Design of Mechatronic Systems
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Lecture Number 38
Mathematical Preliminaries Nonlinear Control

This part of the course will begin with the nonlinear control design. So, right now we will have some small chunks in which I will post this. So, that you can look at this little bit of think in and ponder over. I do not want to kind of continue in one chunk everything. So, we will begin with some preliminaries of design for nonlinear control systems. Let me get to the slides.

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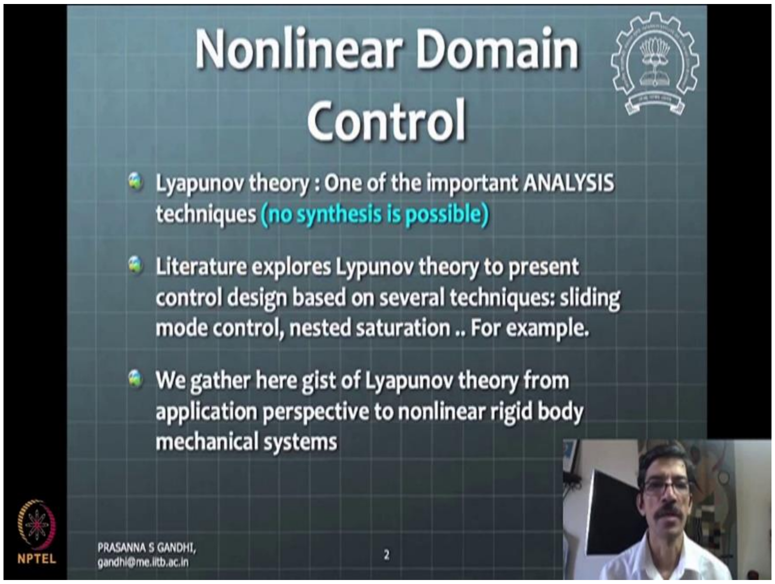

Now in this we will basically look at Lyapunov theory one on various important tools in the nonlinear control analysis. So, you see that this Lyapunov theory is an analysis tool it is not a synthesis tool. So, you need to propose a control and check out whether it will be working. So, this will be the some kind of iterative design you propose a control and find like some kind of stability proof for that if you do not get proposed another control like that till the time you kind of get some stability proof.

And this Lyapunov theory has been extensively used in many different kind of new control techniques designed So, it is useful form some kind of a fundamental foundation for nonlinear control. And here we are looking at mainly application perspective. So, we will present some kind of a preliminary mathematics which is most essential for the development of some of these control strategies.

But mainly our focus is going to be an application rather than the proofs of the theorem for the for what Lyapunov has proposed. So, I will present you these some of these theorems but we will not get into the details of the proof of them because our main focus is going to be an application.

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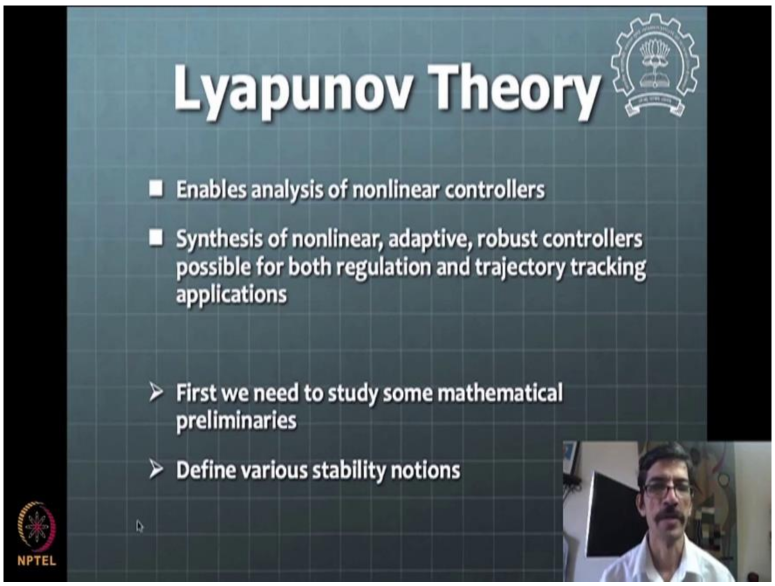

