energy expression. So, this is, you can see that from equations derived here. So, here you can see that the kinetic energy is expressed in this d_{11}. This d_{12} is basically this d_0 by 2 term. This d_0 is given here, expression is given here. So, this d_0, d_{11}, d_{12}, d_{21} and d_{22} these are like four terms in this kind of a quadratic form matrix and then you have this $\dot{q}_1$, $\dot{q}_2$ multiplying here, transpose and the $\dot{q}_1$, $\dot{q}_2$. So, this is like a quadratic form of expression for kinetic energy. And you will find that all the mechanical systems kinetic energy can be expressed in this kind of a form.

Now, this form gives us a clear indication that this matrix which forms this kinetic energy by this quadratic form matrix is going to be positive definite. Can you see that, why it is going to be positive definite?. Pause for a while and check why this matrix need to be positive definite, because this is a quadratic form. If you express for a quadratic form of a matrix, then if the quadratic form has a value which is greater than 0, quadratic for is a scalar value. So, that value is greater than 0, then the matrix is positive definite.

So, the kinetic energy of any system will always be positive. From the simple kind of physical analogy, one can see that this matrix is going to be positive definite. So, this is one of the properties of this matrix for *d* is it is positive definite.

Now, you can see that the same matrix appears in the final expression of the equations. This equation 1 if you see this term here this is $d_{11}\,\ddot{q}_1$ and half d_0 or $d_{12}\,\ddot{q}_2$. These terms appear here as well in 1 equation. And in second equation, it is again half d_0 $d_{21}\,\ddot{q}_1$ and $d_{22}\,\ddot{q}_2$. So, this is a way like this *D matrix* appears in the final equations. So, the moment you write kinetic energy, you know the part terms of the final equation.

Even the second part has some kind of interesting mathematical expression in terms of this D matrix as well and maybe we will have a potential energy terms also coming in there. So, we will see that. So, this can be generalized, in general for any mechanical system. When we say mechanical system, what we mean is that the systems which are expressible in the form of kinetic energy for mechanical systems. So, we will just get little bit details into that now. So, let us go back to our slides.

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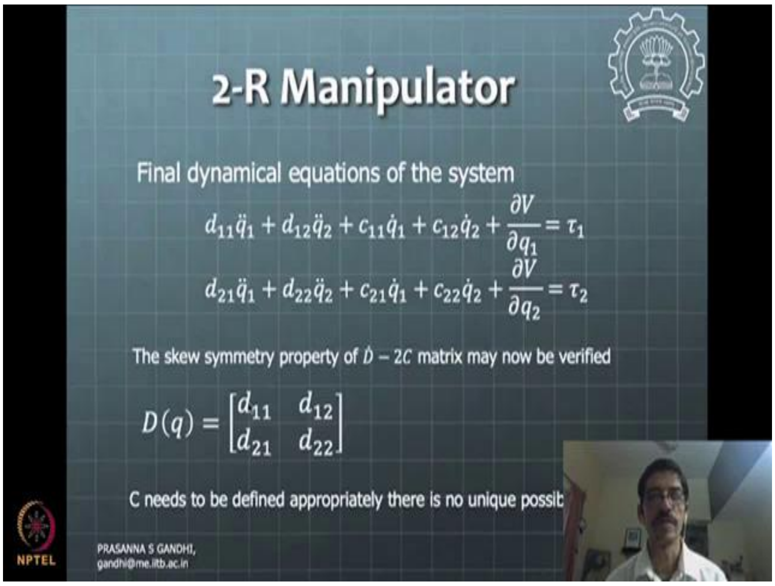

So, as we see this kinetic energy gets expressed in terms of this *D matrix* and this *D matrix* now is positive definite. So, up to that we see like why the D matrix should be positive definite.

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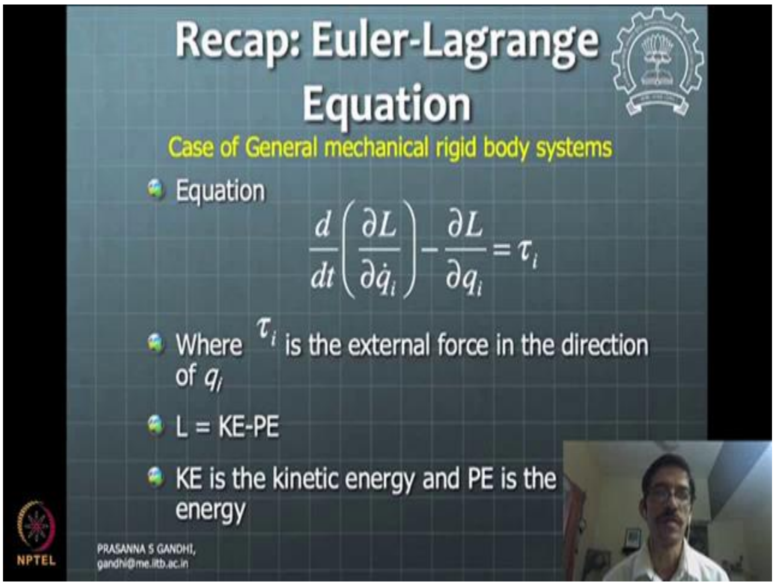

Now, lets generalize this in general for kinetic energy of several body, rigid body mechanical systems. So, to some extent it may be valid for even flexible body mechanical systems if we restrict the discussion about them to a finite number of modes. So, let us not worry right now about flexible body system, but we will just get into the rigid body system dynamics.

