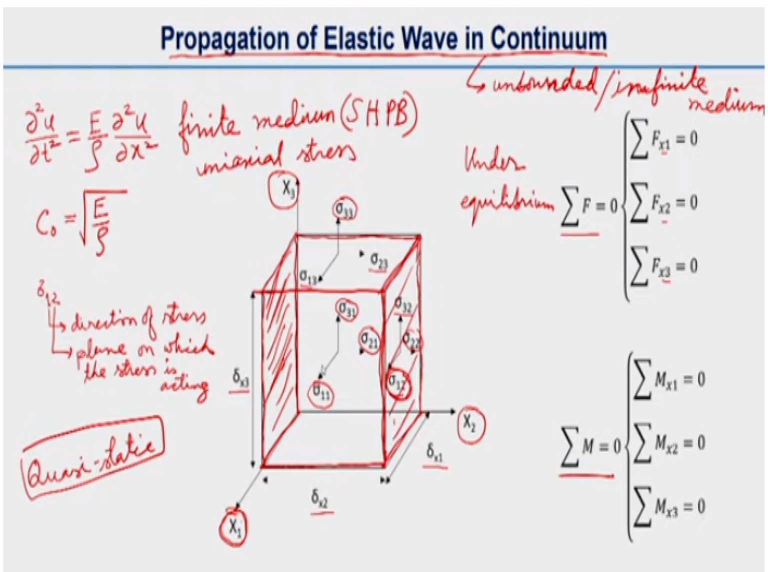
Dynamic Behavior of Materials
Prof. Prasenjit Khanikar
Department of Mechanical Engineering
Indian Institute of Technology – Guwahati, Assam

Module No # 01
Lecture No # 06
Propagation of Elastic Waves in Continuum

Hello everyone so this lecture we will discuss about propagation of elastic wave in continuum that means we will discuss about unbounded infinite medium earlier in previous class we discussed about finite medium like a stricking bar, hitting him long cylindrical bar.

(Refer Slide Time: 00:54)



So the equation for that we have derived as partial of eu second partial u with respect to time = Young's modulus of Rho and divided by Rho second partial of u which respect to x2 so this is for finite medium like Split-Hopkison Pressure Bar SHPB. So which is in this case it was uniaxial stress this is what we discuss in the in a previous lecture and then the velocity of the wave that longitudinal we found out to be the square root of the ratio of Young's modulus divided by density.

So now in this lecture we will discuss about the equation for propagation of elastic wave in continuum which will denote a which means unbounded or infinite medium. So let us consider and a body a unit q in that body so this is the q we have coordinate axis is X1, X2 and X3 this unit q has the length of the sides are delta X1, delta X2 and delta X3 so now the stresses acting on this unit q or we can see phase by phase the first will be we if see the front phase this phase are sigma 11, sigma 21 and sigma 31.

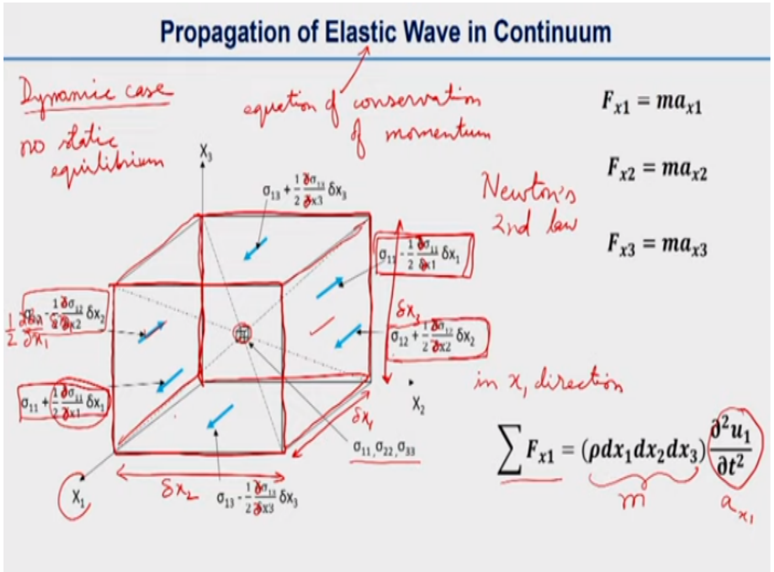
So this is the normal stress and these are the serious stress similarly if we see the stresses in tis phase on this phase it will be sigma 22 is the normal stress and other are serious stress and the similarly that the third phase under the top it has sigma 33 as the normal stress and other two are serious stresses. So we are not going to discuss about the concept of stress I am assuming that all the students have already studied about stress at a point and in this the stress component the suffix suppose we are writing as sigma 12.