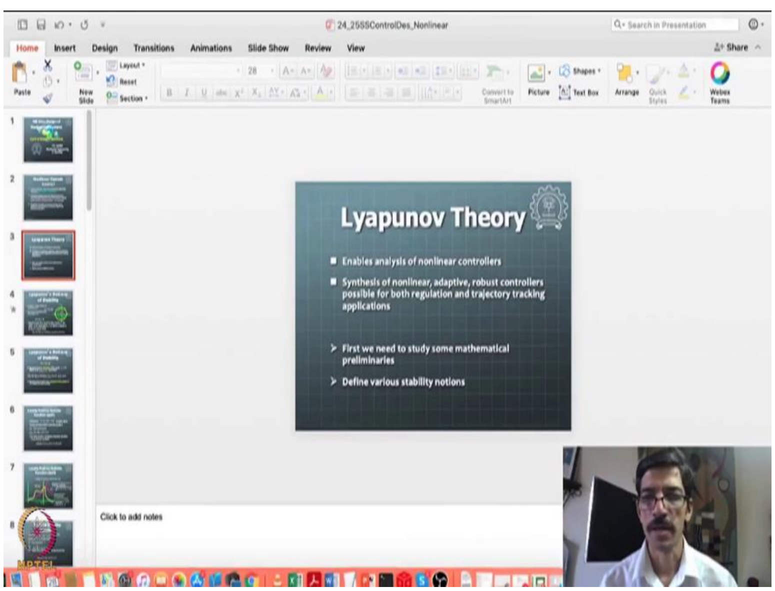

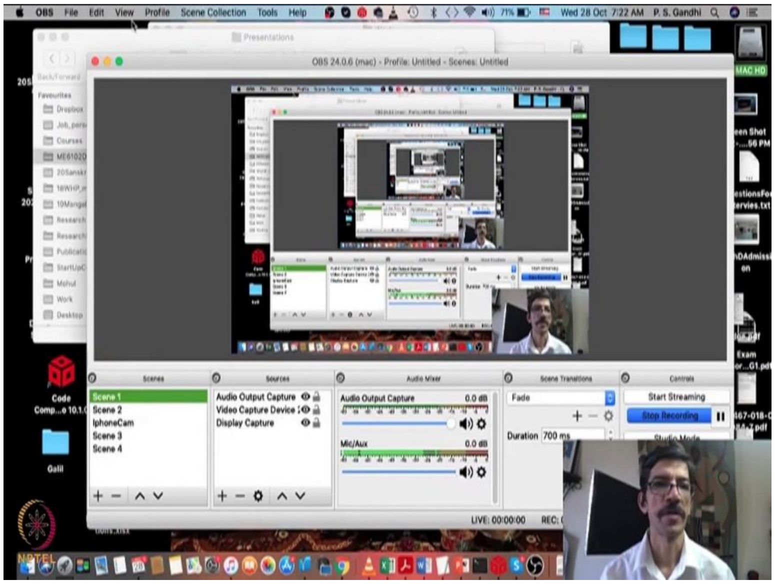

So, this theory as we see it So, first we get into some fundamental mathematical preliminaries and define like various notions of stability as I mentioned earlier also that from linear systems to nonlinear systems there are different notions of stability that one needs to get into. So, let me just check things are getting recorded properly and then come back I think. So, let us see start with these preliminaries now.

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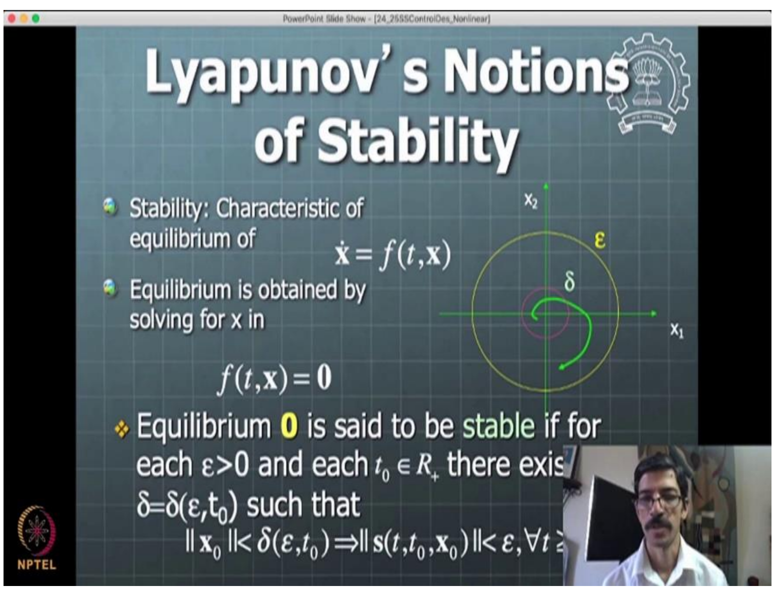

So, first is the notion of stability. So, this Lyapunov if you consider this all these Lyapunov theorems and stability notions et cetera are developed based on the state form of nonlinear equation. So, $\dot{x} = f(t, x)$. Where x is a vector and u if it is a function, it is a function of x or it is a function of time whatever way it is, it is incorporated in this function f here.

Now, the equilibrium of such a system is obtained by setting is $\dot{x} = 0$,is very first important concept at equilibrium you want to know what is the equilibrium of a system you set $\dot{x} = 0$ whatever you get will be equilibrium of a system. And we assume that this equilibrium when you set $\dot{x} = 0$ in this kind of a form is 0. So, if it is not 0 what you do is basically shift the new x you say new x_1 is equal to some x minus that equilibrium state.

And in that when you now see your equilibrium per system that will be that will be 0. So, it is important to make sure that all equilibrium that we are talking about in this when you do this analysis is considered to be 0. So, now we assume that this whatever system where it has been defined has equilibrium 0. That means when we put $\dot{x} = 0$ we get $x = 0$.

So, that equilibrium 0 in Lyapunov notion, first notion of stability is a stable kind of a system is said to be stable if for each $\varepsilon > 0$. And each t belonging to this positive real line there exists δ which is function of ε and t such that this condition is satisfied. This s is a solution trajectory of this equation near equilibrium. So, what it means is if I have the initial state developing in this ball of radius δ. So, what you have here is this norm of x_0. So, this x_0 is the initial condition. So, we start off with x_0 as the initial condition at time t_0.