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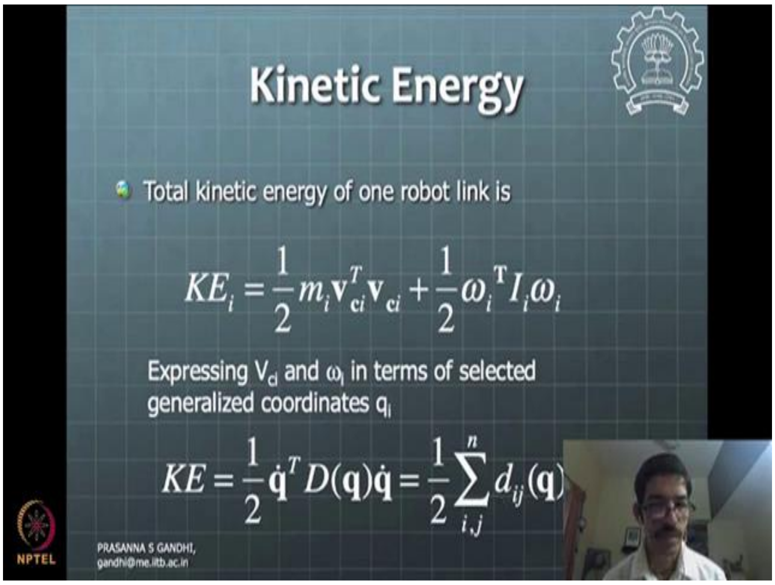

So, for rigid body the kinetic energy is expressible in this form. Kinetic energy is actually summation of all these different mass elements in the system and corresponding velocities of the CGs of the mass elements and then inertia of those elements and corresponding rotation kinetic energy part, so the corresponding ω for that element. And summation of all such elements over, it should be summation here, over all different bodies will give you total kinetic energy of the system.

Now, this kinetic energy we need to express in the form of a generalized coordinates. So, this kind of a quadratic form in the generalized coordinate it can be expressed. So, this is possible by expressing as we have seen in 2-R manipulator the velocities of CG and ω they should be expressed in generalized coordinates and you will have this expression possibility. Once we do that, you get this kinetic energy as some summation of this $\dot{q}_i$, $\dot{q}_j$ elements and d_{ij}.

Now, we start off with our Lagrange formulation based on this expression. Let me give a check once if we are okay with the recording. Just pause for a while. And you start doing this, using this kinetic energy expression and deriving some terms in the Lagrange formulation. We will come to that immediately.

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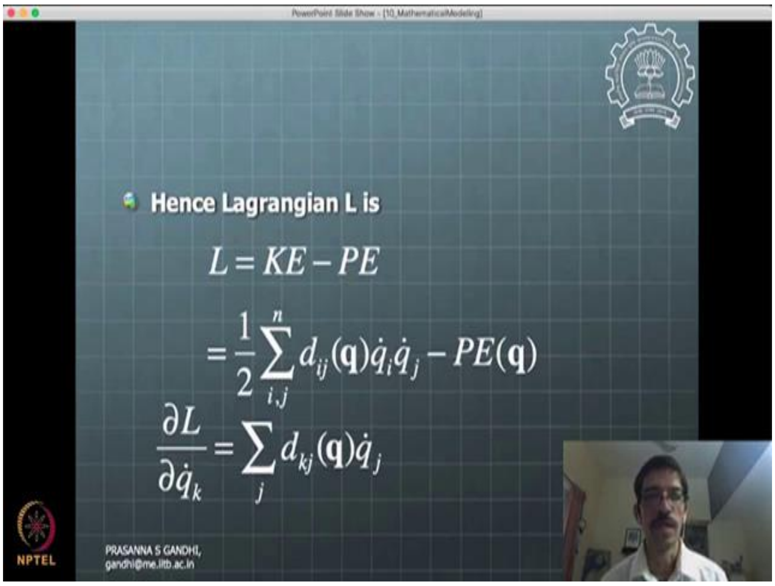