So the first suffix will tell us direction of stress and the second suffix will tell us the plane on which the stress is acting. So if you see here sigma 12 so the direction of stress is 1 along this axon direction and then 2 either second suffix will give us the plane that means that plane is this plane is perpendicular to or normal to X2 so that 2 is second suffix. And under equilibrium so we are assuming that this case is a Quasi static case let us say and we think that this is under equilibrium.

So under equilibrium the forces are summation of forces will be equal to 0 and summation of moment will be equal to 0 so these are the X1, X2 and X3 are as you can see the different three different coordinate axis direction. So under equilibrium the static equilibrium conditions so we will get summation of forces = 0 and summation of moment = 0 and these planes are as you can see these planes are in a unit qr taken as the normal to the coordinate axis X1, X2 and X3.

And if you want to get the forces on the other phases here we are only showing three phases if you to get the force in the opposite phases let us say we have a right hand side right side phase is this one and then if you want to on the stresses on the phase on the left hand side so this phase then the stresses will be equal on that phase so these are equal on that phase if you compares this phase and this phase.

(Refer Slide Time: 07:20)



However if we go for a dynamic case so we do not have any static equilibrium so we have no static equilibrium. So we cannot assume like the case at the previous case now what will happen is there will be variation in the stresses of this phase and that phase. So now we will see that we will assume a center point of that unit q and let us this stresses are sigma11, sigma 2 and sigma 33 here.

So if you see in the front phase now this front phase so the stress acting here are sigma 11 + half of sorry this I am sorry about that these symbol are all somehow mistakenly take is this these are all partial of sigma 11 which respect to x1 so because this is our del X1 delta X1 this is delta X2 and this is our delta X3. So similarly if we want to talk about the back side plane this sigma will vary so this sigma and this sigma so if you see there is a difference.

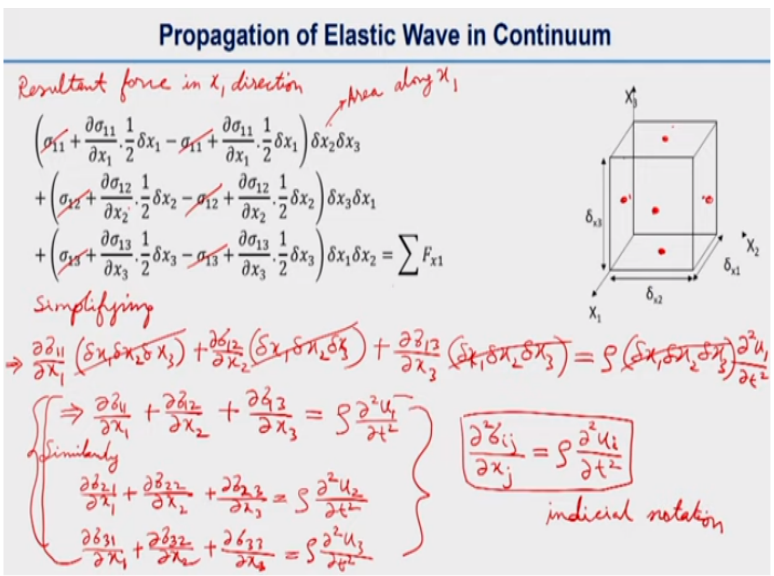
So this term half of partial of sigma 11 which respect to X1 delta X1 means the variation of stress when we move from center point to on this front surface so this means the variation and similarly this has negative value the same term with negative value that is in the negative direction. So similarly if you want to talk about the stresses on this phase right hand phase and left hand sorry left hand phase and right hand phase.

So this will be this one and here this one all of this very mistake so these are all partial derivative and similarly on the top and bottom phase we will get the similar trend. So now we will because this is we are talking about a condition that is that as no equilibrium no static equilibrium so we will use the Newton's second law. So basically to get the elastic wave equation for elastic wave equation.

So we want to use equation of conservation of momentum for shock wave later we will we may discuss about that for shock we may need conservation of mass and energy as well. But in this case so we will go with equation of conservation of momentum so if we have this Newton's second law the forces = force in different 3 different coordinate exist directions r = mass into acceleration into direction we can have in one direction let us say in X1 direction only X1 direction this direction in the figure and the X1 direction.