And we see the evolution of this trajectory. Now this evolutionary trajectory is less than ε for all t greater than t_0 then we say that the system has a stability function, this is stability notion. So, but it says that if for each ε you are able to find δ. So, you are first given this ε bound on the trajectory as it evolves in time. So, once you have given this bound on the trajectory norm. So, this is when you say this like of second norm or you can consider this as a square root of a all squares of the term of the vector both the norms.

Norm 2 kind of you can use whatever norm you want to use but typically norm 2 is used. Now, So, you start off with this ε radius So, this is a example on the in some kind of planar or a two state kind of system. So, x consists of a two state x_1 and x_2 kind of states. So, you have this ball of radius ε around the equilibrium. So, these ε ball is vary a solution trajectory should restrict then you are able to find some δ which is greater than 0. So, their δ should not be 0.

So, if you are starting at 0 state then you know that whether definition of equilibrium will remain at equilibrium only. But you need to have a little bit kind of away from 0. So, this δ is greater than 0 such that for x 0 beginning into the a. So, δ is some smaller kind of region. So, suppose my x 0 evolves in this starts off in this δ region then my solution trajectory all are constrained to this ball of radius ε.

If you are able to find such δ this is like a base fundamental definition of stability for the Lyapunov's sense. So, but a nonlinear system this kind of a base stability definition is used. And then you derive many other kinds of conditions based on this fundamental definition of stability.

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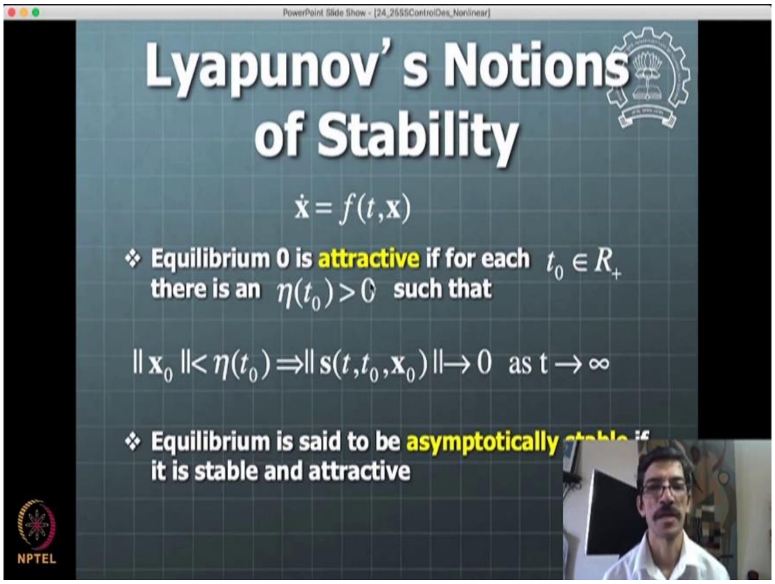

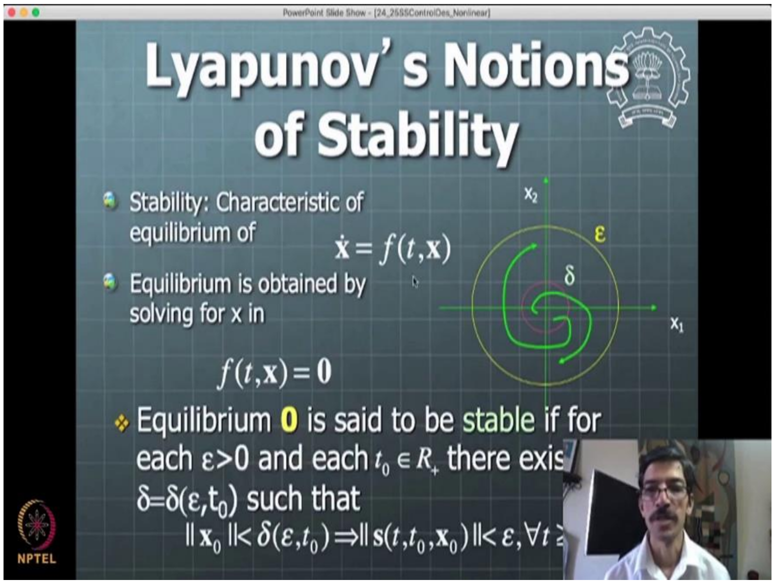

Then we have other kind of definitions coming up here. So, for the same system this equilibrium same now is 0 is our equilibrium is attractive if for each time greater than trajectory time belonging to R plus real line. There is η which is greater than 0 such that this solution trajectory tends to 0 as t tends to infinity. So, what it means is like the solution trajectories not only remain now in the in the ball of ε but they actually tend to go to 0 as t tends to infinity.

Then the equilibrium is attractive. So, x_0 constraint in some kind of ball of radius η So, you begin with the this η is greater than 0 So, again you are not supposed to be on the at the equilibrium point you can go a little bit So, your initial state should be away from equilibrium point by some small amount. And if under that situation your solution trajectories actually go to 0 eventually t tends to infinity then your equilibrium is attractive. So, that is a definition.

So, here like you can construct some kind of a ball of radius η and you start all those solution trajectory they will do not come out of this η but they actually go to 0. So, these solution trajectories are actually attracted to the equilibrium and then the system becomes attractive. So, it is said here it is said that the solution trajectories not necessarily remain in the ball of radius η. So, they may go out of that ball but they may come back eventually and go to 0 that is what is this definition says.