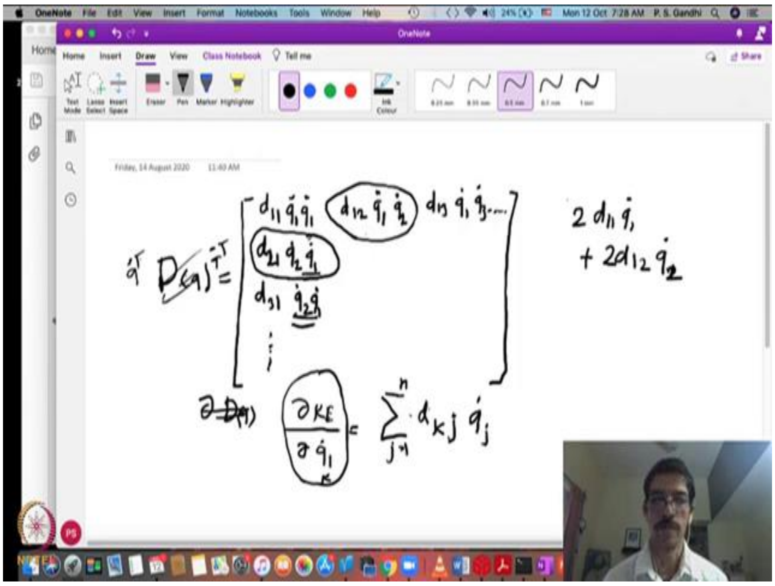

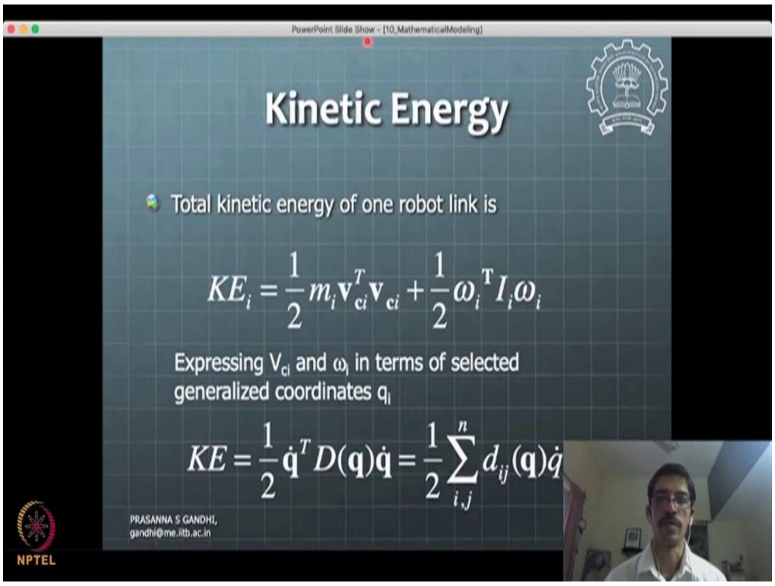

So, if you see this terms. First term, so what we do is see potential energy in the system is a function of the q_s only, only the generalized coordinate or degrees of freedom variable and not its derivative or $\dot{q}$, $\dot{q}$ is not appearing in the potential energy term. Notice, this is important. This q is a vector. It may be like all, dependency on all generalized coordinates, but the derivative of q will not appear in potential energy terms in general for mechanical rigid body systems.

Then, now, if we see, we start taking first partial derivative of L with respect to $\dot{q}$. So, this will have this form. Now, how this form comes. You need to think about that. It is not like straight forward or trivial tool, like this reason being there are no, no. So, what we are doing is we are expressing this energy terms in the, in d_{ij} $\dot{q}_1$, $\dot{q}_j$,. Now, if you form these terms or look at these terms in the matrix where are they. And then like you see if I take the partial with respect to say $\dot{q}_1$ here, any k $\dot{q}_1$ then what terms are going to appear is what we are looking in this expression.

So, if you see this *D* of *q matrix* that has these terms. Now, I will write these terms with the multiplication so that we have summation of all these terms coming in the derivation, $\dot{q}_1$, then d_{12} $\dot{q}_1$ and $\dot{q}_2$, like that there will be terms d_{13} into $\dot{q}_1$ $\dot{q}_3$ and here d_{21} again $\dot{q}_2$ and $\dot{q}_1$ is again coming here. So, notice that this $\dot{q}_1$ is coming here, like that d_{31} $\dot{q}_2$ and $\dot{q}_1$is coming here. So, this is how like the terms are going to come for $\dot{q}_1$ part of it.

So, when I take, so this is, this will continue and so on and so forth. And we will have, all these terms are actually summed up in the final expression for the, so we are not saying just *Dq*. This is not really just a *Dq*. So, this is misnomer. It is a *Dq* and like there is some kind of quadratic part multiplication. So, and this is summation of all these elements which will happen in the final expression.

So, you will have also here $\dot{q}$ transpose *Dq* $\dot{q}$. That kind of expression we are writing, but we are not writing as a summation, but like, we are not writing the plus signs, but we are just putting these elements in place so that we know which of the terms are going to appear when I kind of do the partial derivative with.

So, if I take this partial now, this partial of kinetic energy with respect to Dq_1 dot. So, this 1 is actually k now from our previous expression generalization purpose and in here there are going to be terms which are based on, so the terms will come from both places from here also and here also.

So, this d_{21} and d_{12} are same, because this is symmetric matrix. So, this matrix is not only positive definite, but it also is a symmetric matrix. So, why it is symmetric, like you will see that any quadratic form can be rearranged to get into a symmetric kind of a form. So, one can kind of see from there this will get into a symmetric matrix.

So, if it is not symmetric, we can kind of like turn it into a symmetric matrix. So, these are like cross, these are like what do you say cross diagonal terms. They are not like non-diagonal terms. And this is diagonal term. Diagonal term, if you take, see this is derivative with respect to $\dot{q}_1$it will be like 2 times like, so this derivative is going to be, so this diagonal term will have 2 times d_{11} $\dot{q}_1$.

And off diagonal terms we will have 2 again because if you see here this kind of a term and this term will get summed up and because they are symmetric with respect to each other, because the matrix is symmetric like these both the terms are equal and that is why this is going to be $\dot{q}_2$.

When I take a derivative with respect to $\dot{q}_1$, the $\dot{q}_2$will appear here and here it is $\dot{q}_1$ because right now it is $\dot{q}_1{}^2$. So, now if there is a half in the kinetic energy expression. So, what we are writing is only part of this term ∂qe, ∂ kinetic energy by ∂q_1 dot which is now corresponding to first diagonal, first term and its derivatives that will come.

So, because now we are considering with respect to say q_1 which is k is now equal to 1. So, this will be, in general, summation of, now, you can see that the half will get canceled out. So, when half gets cancelled out you get this d, now d_{kj}, so k is my like term which is here, d_1 d_{12} like writing in terms of d_{kj} and $\dot{q}_j$.

So, if you see this, where J is equal to 1 to n. So, for k is equal to 1, if I see, then J is equal to 1 to n like this $\dot{q}_j$ will give me like this expression for first partial derivative of kinetic energy with respect to $\dot{q}_1$. So, clear this part. So, like that you can, you need to see and move on from here. So, let us see this expression.