So summation of forces in X direction will be equal to mass of this in itself which is density into volume and multiplied by the acceleration X1 along X1 direction. So this is we are applying on Newton's law along the X1 direction so if we want to have the force's now we want to have the forces in the previous slide we have seen that there are stresses these stresses are on the phase that means actually stresses are defined at a point that is at the center of the phases.

(Refer Slide Time: 13:43)



So this stress this one or maybe this one which a negative sign if you find in whatever we have found in the previous slide is negative one because we have a negative here that is why we used positive sign here. So this one and this one is the stresses at the center of the phase suppose the front phase. And if you want to know the forces along this direction then what we need to do is we will multiply the stresses with the area so this is the area of this area along X1 direction area along X1 and this is the resultant force.

So we have resultant force in X1 direction sigma 11 and sigma 11 here and front phase and back phase and then similarly sigma 12 and sigma 12 from the left hand side phase and the right hand side phase. Similarly sigma 13 this entire term actually sigma 13 this term and this term similarly in the second case also this term and this term. So this term on the right hand side phase and this term is from the left hand side phase and then and the second last term this is on the top phase and then last term of the stress is from the bottom phase.

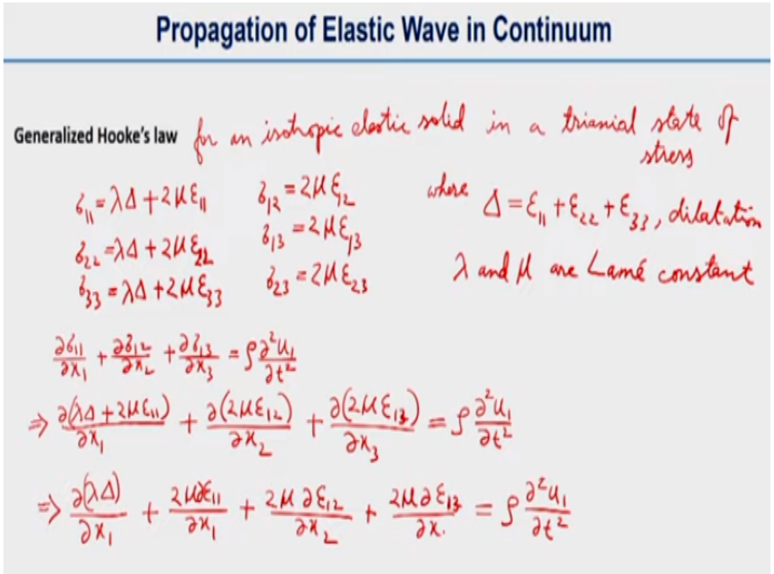
So now we multiplied area and then that is equal to the summation of all the forces resultant forces in X1 direction. So we want to simplify this equation now what we will do is we want to simplify this equation how we will simplify so we know that this sigma 11 and sigma 11 cancel out sigma 12 sigma 12 will cancel out sigma 13 and sigma 13 will cancel out. So this will give us so what we need is here simplifying partial of sigma 11 which respect to X1 multiplied by delta X1 delta X2 delta X3 + partial of sigma 12 with respect to X2.

So multiplied by delta X1, delta X2, delta X3 + partial of sigma 13 with respect to X3 delta X1, delta X2, delta X3 is = Rho delta X1, delta X2, delta X3 and second partial of u1 with respect to t. Now if we simplify we can cancel this volume term and that will give us partial of sigma 11 which respect to X1 partial of sigma 12 with respect to X2 partial of sigma 13 which respect to X3 which will be equal to density Rho times second partial of e1 which respect to time.

And similarly for other two direction sigma 21 X1 partial of sigma 22 which respect to X2 partial of sorry 23 which respect to X3 = Rho second partial of u2 which respect to t and partial of sigma 31 which respect to 1 partial of sigma 32 which respect to X2 partial of sigma 33 which respect to X3 and Rho multiplied by second partial of u3 which respect to time. So these are the three equations we are getting and these three equations in this initial notation we have not discussed about initial notation by the way if times permits we will discuss that in subsequent lecture some of you may have already some knowledge of initial notation.