And for asymptotic stability this is a we have to define some notions. No there are these different notions are for next thing but for asymptotic stability it needs to be stable and attractive. So, the this is another notion of stability where it is not only stable but it is attractive also the equilibrium is attractive also. And equilibrium is stable in this sense So, stable and attractive both are properties are there. Then you have an asymptotic stability. So, these are the basic fundamental notions here.

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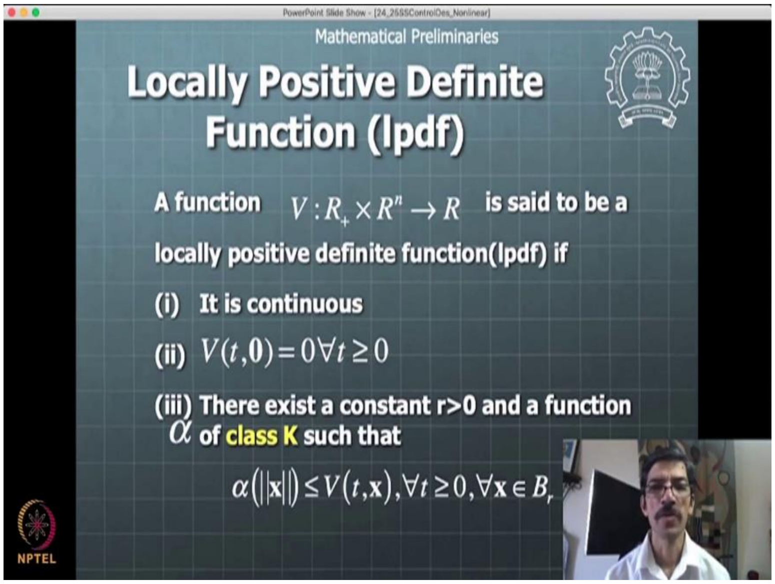

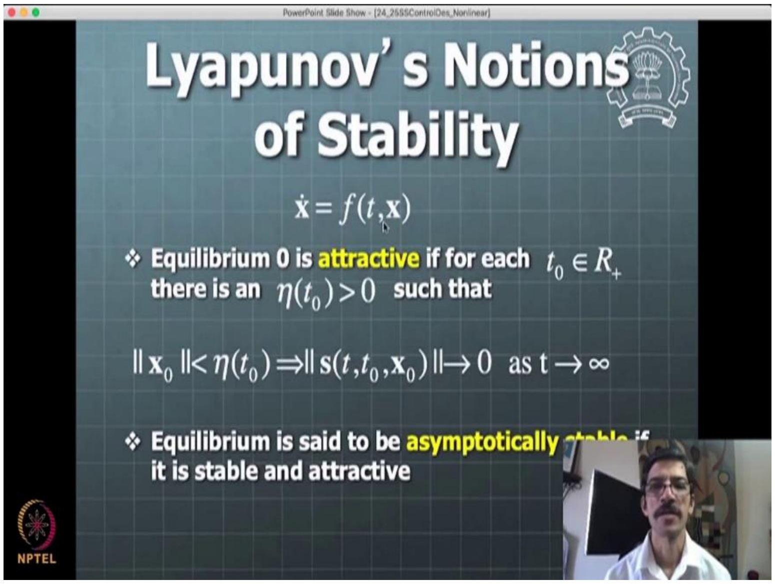

So, we will introduce some more notions for the definition and some financial analysis purposes. So, that we will be able to apply Lyapunov theory. So, this some of you may be familiar with,the function this function is defined on the time and vector R n taking to R. So, this particular kind of a function is coming up in the Lyapunov theory later as a Lyapunov candidate they call it for stability purposes.

So, this is defined over time and this vector R^n So, same as like know what we have in this definition of f time and x. So, is a function of a general function of time and vector R n and it said to be locally positive definite function. Namely like a LPDF if it is continuous then it needs to have the property that v(t,0) = 0. So, whenever this vector state is 0 you give the get a value of the function to be 0 for all t > 0 this should be valid.

So this is other property and then there exists a constant r > 0 and a function α of class K. Now we will see what is a class K function. Such that the this function is bounded by from below it is like the lower bound for this function bounded from below by this function of class K. So, this class K function we will see first and then like now we will come back to this definition.

