Design of Mechatronic Systems
Professor Prasanna S Gandhi
Department of Mechanical Engineering
Indian Institute of Technology Bombay
Lecture Number 39
Fundamentals of Lyapunov Theory

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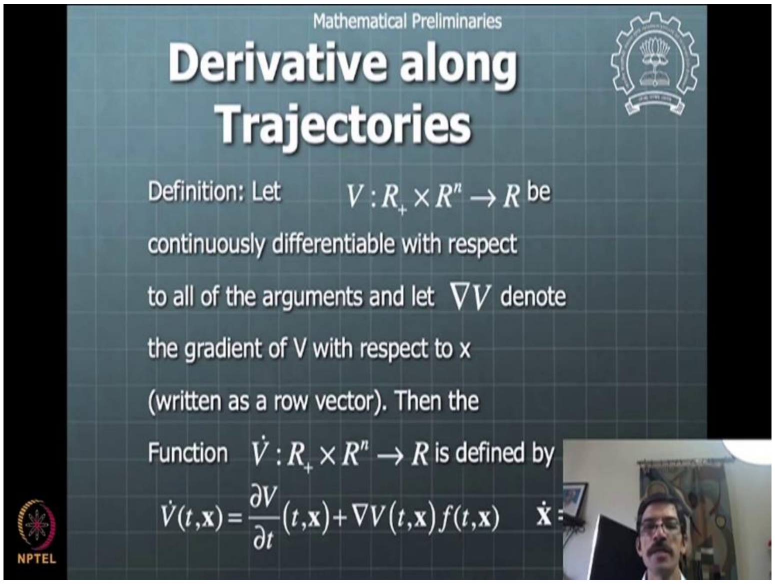

So now we will see this other part some more kind of a fundamentals and then we start application. So, next little bit of important concept is this derivative along the trajectories. So as you can see here the derivative along the trajectories is in like it is a concept for developed for the same function V. So V if V is given as the function, let me keep the mouse, of time t and vector R^n, vector belonging to R^n, space.

And it is a real, I mean, it is a real valued function here, taking V finally getting the value to be in the real space it is continuously differentiable with respect to all arguments. This is one condition that is necessary here that it should be continuously differentiable with all arguments and ∇V is denoting its gradient with respect to x. This is written as a row vector, it is a other kind of a small thing, then the function $\dot{V}$ defined by this way.

So, what we are doing here if you see the V being function of time and this vector is differentiation will have a time part plus this vector part and multiplied by ∇V by ∇ x dx by dt. This is a complete derivative so ∇v by ∇ t plus ∇v by ∇x into dx/ dt and ∇ V by ∇ x is actually this ∇V the gradient ∇

V. So, this part is fine but what we do here is for dx/dt which is $\dot{x}$ we are actually substituting our system.

So, this is a very important part here that we are taking the derivative along the trajectories of a of a system. So, if you generally see this V function it can have derivative in many different directions. This is a directional derivative. So, since V is a function of this vectors R^nspace this derivative is a directional derivative it is a vector function so it will have a derivative which is directional derivative.

And we are taking it in the direction of the of the trajectories direction given by these trajectories of your function or of your system. The system is defined as $\dot{x}$is equal to f (t ,x). Because we are substituting that $\dot{x}$ here we get this derivative which is along the trajectories of the system for this function.

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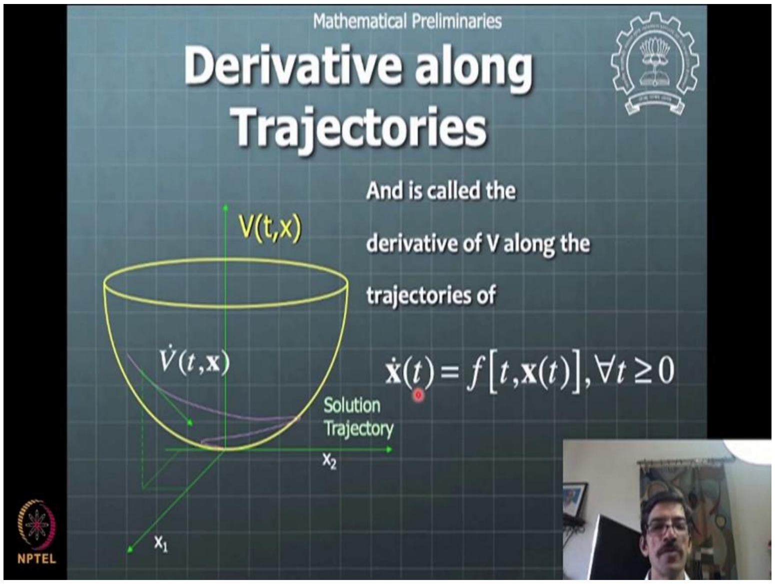