This is d_{kj} $\dot{q}_j$ that is what we have seen. So, this is how you need to, it is not, do not consider this to be a trivial exercise. This is, one has to kind of, think about in terms of the expression for *D matrix* and then the terms coming from the quadratic form and then take them further to differentiate.

Like that we do now further for other terms now, like say d by dt of ∂L by ∂ $\dot{q}$ and ∂L by ∂q these types of terms you can express now in a similar kind of fashion and try yourself first doing that and then only you can go ahead and look at the further part of the lecture.

So, please do this. That is an important part of the exercise to get to the real understanding of what is happening. And even if you have a difficulty, no problem like you can get stuck no problem. Getting stuck is a very important step in understanding. And once you get stuck then you come back and look at how things are done and then maybe things would be better clear.

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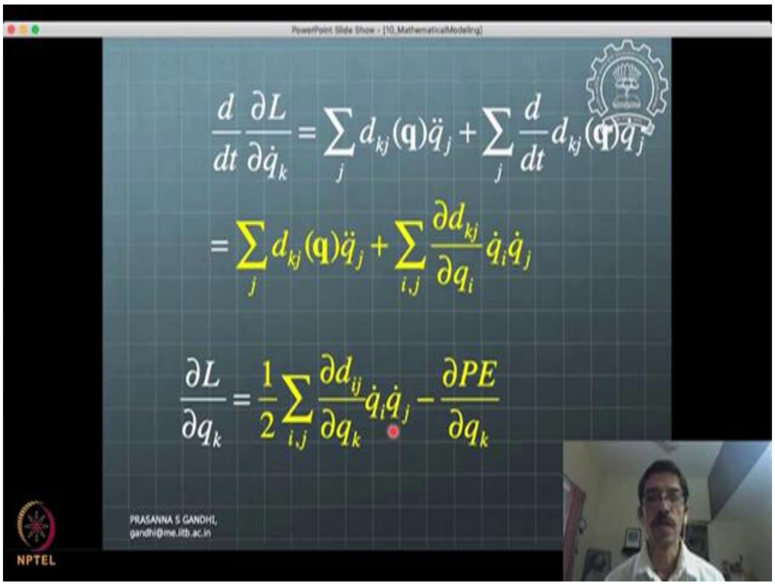

So, if we move on for further differentiation, so you can see this is a term coming because of the complete differentiation with respect to time for the same term. So, you can see here, this part first like there are two derivatives that will come because of the multiplication. So, first is the derivative of this $\ddot{q}_j$. So, $\dot{q}_j$ term will get differentiated to give you $\ddot{q}_j$ and then you come to the differentiation of d_{kj}.

Now, you can see that this first term used to directly like a term in the final equation $d_{kj}\ \ddot{q}_j$. This is a term that will appear in the final equation, because now we will assemble these terms when this complete derivative. This is, no more derivatives will be executed after this. So, after this there is just assembly of the terms. And then if you see this term, this term can be expressed as under summation now.

So, this d by dt of d_{kj} of like in general generalized coordinate vector q function of that. So, we take partial with respect to q and then $\ddot{q}$ will be multiplied and that is how, these summations still remains, but inner, this derivative can be expressed as a summation over i and the summation over j is already there. So, like this becomes like a summation over both i and j d_{kj} ∂kj ∂q_i and $\dot{q}_i\ \dot{q}_j$. This kind of a term will come.

This we can work it out. And again, do the way that I have expressed in the form of like in this matrix form with all the terms written there, and then things will be clear for you. So, we will consider it that way. Now, for ∂L by ∂q, you have these terms coming up. And again, these terms are quite straightforward to see probably, because this is already from the kinetic energy, we will have this term coming.

We are just taking partial with respect to q. So, this is a partial with respect to q kind of a term and then this ∂P by ∂q again here. So, I want to request you to kind of go through clearly in all expanded terms in the matrix kind of a form and then things will become a little more clear, how we are expressing this.

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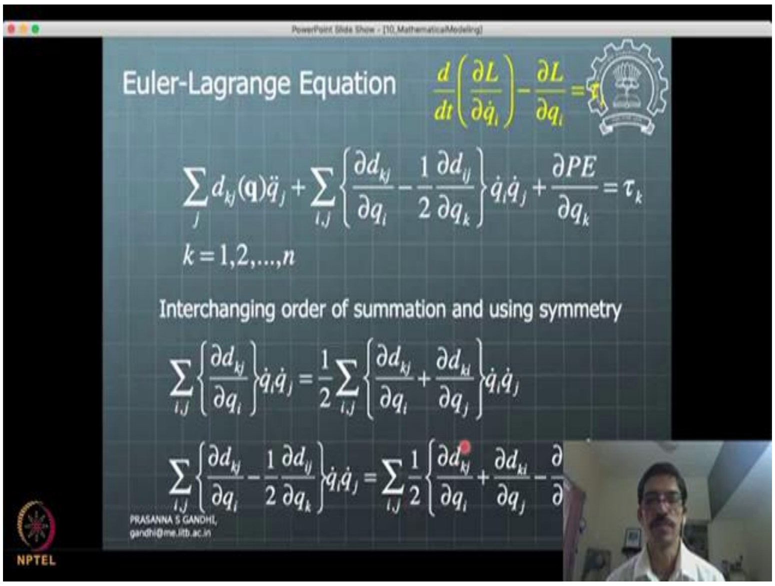

So, now, if we move to assembly of these terms, we get this kind of assembly. You just put these together, here in this complete equation and you get these terms assembled in this form here. So, by interchanging the order of summation and using some kind of symmetry in the system, again this is not very trivial to see directly. One has to kind of write these terms in ij proper, consideration of ijk and then you will find that this can be written by rearranging.