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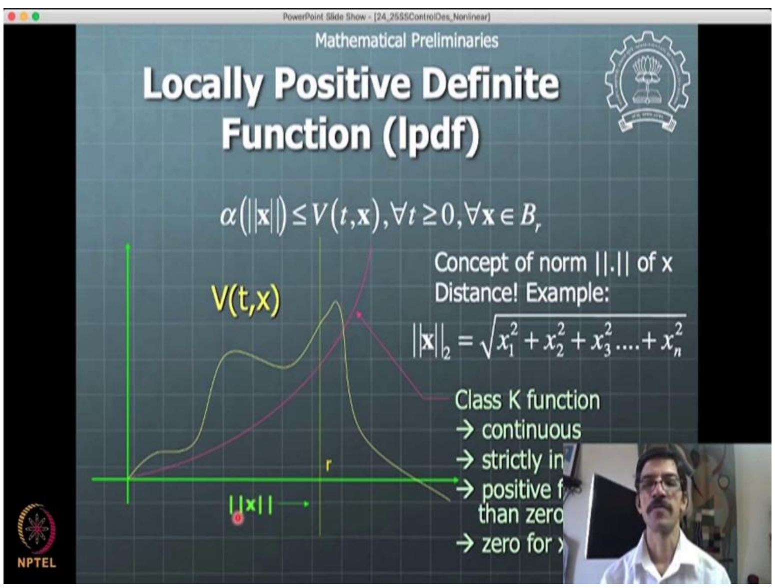
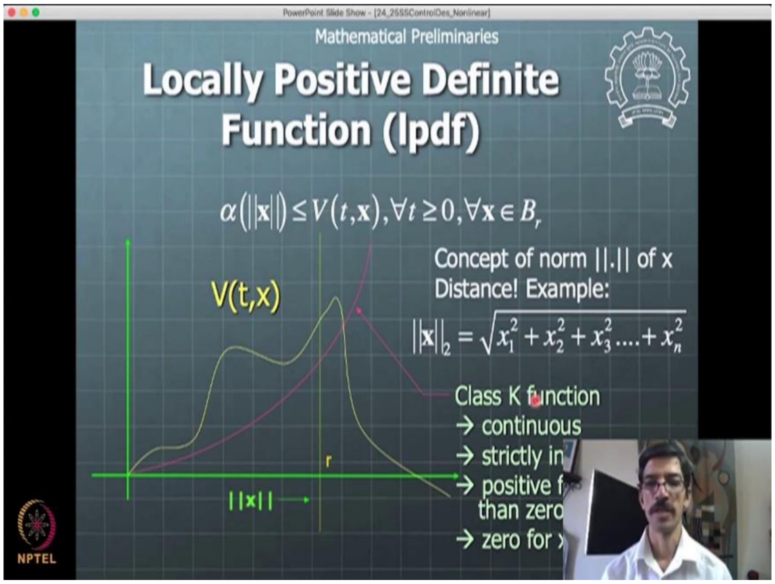




So, this class K function is a function which is monotonically increasing function. So, it is continuous strictly increasing this class K function if you see this it is strictly increasing sorry I get a pointer right here. So, this is strictly increasing function here continuous function and positive for x other than 0. So, it is 0 for x = 0 same as like V function and this has no time dependency class K function is only function of state.

There is no time dependency of classical function. If you see this α is only of x there is no time coming here. So, this for such a this α is a function by which this V is bounded from the lower side. So, V is always greater than α of class K. Then this function is called locally positive definite function. This is again most fundamental definition of this Lyapunov function candidates LPDF function.

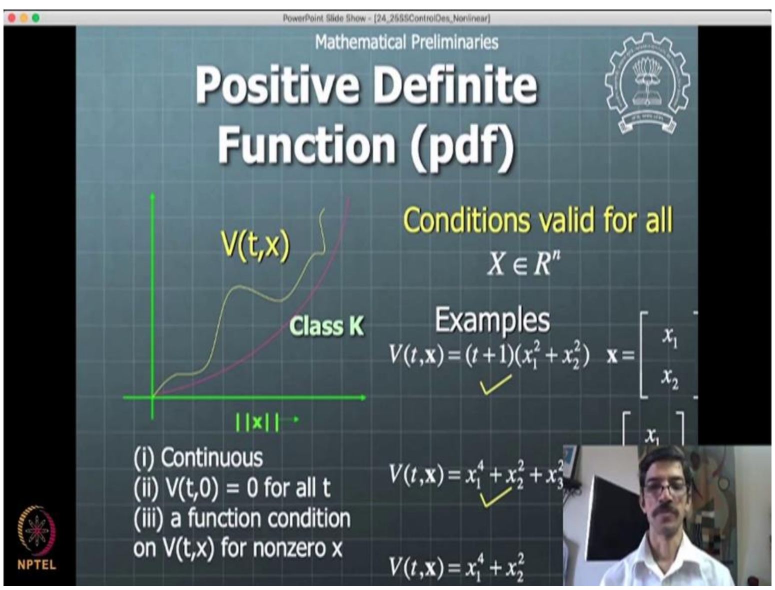
For function to be LPDF you have to have these conditions so, that means this function v of t comma x cannot touch the x axis here. It cannot come to 0 again. Once it is 0 only add x is equal to 0 and now you see for any other value it cannot come to 0 again within the ball of radius r. So, this is important. So, this is for locally local definition you need to have this ball of radius r at least for some kind of a norm of limit on the norm of x it is valid it is okay.

Then that is a locally positive definite function. So, this is a base definition for the locally positive definite function and if this r is extended now to the entire space r tends to infinity then this will become positive definite function. We do not say globally but it is positive definite function. So, that is a main kind of a definition here. So, again the same thing the positive definite function the function is said to be pdf if it is continuous same condition as before.

Then it has a V is equal to 0 for all t when x is 0 this R n vector which will I mean it function of and then there exists a function of α of class K such that now this function is bounded from below by this function α of class K for all R n space. And for all t greater than 0 of course which was there already for the LPDF case.

So, if you just plot I mean this is kind of this showing these plots some kind of schematics to kind of make sure that understanding happens really value it is actually R^n space. And we may be able to see this function like this only if it is a function of x alone, if you have some kind of a time axis we are not putting the time axis on this plot. So, you may see that the function may have some kind of a time variation also at the same x norm possibilities there.