So, let me explain you this concept a little bit more in detail here. So, if you have this V x, now like let us consider the two state systems the x1 and x2 are only two states for the system. We have defined this V to be positive definite function in some way. So, I am considering this kind of a bowl nature for this V and if I am at any point on this V, whatever point I am, then at this point I can define derivative in many directions on this x space here.

So, along some direction, so if you map these points on the plane you will get some kind of a directions for the derivative. So, I can ask what is my derivative along, derivative of this function along some direction here in the x1, x2 space. Now this particular direction that I am choosing to define is trajectory evolution of this system on this space plane. So $\dot{x}$ t is actually giving me a derivative of this is kind of trajectories which are evolving.

So, when we put, we take a derivative along this we substitute this $\dot{x}$ into our equation for V, then actually we are kind of getting the derivative this way the function is evolving or the derivative function is going or the tangent to the to this V surface is a tangent to the surface along the direction of the trajectories. So, that is a kind of a notion that it gets into.

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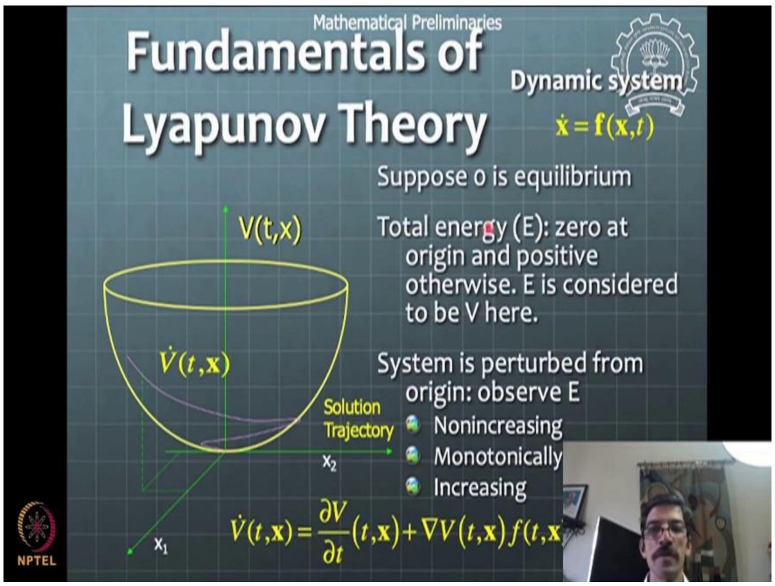

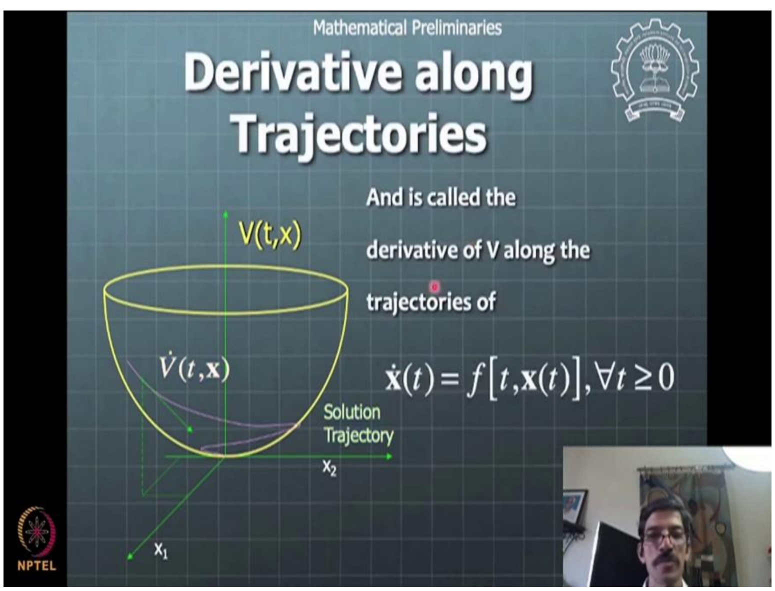

And if you see for these things in terms of so this so in terms of system dynamics if you see the system dynamics evolving here in some kind of a way when that system dynamics trajectory is mapped or projected on this V surface then along the system trajectory direction what is happening to V is what we are actually eventually getting. So, that is a kind of a concept for derivative trajectory.