So, the symmetry is used to kind of get this split part of the terms and this kind of helps in some definitions later that is why this is done. So, you see, actually this is your C part of the term. *C matrix* if you want to say or like some kind of a coriolis and centrifugal kind of components which are coming in these terms, their derivatives and things like that.

So, kinetic energy derivative will actually give you this centrifugal and coriolis kind of a post terms here, when q_i is equal to q_j you will have a $\dot{q}^2$ term that is coming which you relate to somewhat like a centrifugal force term and when i is not equal to j they will be like a coriolis terms. Now, by doing this, like some kind of a readjustment is done for this particular term only and you get this expressed in this form.

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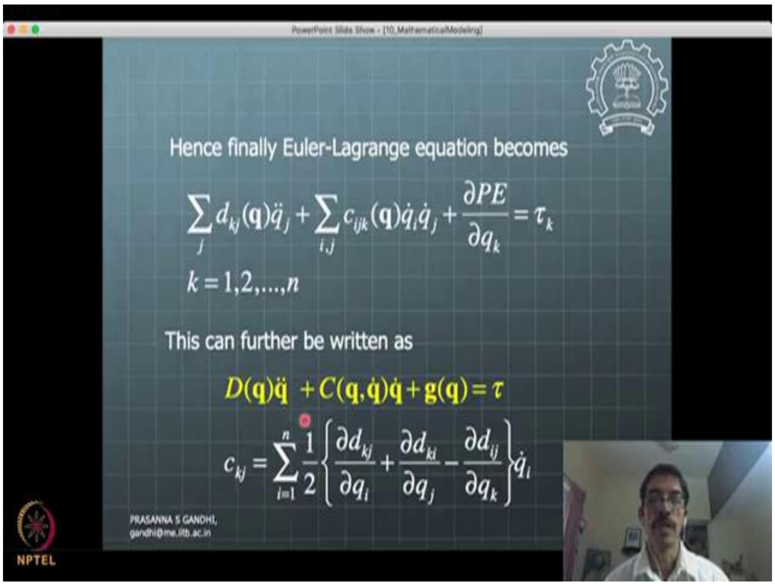

And further, you can express that in terms of this C_{ijk} kind of coefficients. These C_{ijk} coefficients sometimes are called Christoffel symbols. So, there is a mathematical kind of a jargon there, but let us not worry about that. It is basically like you get this kind of a C matrix multiplying some $\dot{q}$ and this *D matrix* we know already and $\dot{q}$ and these potential energy terms are clubbed into this *G matrix* and then you get this complete expression for your final equation of motion. So, this is how things happen here. So, lets now see, this form has some certain other property.

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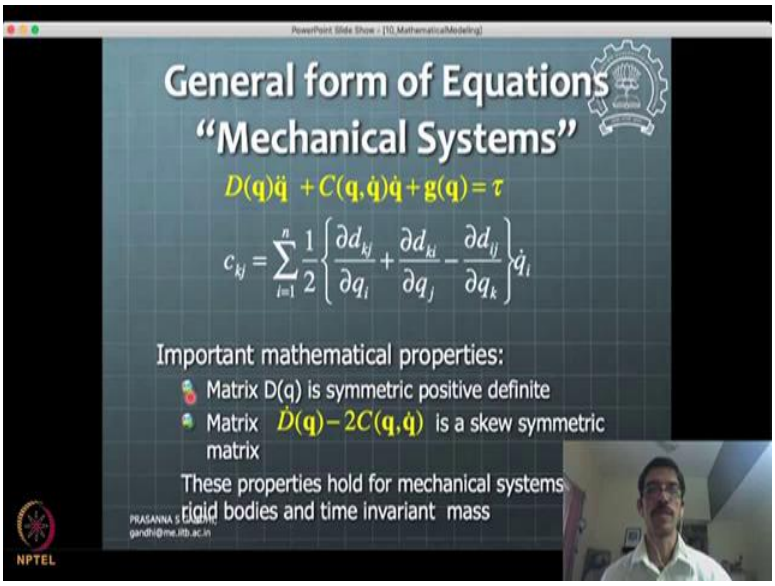