So we will I will show you how we can express it indicial notation its sigma partial of sigma subscript Ij with respect to Xj = Rho density second partial of ui with respect to time. So that I called an indicial notation so these are easier way to express equations like this so we can discuss this later. But even if we do not use the initial notation we can work out this derivation so these three equations is can if we can solve and it will give the equation of elastic wave so we will be working on that and for that by the way stresses need to be replaced by strains we will see how to do that.

(Refer Slide Time: 22:02)



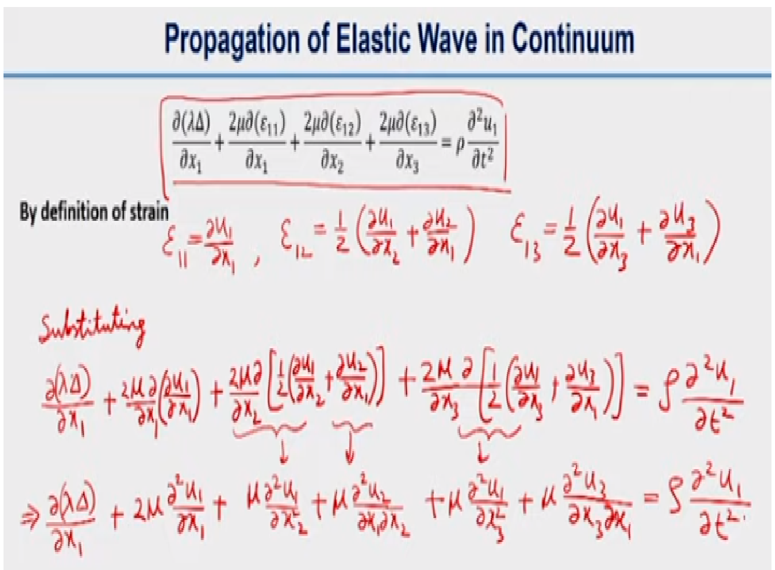
So now for generalized Hook's law for an isotropic elastic solid in a tri-axial state of stress it is not uniaxial, tri-axial state of stress can be expressed with this equations like sigma subscript 11 is lambda delta I will tell what this symbol means twice Mu epsilon 11 similarly sigma 22 is lambda delta twice Mu epsilon 11 and sigma 33 lambda delta twice Mu epsilon 33 sorry this one is 22 and similarly the shear stress is 12 = twice Mu epsilon 12 sigma 13 = twice Mu sigma 13 sigma 23 = twice Mu epsilon 23 I am sorry this say 13.

So here that delta is nothing but epsilon 11 + epsilon 22 + epsilon 33 and it is call as dilatation and both lambda and Mu are Lame's constant. So they are the material constant I will tell later the relationship between lambda and Mu with Young's modulus and partial ratio. So these are this 6 equations are generalized Hook's law and then we will be using these equations in our previous previously found equation.

So whatever we found earlier is partial log sigma 11 with respect to X1 + partial of sigma 12 with respect to X2 + partial log sigma 13 with respect to X3 = density multiplied by second partial of e1 with respect to time. Now if we substitute these stresses with strain what we get is partial of lambda delta + twice Mu epsilon 1 with respect to the whole thing the partial derivative of whole thing with respect to X1 + partial log twice Mu epsilon 12 with respect to X2 + partial of twice Mu epsilon 12 with respect to X3 = Rho second partial of e1 with respect to t.

And if we simplify more partial of lambda multiplied by delta with respect to X1 then twice Mu epsilon subscript 11 X1 so (()) (27:19) partial sign here + twice of Mu partial of epsilon 12 with respect to X2 twice Mu. So as you can understand the Mu is the constant material constant so that comes out of this partial derivative and that will be equal to Rho per second partial of e1 with respect to t sorry I am I have mistakenly written this as epsilon 12 this should be epsilon 13 here also epsilon 13 not 12.