Now, how do we see the stability of a system based on this condition. So, suppose we are considering the same kind of a system $\dot{x}$is equal to f (x t) and then the equilibrium for the system is 0. And total energy is 0 at origin and positive otherwise. So, that is we know this energy function has these properties that we want for this say function candidate v. So, V is considered as important for the function in the Lyapunov theorems eventually.

So, you say that okay there exists such a V in such a way that some conditions will come. So, this V Lyapunov function candidate is important and energy is one of the possibilities for considering for this V. So, we see this kind of idea in first in terms of the energy and then we can argue some of the things. Now, system is perturbed from origin and we start observing system is perturbed at some point from origin at this point.

And now we see the system kind of trajectories evolve and we see what is happening to my v or what is happening to my energy.

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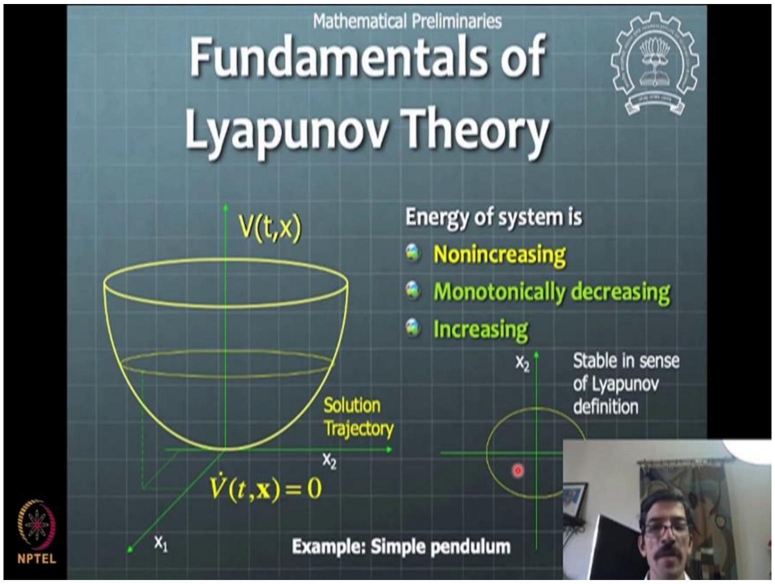

So, in first case the energy is non increasing the energy remains there so we are I am perturbing here and I am right now the energies is same. So, V is equal to 0 what it means is that the system trajectories will not leave this V level. And they will keep on kind of the system will keep on oscillating in the as projected on the this face plane. And see that the trajectories are going in the seconds.

Then if you see the other condition this is kind of a stable also in the sense of Lyapunov definition you remember that epsilon delta definition it is this is this system is stable in that sense. Although we may consider from the perspective of linear system this is like a marginally stable kind of behavior that one can see. Although, other thing is this what we are talking is of the equilibrium stability of the equilibrium.

In the Lyapunov we do not says the system stability because the system may have many equilibrium so, in the nonlinear systems. In a linear systems we have only one equilibrium. So, we do not have any problem. But here in the nonlinear system there are many equilibrium that are possible and we talk of the stability of the equilibrium rather than the system that is another important point to make sure.