But we are not kind of capturing that part in here. So, this is just to kind of see some kind of a graphical illustration of what it means. (Refer Slide Time: 18:35)



So, now let us see some examples. So, we need to check whether these conditions are valid for these functions or not. So, you remember the definition you need to have a function of class K and V should be continuous. So, these are the conditions continuous and V of 0 is 0 and function condition on these for non 0 x, class K function condition. So, we need to like strictly follow these properties for the let me checking whether these functions are positive definite or not.

So, if you see this function for example, what do you think just do a little bit of a thinking whether this is continuous yes it is continuous whether V this if I put x equal to 0 state here. I get V is equal to 0 for any time yes that is valid here. Now, it is bounded from below by class K function and you see that. So, you find some function by which it is bounded from below and I think this is 10.

So, this is satisfying all the conditions and that is why it is a positive definite function. Now, if I see this function this is a function of x_1 x_2 x_3. And this state is also given as x_1 x_2 x_3. So, when you have your, first thing you need to check whether all the state components are showing up in the function are not. So, if you see these conditions is it continuous yes it is continuous, then this is suppose I put x_1 x_2 x_3 0 x equal to 0 in the V then V is indeed is equal to 0.

And now this class K function condition is satisfying and for all other values of V other than 0 x this function is not taking value 0 any time. So, this is also positive definite function but now if I define this function ${x_1}^4 + {x_2}^2$ and I have this as a state. Then we need to see whether this is positive definite function. So, see although the form looks similar to this function first condition gets satisfied second condition gets satisfied but the third condition you see there is some problem.

See this x_3 component is not showing up here. So, for even if x_3 is positive. Then the $\|x\|$ is not equal to 0 , $\|x\|$ is some kind of a finite value here. But if x_1 and x_2 are 0 then this function is taking value 0 this is not allowed. So, that is why this is not a positive definite function if the state is x_1 x_2 $x_3$if the state is given to be x_1 and x_2 alone then this becomes a positive definite function.

So, that is a catch here that way we need to be careful about when x_1 is defined as the state for this function then it is not positive definite function but if x_1 is defined as a state consisting of x_1 and x_2 alone then this is a positive definite function clear this is what is the main crux we need to look for in the examples. Typically we tend to miss because we know these are all kind of terms which are positive and then we do not see really what two state we are looking for. So, that is a kind of important condition.

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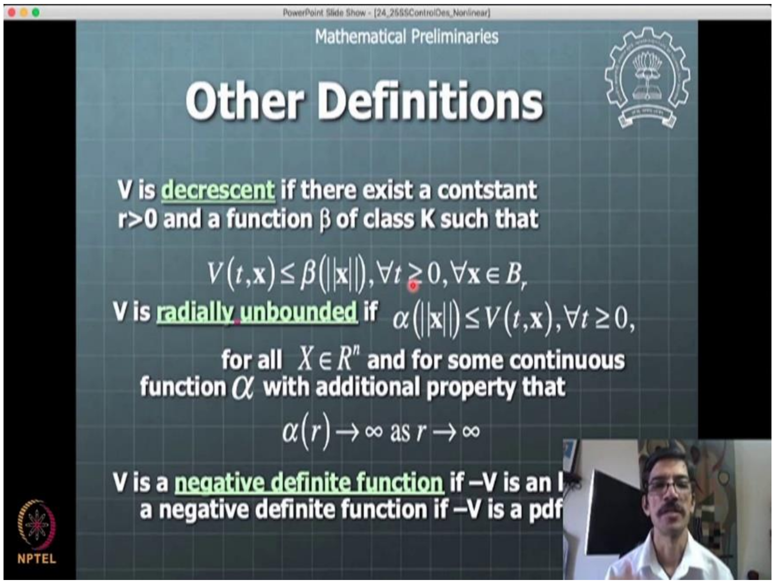

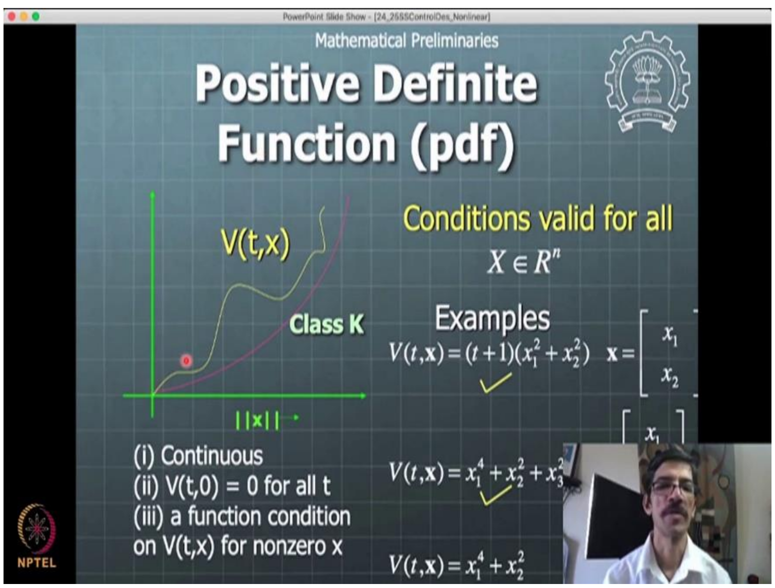

So, these other definitions are again based on if you understand class K functions and this is putting some more conditions on this. These are the conditions that typically Lyapunov theory it puts on some on these. So, these are more from the mathematical perspective but many of the examples we do not need to really worry about these conditions, most of the application examples in the actual control.