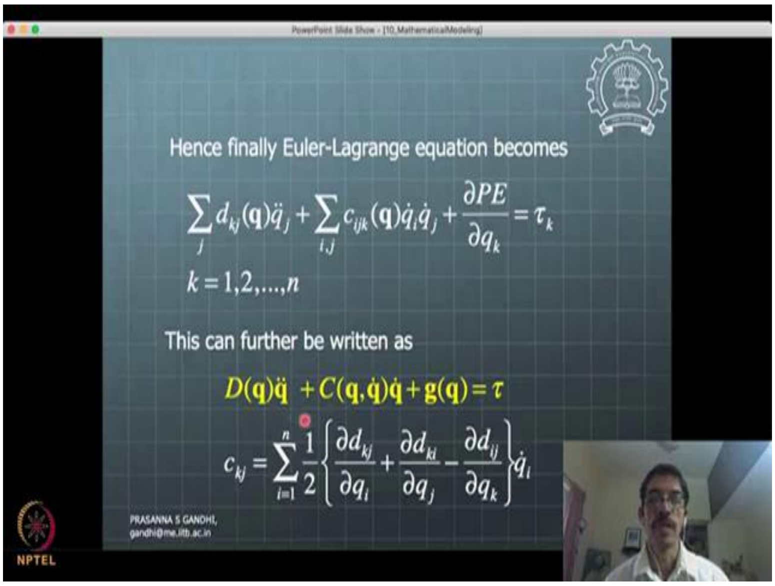
Hence finally Euler-Lagrange equation becomes
$$\sum_j d_{kj}(\mathbf{q})\ddot{q}_j + \sum_{i,j} c_{ijk}(\mathbf{q})\dot{q}_i\dot{q}_j + \frac{\partial PE}{\partial q_k} = \tau_k$$
$k = 1,2,\ldots,n$
This can further be written as
$$D(\mathbf{q})\ddot{\mathbf{q}} + C(\mathbf{q},\dot{\mathbf{q}})\dot{\mathbf{q}} + \mathbf{g}(\mathbf{q}) = \tau$$
$$c_{kj} = \sum_{i=1}^{n} \frac{1}{2}\left\{\frac{\partial d_{kj}}{\partial q_i} + \frac{\partial d_{ki}}{\partial q_j} - \frac{\partial d_{ij}}{\partial q_k}\right\}\dot{q}_i$$
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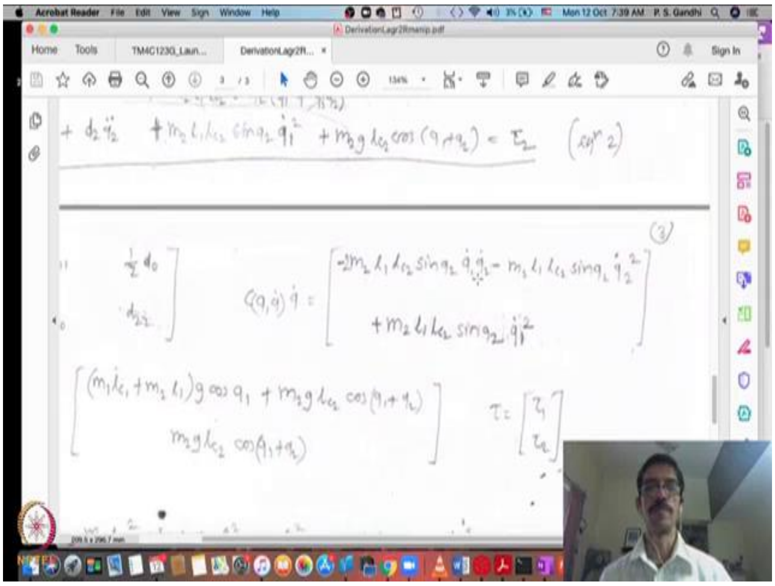
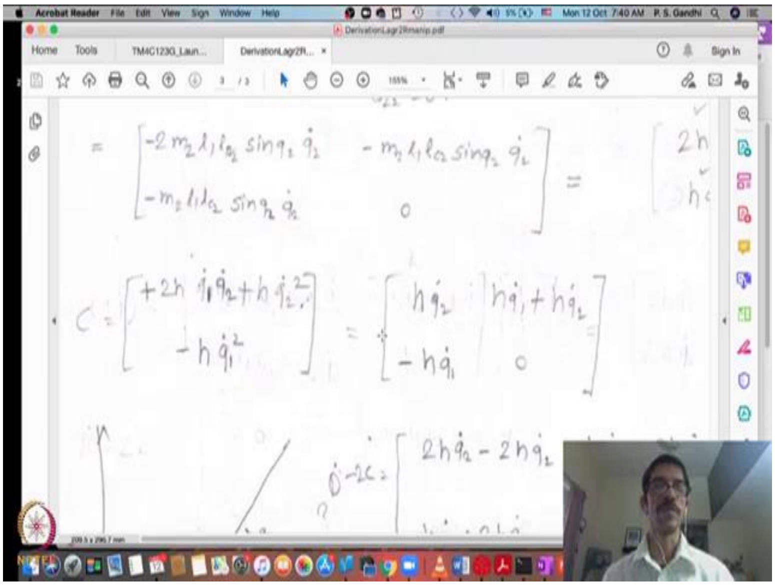
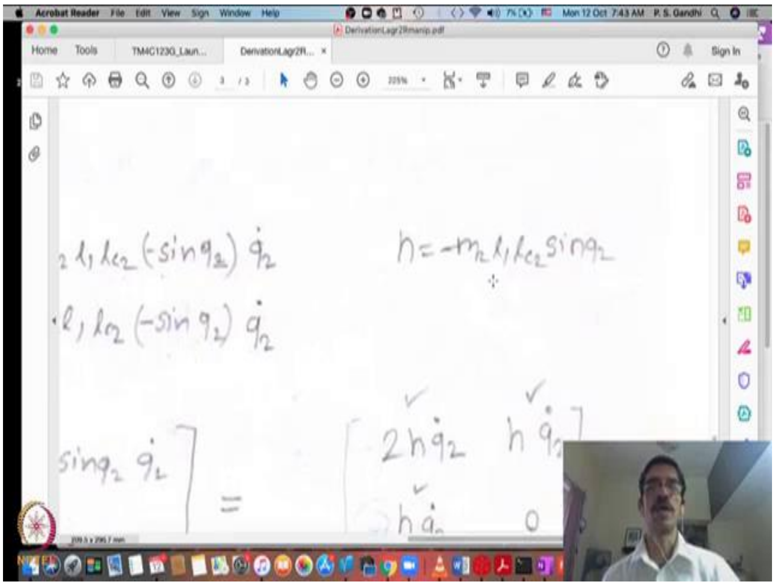
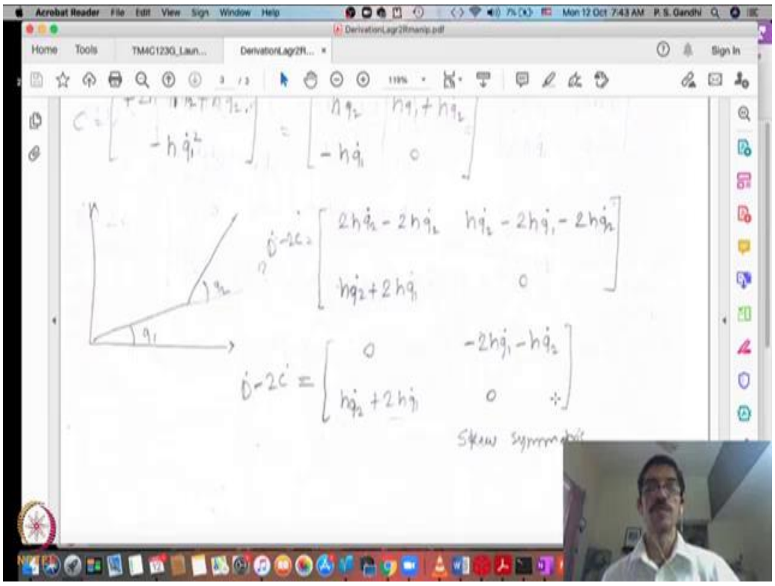