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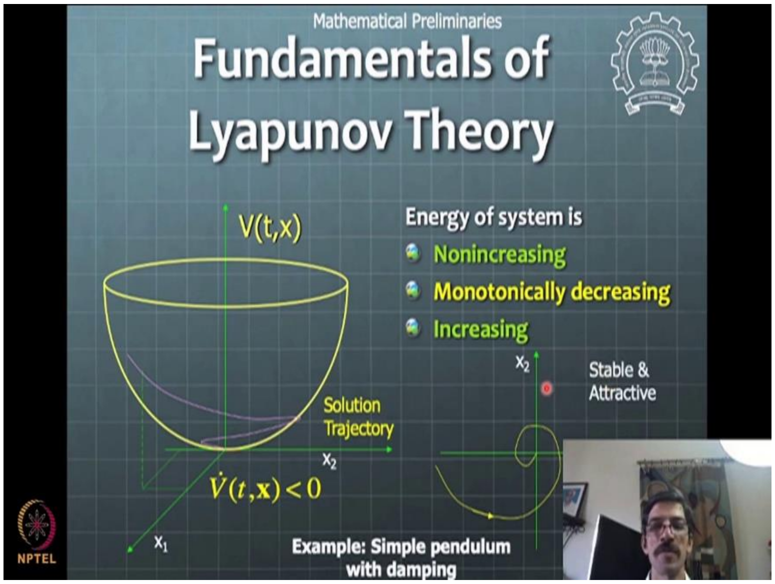

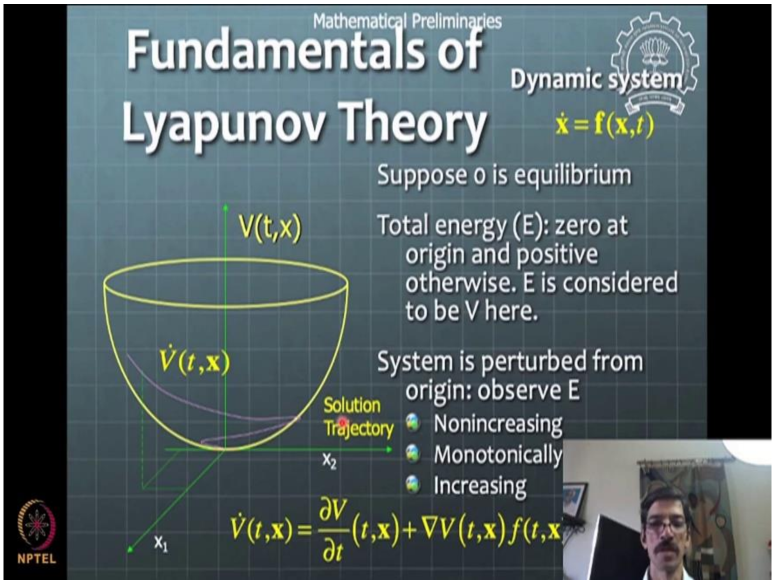

Then if you have an energy of a system monotonically decreasing. So from here energy at some So, these are longer trajectories and it is decreasing along the trajectories the energy of the system is kind of decreasing and eventually going to 0 as you see this is trajectories are so what we are doing is we are actually getting conditions from without the solution in place. So, what we are looking at is just a $\dot{V}$ is less than 0 or not.

So, we are not really looking at it as if it is solution is available and we are predicting the solution and we are seeing that, what we are getting these conditions directly based on the $\dot{V}$ computation as per this previous formula. So, this is a $\dot{V}$ we can compute to $\dot{V}$ directly and conclude something based on these about $\dot{V}$ right. We do not need really the solution of the system to be there.

That is the beauty of this theorem that without solving a system we can kind of do some kind of a predictions about the stability part. So, this I am showing you the solution just for the sake of understanding here that is if it is monotonically decreasing energy along the trajectories. So, some kind of a evolution will happen here, we do not know that, we do not we need not know as long as we know that $\dot{V}$ is getting less than 0 happening along the trajectories we are fine.

So, if that is the case that means some way the trajectories are kind of approaching this equilibrium with position 0. And then if it is increasing function, so this is becomes then stable and attractive kind of equilibrium and that is what gives you this asymptotic stability.

And when this is increasing, you can see that like if this is increasing in the direction along the surface of this V, then we are your trajectories are going away from, like going out of the bounds from the bounds of whatever, they are not coming back to the equilibrium 0 and the system is having that unstable kind of behavior.