So, there are some kind of academic examples one can use which need this to be satisfied and things like that. But from an application perspective we need not bother too much about this condition. But we will see this for the sake of completeness. So, V is decrescent function if there exists a constant r greater than 0 and function β of class K such that V is lesser than β. So, you remember the α of class K function α was bounding this V from below.

And now this is bound from above. So, V is always less than this bη is guaranteed for any time t greater than 0. So, in time the function cannot go to infinity that is what is condition that it is bounded from above also by class K function. So, it I think it is we have done pictorial thing no I think there is no tab. So, that is a condition put for V for bonding from above also now by function of say function β of class K.

So, there is a bound coming from above also that is what is a decrescent function. Then V is radially unbounded if it is bounded from below by function α and for some continuous function α with the additional property in that α r tends to infinity as r tends to infinity. So, now this function α actually tends to infinity it is not a function which is so if you see this function here α of class K is actually going to infinity as x tends to infinity.

So, it is not getting saturated here somewhere. So, it has to go to infinity as t tends to infinity and it has again these bounds from the top and bottom. So, this is an additional condition this is a additional property. This α with an additional property but this radially unbounded condition will have this derescency also not I am not very sure about so we will check that. But radially unbounded function will have definitely this property it needs to be decrescent probably not.

So, function is decrescent bound from above and radially unbounded if this bound from below goes to infinity as t tends to infinity. And negative definite function if it is if minus V is an positive definite function or negative LPDF if the minus V is LPDF. So, these are the kind of conditions that we have more definitions.

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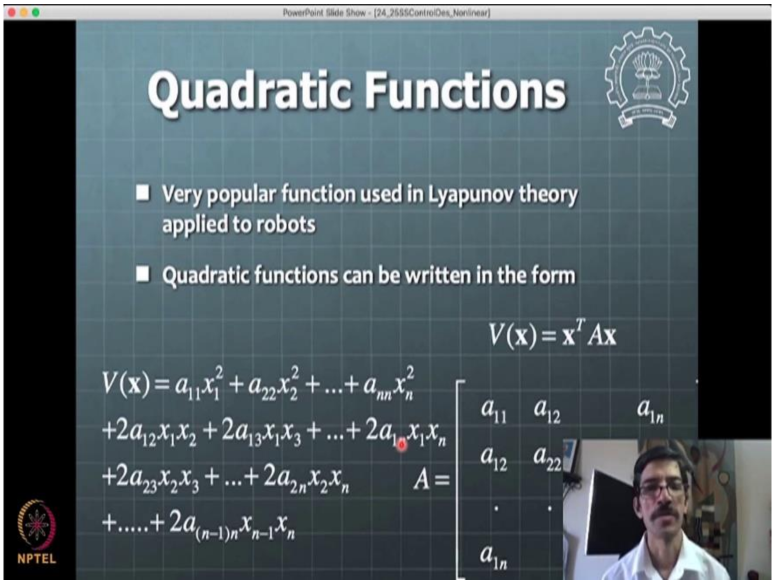


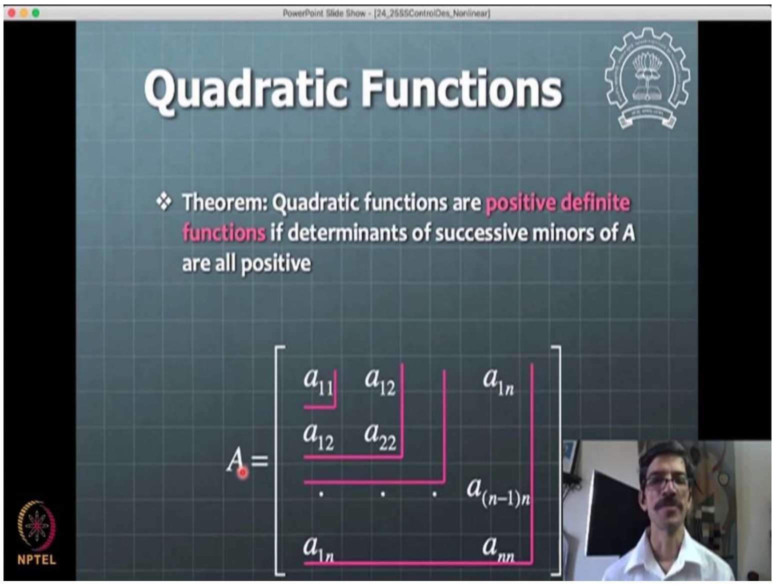
So, I think these are the main kind of fundamental definitions here. And then we start using the other concept of quadratic functions. So, this quadratic function you may be some of you may be aware about this concept. And we have partly seen it when we did the proof that your kinetic energy was considered as a quadratic function right in terms of the velocities or the generalized quadratic velocities.