$+ d_2\ddot{q}_2 + m_2 l_1 l_{c2} \sin q_2 \, \dot{q}_1^2 + m_2 g\, l_{c2} \cos(q_1+q_2) = \tau_2$ (eqn 2)
$C(q,\dot{q})\dot{q} = \begin{bmatrix} -2m_2 l_1 l_{c2} \sin q_2 \, \dot{q}_1\dot{q}_2 - m_2 l_1 l_{c2} \sin q_2 \, \dot{q}_2^2 \\ + m_2 l_1 l_{c2} \sin q_2 \, \dot{q}_1^2 \end{bmatrix}$
$\begin{bmatrix} (m_1 l_{c1} + m_2 l_1) g \cos q_1 + m_2 g\, l_{c2} \cos(q_1+q_2) \\ m_2 g\, l_{c2} \cos(q_1+q_2) \end{bmatrix}$
$\tau = \begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix}$


$= \begin{bmatrix} -2m_2 l_1 l_{c2} \sin q_2 \, \dot{q}_2 & -m_2 l_1 l_{c2} \sin q_2 \, \dot{q}_2 \\ -m_2 l_1 l_{c2} \sin q_2 \, \dot{q}_1 & 0 \end{bmatrix}$
$C = \begin{bmatrix} +2h\,\dot{q}_1\dot{q}_2 + h\,\dot{q}_2^2 \\ -h\,\dot{q}_1^2 \end{bmatrix} = \begin{bmatrix} h\dot{q}_2 & h\dot{q}_1 + h\dot{q}_2 \\ -h\dot{q}_1 & 0 \end{bmatrix}$
$2h\dot{q}_2 - 2h\dot{q}_2$





So, these important mechanical properties that we have seen, important mathematical properties that we have seen for this mechanical system, first is *D matrix* is symmetric and positive definite. Now, this is the other property. This matrix $\dot{D}$ minus *2C* is skew symmetric.

So, you see that this *C matrix* can be developed in many different kinds of ways. Even this as a quadratic form here, I can have like different combinations of the terms coming to get it out to kind of find, this is quadratic expression, but we are not considering here a quadratic form.

So, here the form is just *C matrix* multiplied by a vector. So, for this, by rearranging terms in *C* we can have different possibilities for C matrix. It is not a unique. So, maybe we can see that with respect to our example of 2-R manipulator. So, we go back to our 2-R manipulator case. So, let us get here. If you see this C $\dot{q}$ term, so this is a C $\dot{q}$ term as a vector form. So, we say this has some terms as $2m_2$, this $q_1\dot{q}_1$ kind of $q_1\dot{q}_2$term and $\dot{q}_2{}^2$ term and $\dot{q}_1{}^2$ term.

Now, I can say like, so I can choose to express like this is like C $\dot{q}$. So, what should be the *C matrix*? I can have different possibilities. So, one of the possibilities you can see up here. I can say *C* can be expressed as, so I have used some symbols. So, just to kind of get rid of these complicated terms coming there, I am just using this h as a symbol for some term.

So, this $h\dot{q}_2$, so this is like $2h\dot{q}_1\dot{q}_2$ $h\dot{q}_2{}^2$and minus $h\dot{q}_1$, $\dot{q}_1{}^2$ is like C times $\dot{q}$ term, sorry, this $\dot{q}$ is missing here. So, I am kind of trying to get my *C matrix* from here. So, this is my $C\dot{q}$ term and this is actually *C matrix* one of the possibilities. So, what I have done is like we can see that now this term is going to multiply in the first row by $\dot{q}_1$.

So, this $h\dot{q}_2$ $\dot{q}_1$ plus now this $h\dot{q}_1{}^2$and $h\dot{q}_2$, $h\dot{q}_1$, this is the second term of the matrix, so this $h\dot{q}_1\dot{q}_2$ and $h\dot{q}_2{}^2$. Now, this $h\dot{q}_2\dot{q}_1$ and $h\dot{q}_1\dot{q}_2$ is the same term. They will get added to 2 $h\dot{q}_1\dot{q}_2$ here. See the way it is split. So, this term is split into two parts. One part is put in this side and other part is put on the other side. And then this is getting multiplied by $\dot{q}_2$ where I have a multiplication $\dot{q}_1\dot{q}_2$ as a factor here.