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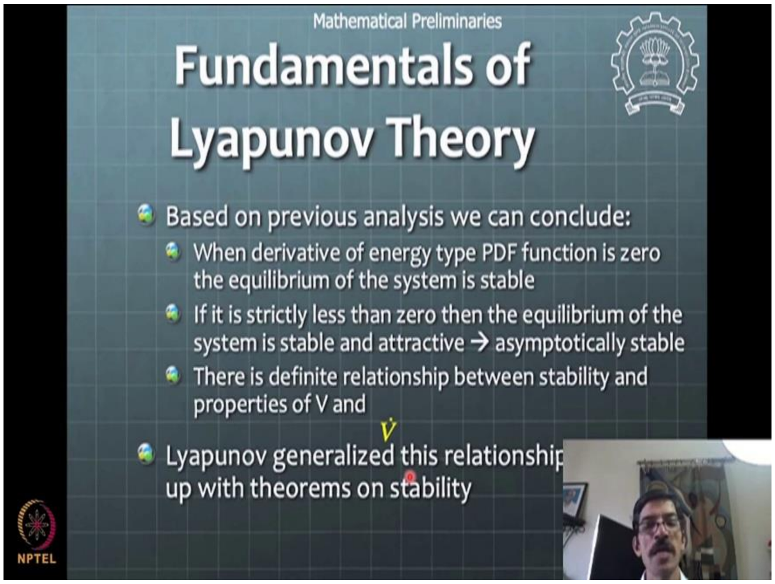

So, these are the basic foundations the fundamentals for this Lyapunov theory. So we can so this derivative of energy type positive definite function is 0 the equilibrium is stable. Then if it is strictly less than 0 then this derivative of energy type function if this is strictly less than 0 then the equilibrium of the system is stable and attractive which means it is having asymptotic stability. And there is this so this there is definite relationship between the stability and the properties of V.

So, this is some kind of sketchy kind of analysis to say that and there is more formal kind of a mathematical formation Lyapunov has proposed and come up with the theorems on stability. So, the idea here is not to restrict ourselves to only energy type of functions you can define any function which is pdf is another kind of a beauty of Lyapunov theory that you do not restrict yourself to only energy type of function any function you can take if you are able to find this.

So, this are so Lyapunov is all the necessary conditions. So, if you are able to find l whatever they satisfying the conditions then you have a result. So, if you are not able to find we cannot see those are kind of conditions that typically the theorem has so these are like one way kind of arguments.

So, if you are able to give me these conditions satisfying V or other kind of a function or whatever it is then I can guarantee you that your system will be stable or asymptotically stable or whatever that stability definitions will come for the system. So, that is a idea that Lyapunov has proposed so we will see one or two theorems and then we will close for now.

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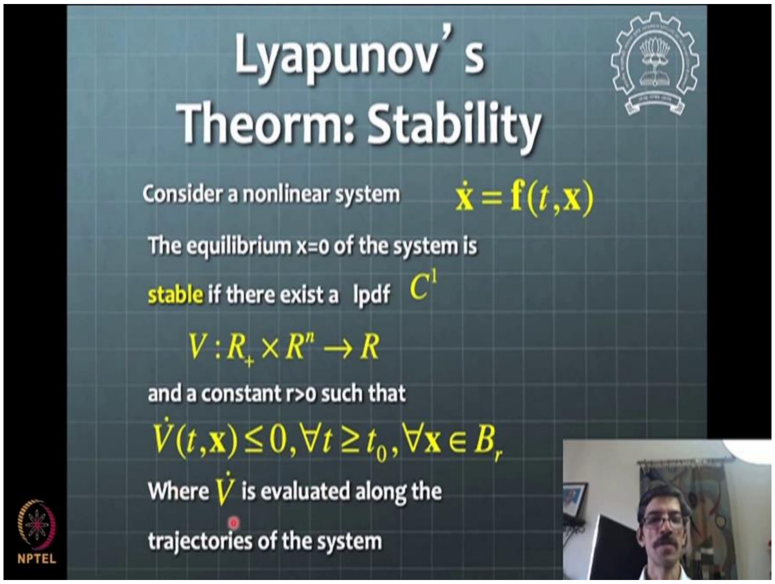

So this first theorem of stability proof the proof here we are just going to see these theorems now. The proof is there in the books if you want to get into but as a part of this class we are see this otherwise this again Lyapunov theory and it is all these theorems becomes like a matter of discussion, long discussion in maybe a half a semester kind of a course can be there on all the details nitty gritty of definitions and mathematics.