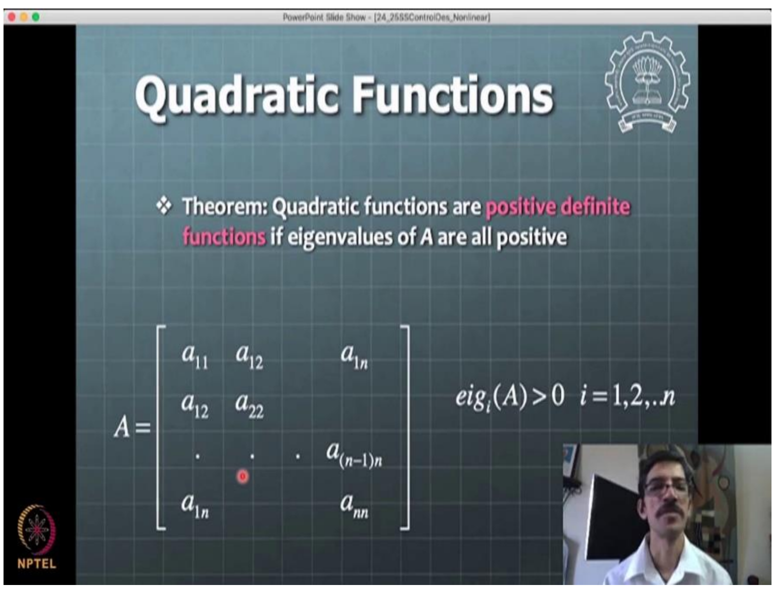
So, this concept we will see and probably will stop for this mathematical preliminaries part here. And then next part we will look at a proofs not proof like the theorems of Lyapunov and his application. So, these quadratic functions typically will have this kind of a quadratic form. So, you are familiar with this form where you have n vector state like say a vector belonging to R n space. And you are developing this form for all these n components coming up in this form.

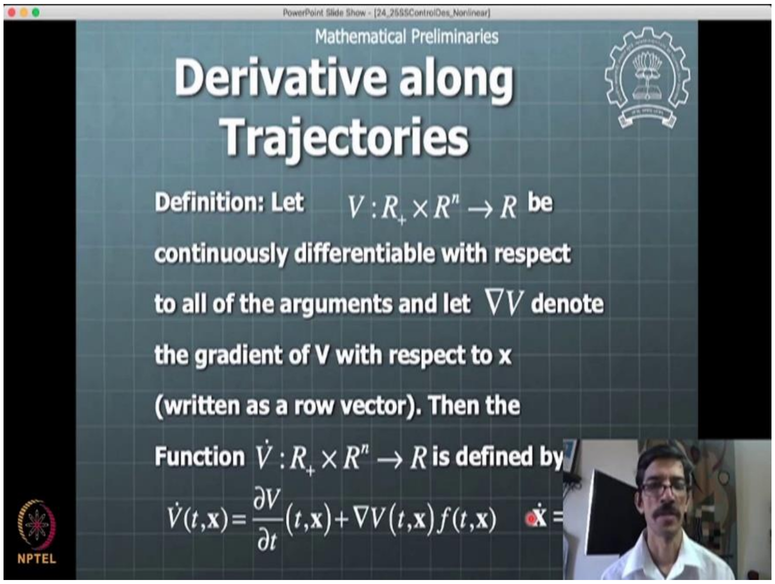
If this is a quadratic terms only here we do not have cubic and other kind of terms linear terms also are not there only all the terms are quadratic in nature. So, x_1 x_2 you can have multiplications of whatever components but degree no more than 2. Then this form can be expressed as a x^Tx and where A is this matrix of the coefficients here.

So, you see this cross coupling terms a_{12} a_{12} there kind of going to be same here I can split these two a12 into one part which will go up here and other part which will come up here. So, this is a kind of a quadratic form of so for matrix or given a matrix I can write a quadratic form in this kind of a fashion. So, either way and for such a form to be positive definite or those kind of properties it can hold there are certain conditions that mathematically have been derived.

So, we can one can talk about the positive definiteness of such a form based on there are many ways one can see one of the ways is look at minimum value of the quadratic function. And see that value if it is greater than 0 then the form is a having this property of pdf or LPDF or things like that we can talk about. So, for that there are by using that such kind of a concept of mathematical optimization or things like that. (Refer Slide Time: 29:37)







People have come up with this some kind of tangible conditions based on these theorems. So, they propose these theorems to give some kind of tangible conditions for this form of quadratic nature. Now, why we are bothered about this quadratic form because we will use these kinds of form many places for proposing the V's that we talked about in the definition. So, and then use them for some kind of Lyapunov theorem applications.

So, these successive minors if you take determinants of successive you minor. So, this is a first minor of matrix A then there is a second minor of matrix A and like that you can develop successive minors of this matrix A. And get a determinant and if the determinants are positive then you say this quadratic form is positive definite.

This quadratic form is a positive definite function. So, it has continuity we can have we see that continuity property is existing when it is the state is 0 you will have this form is equal to 0 that is also existing. So, only thing that we need is this taken on the positiveness or α function kind of a property that will come if the minimum of this function is guaranteed to be having a value greater than 0.

Then minimum with respect to x state of course then this form will be positive definite. So, using that condition one can get these other more tangible properties we not get into the proofs of these things in too much details right now. So, you need to see all the successive minors of A to be positive to get the positive definiteness property. Then this other kind of theorem which says that quadratic functions have positive definite functions if the Eigen values of A are all positive.

So, these are all Eigen values of A you can find out and if they are a positive then this quadratic form has a positive value. So, we now see this thing in the next part of the course. So, derivative along the trajectories is under a concept that we develop and that is where like know our function will come into picture. And then we will see what is a Lyapunov theorem for a stability and things like that. So, we will stop at this point here.