So, this term is getting multiplied by $\dot{q}_1$ alone and this term is getting multiplied by $\dot{q}_2$ alone. And then like when I add these two terms as a first term of the matrix product, matrix vector product then like I get 2 $h\dot{q}_1\dot{q}_2$ as we want up here. Like that you can express similarly like $h\dot{q}_1$ term, $h\dot{q}_1{}^2$ term, $h\dot{q}_1$ here and nothing here. So, this is, there is no $\dot{q}_2$ term given there. So, like this is one possibility of arrangement.

There can be other possibilities of arrangement also. I could have just said, this term, this part of the term to be 0 in here I would say this is 2 $h\dot{q}_1$, that also could have been a possibility. All these diagonal terms are called to be 0 and like I combine these 2 $h\dot{q}_1\dot{q}_2$ on this side or I could say I will use 2 $h\dot{q}_2$ here and I do not use this term here. I just have this term to be equal to $h\dot{q}_2$. So, there are multiple possibilities for assembling this *C matrix* to get the same expression C$\dot{q}$ as a vector.

So, my choice of this is going to be in a way, so that like I use it for this property $\dot{D}$ minus 2C to express. So, by expressing in this kind of a form and I see this $\dot{D}$ minus *2C*, so D expression is here, so $\dot{D}$ expression is this. Again, by using this *h* as this value here, I have used *h* to be equal to say minus $m_2 l_1\ l_{c2}\ sin q_2$. If I use that as *h* value, then I get $\dot{D}$ is equal to some part and when I take $\dot{D}$ minus *2C* it gets into the skew symmetric form.

So, you just do it yourself, starting from scratch and then like come to understanding of this. It is very important to get to, because many times you will need to do this kind of a little bit mathematical manipulation to see how to get, make sure that, once you have $\dot{D}$ using this $\dot{D}$ in some way to make sure that C is expressed in such a way that $\dot{D}$ minus $2C$ gives you skew symmetry. So, this exercise would be interesting to do to understand like, how this can be further generalized.

So, I leave this to you for generalization kind of a case. I will not kind of get into generalization. It is just a little bit of mathematical jugglery that I think you guys can do really nicely. If at all you need, maybe we can go through that or maybe I can post the expressions or the derivations. So, lets get back to our slides again.

So, this way we defined this *C matrix* in this. So, this is one of the kind of a rearrangement is done to kind of get to this form. And in this form, you will find that when you define *C matrix* in this kind of a form you get this property $\dot{D}$ minus $2C$ to have a skew symmetry. That can be derived and shown. We will not get into the depth. We have been seeing now how we can use this form to further get, to a control for these kind of systems. So, rigid body systems control when we see that time, we will be using this property of skew symmetry and the property of D to be positive definite matrix.

And one can see now, if you go for the simulation, then for simulation you need this $\ddot{q}$ to be expressed as a function of q_s and $\dot{q}$. And for that, you will have to push these terms like on the right hand side, C term and G term on the right hand side and then entire expression on the right side you need to multiply by D^{-1} to, isolate these $\ddot{q}$ terms. Once you isolate them then only you can use them in your simulation for getting the ODE45 simulation going. For the simulation you need this $\ddot{q}$ isolated terms.

So, vector, isolated vector you can get by this D^{-1} *matrix* multiplying all the terms which are pushed on the right-hand side of this equation. And that is how one can get going with the simulation of such complex systems which are in general like having this cross-diagonal terms or off diagonal terms to have cross coupling between different-different variables that are there as a generalized coordinates in the system.

So, this is how things go for the general mechanical system and these properties are going to hold for mechanical systems for sure with rigid bodies and this is important time invariant mass. So, the mass is not changing with respect to time. There, these properties will be in hold true. So, when the mass is changing with respect to time, then we will have to again recheck and make sure that this will happen or not, because we have so far in the thing, we have like no dependency of D on the time is considered explicitly. Implicit in q is fine, no problem.

But explicit dependency of *D matrix* on time is not considered in the whole derivation, explicit dependence of kinetic energy on the time. *D matrix* is just a part of the kinetic energy. So, this kinetic energy expressed as a time, explicit function of time is not considered in this derivation. That is why, we cannot say these properties will hold for time invariant, time varying mass. These are like so far valid for time invariant mass. So, one can see what changes it may have announced and so forth that can be a matter of further discussion.

So, this is how we proceed with the base development for general mechanical systems. So, all the mechanical systems with n rigid body, so robotic systems as a part, any mechanism is a part, so everything is covered under these, edges of this Lagrangian formulation. And what we have, we do not have so far here is the damping terms and any other kind of forms of a term which are energy consuming. So, those terms are going to be forming like these external forces in the system. So, as the damping, as we saw, the damping will form this external torque in the system.

And then come appear here, external photons that you are applying or external forces that you are applying so this τ is just term but it can be in force or it can be in a torque form or moment form or whatever form in the direction of this generalized coordinate. If the generalized coordinate is a linear kind of a quantity like x, then this will be force.

If the generalized quantity is θ, the rotation kind of quantity, then this will be actually a torque. So, these torques in the direction of generalized coordinates, they need to be added, the damping and other terms can be added into those parts. So, that is how we will proceed with the mechanical system derivation.