So, we are we are right now looking at more mainly application perspective for this is this where we are not getting into too many mathematical details but just the essential kind of details so that we can look at these theorems and apply them for a case that it is at our hand. That is the whole idea of development here. So, this theorem on stability says that equilibrium 0 of the system.

So, this system as we saw earlier that we where is free define the system such that the equilibrium is 0. And basically when 0 of the system is stable if there exists LPDF like locally positive definite function which is continuously differentiable at least once V which is again defining the based on this t and x here and real valued function.

And constant r greater than 0 such that this $\dot{V}$ of x is less than 0 less than or equal to 0, this sign is not less than 0 is less than or equal to 0 and for all t greater than t 0 and for all x belonging to ball of radius r. So, these are way constant is defined in the way that we have this condition valid at least for some distance or some kind of a norm around origin for that this is valid.

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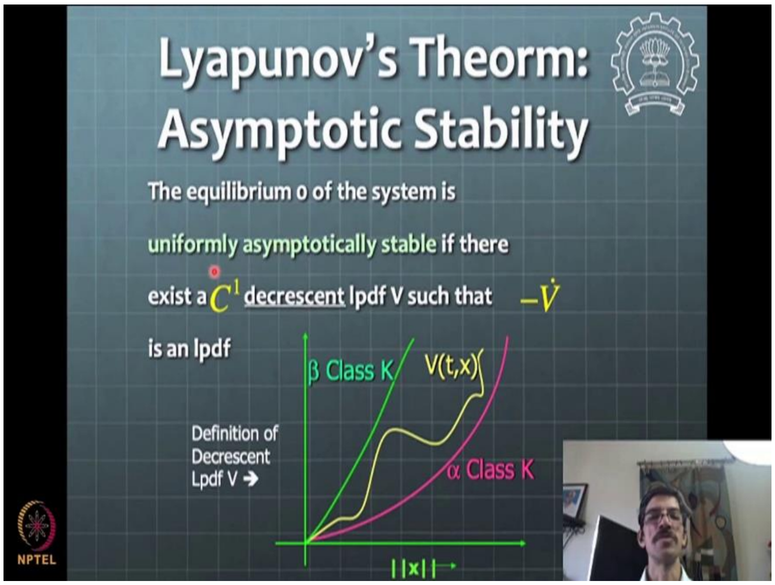

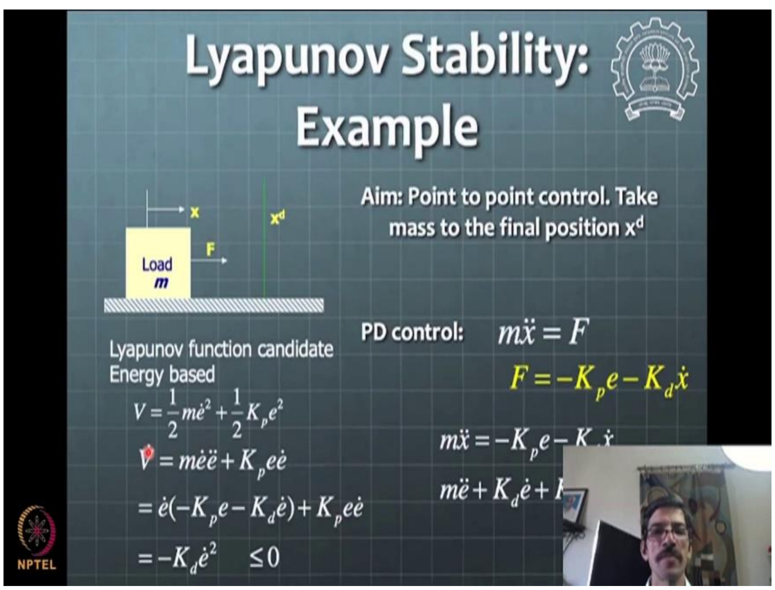

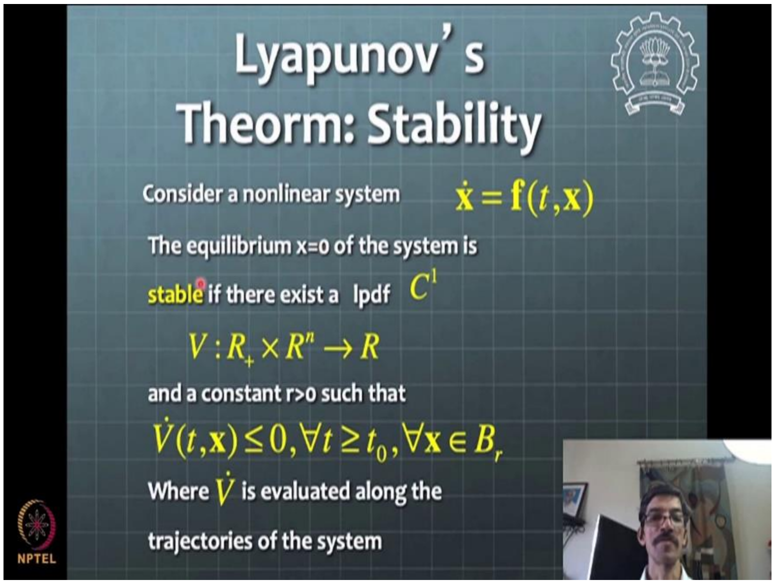

Then this equilibrium of the system is uniformly asymptotically stable. This is equilibrium stable. So, V dot is evaluated along the trajectories of the system. then this equilibrium is stable. So, there are this conditions which are giving putting a condition on $\dot{V}$ to be less than or equal to 0. So, this is important condition here. So, you define some kind of a V which is positive definite function of so it needs to have all the components of x represented in the function to be a LPDF as we saw in some examples.

So, you define some function it may not be energy function energy functions is typically the first candidate that we know we should try but it is not mandatory that it has to be an function. So, when we do our Lagrangian formulation already we have energy expressions available so that becomes like a naturally the first kind of candidate to consider that. So, that is why this is connected the V matrix is kind of very important for this Lyapunov theorems when we start using V as a as an energy function for our mechanical systems.

Then we have this definition of stability when $\dot{V}$ is less than or equal to 0. So, you need to establish that V dot is less than or equal to 0 by same kind of arguments of say either quadratic function or your basic definition of LPDF and things like that. All right now we have this uniformly asymptotic stable kind of a definition coming up here where now we have a this exist the crescent LPDF V.

So, now, this is decreascency condition is coming in the addiction. So, this function beta of class K is an upper bound you remember that decrescency we talked in the last part is about existence of this function beta of class K such that the V is bounded from above also by this function. So, that is bounded from above by this function and the bound from below is actually coming by the definition of a LPDF less or.

So, if that kind of thing exists then it is uniformly so this is a condition on V and for asymptotic stability we need this minus $\dot{V}$ is an LPDF. So, this is now a little stronger condition then $\dot{V}$ is less than or equal to 0. So, now we want this $\dot{V}$ - V dot to be an LPDF function. So, you checked in check this minus $\dot{V}$ function and see that it has his property of LPDF where you remember the properties like we have V so this minus $\dot{V}$ should be 0 here.

It should have some evolution such that it is bounded from below by some function of class K that is what will these properties of this $\dot{V}$ will be minus $\dot{V}$ V. So, you need to make sure that LPDF properties are satisfied or these minus $\dot{V}$ LPDF at least in some ball of radius r where r is greater than 0 from the origin you take some kind of a norm up their point r. And then in that thing it should be valid. Then that equilibrium means is all uniformly asymptotically stable.

So, there are more kind of these definitions so what we need to bother about is this asymptotically stable system and a stable system. These are the two main kind of points we take from here that are more kind of a finessy to these definitions asymptotic stability, uniformly asymptotically stable system or exponentially stable system all these kind of different definitions are theorems are there which we are not getting into more details.

What we are interested in to see is that this system is stable or asymptotically stable the asymptotic stability is a stronger condition to have because in asymptotic stability we actually look at the system trajectory is going to 0 so like that we will see. So, we will talk about this and more examples from the next class onwards and application from the next class onwards.

So, you can meanwhile try to apply to this case. And to see whether we are able to kind of see through the so do not look at this directly you can apply yourself to our simple kind of a case of mass lying on the surface to get a hold of get a hang of how things work here. And we will anyway discussing the next class.