(Refer Slide Time: 28:17)

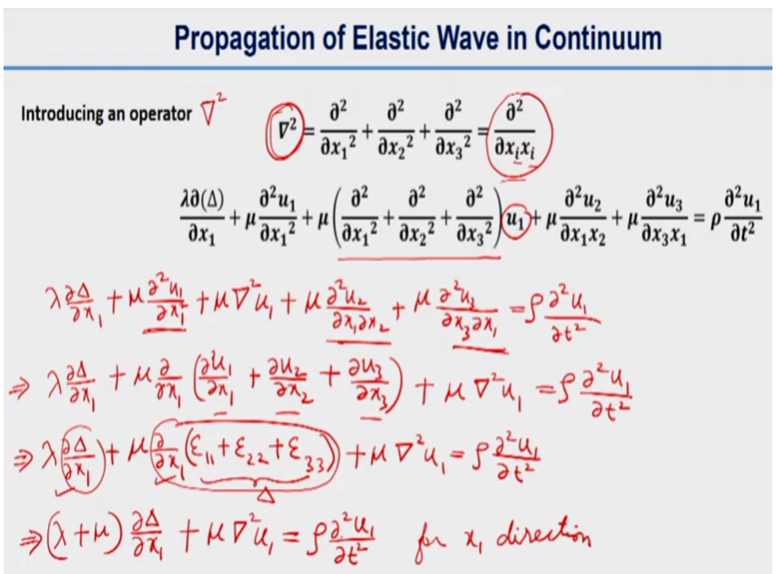


Then the whatever expression we found in the previous slide is this one so we will use the definition of strain to convert this strains to displacement so the definition of strains is epsilon 11 = partial of u1 with respect to X1 then similarly in the shear strain epsilon 12 is half of partial of u1 with respect to X2 partial of u2 with respect to X1 and then epsilon 2 13 sorry we are talking about only X1 direction now. So half of partial of u1 with respect to X3 + partial of u3 with respect to X1.

And then if we substitute substituting on the above equation this equation then we will find that partial of lambda delta with respect to X1 + twice Mu here replacing epsilon 11 with displacement and there is with respect to X1+ twice Mu partial of the whole thing half of partial of u1 with respect to X2 + partial of u2 with respect to X1 then the third term will be twice Mu V half of partial of u1 with respect to X3 and the partial of u2 with respect to X1 and will be equal to Rho second partial of u1 with respect to time.

So now simplifying more the first term will remain same the second term we will simplify as sorry the second term we can simplify as second partial of u1 with respect to X1 + second term the first this will not be true so from this term we get Mu second partial of u1 with respect to X2 and then from this term we can get Mu second partial of u2 X1 with respect to X1 and X2 + from this term Mu second partial of u1 with respect to X3 + Mu second partial of u2 with respect to X3 and X1 which will be equal to Rho multiplied by second partial of u1 which respect to t.

(Refer Slide Time: 33:00)



So now we will be using an introducing an operator and Nabla 2 so this is the operator so this is with initial notation will not discuss much but this indicial notation if you see there are two i's here that shows the summation is implied. So these repeated index shows summation so we can discuss this later if we if time permits but without this indicial notation we will be working out.

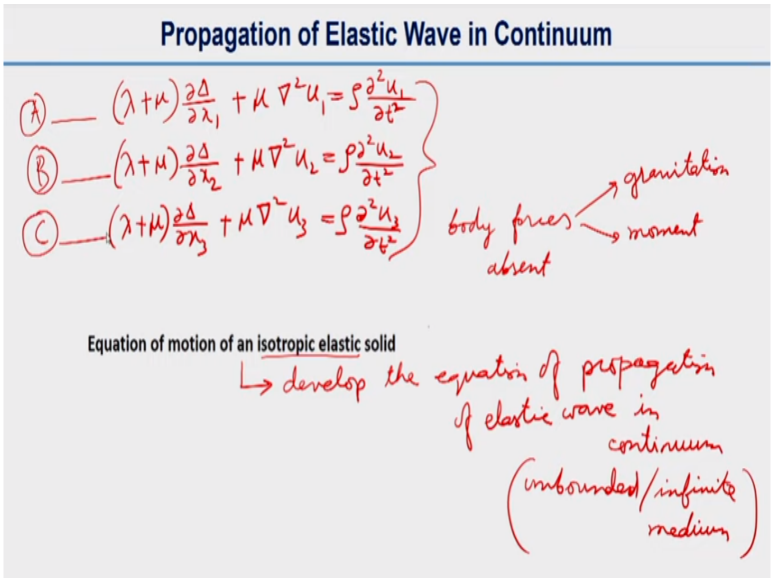
So in the previous slide we ended with this one and so we will start from there so here so if we use the Nabla in this part so what it will give is lambda partial of delta with respect to X1 + Mu double partial of u1 with respect to X1 and then Mu and the operator with u1 + u1 + Mu double partial of u2 with respect to X1 X2 + Mu double partial of u3 with respect to X3 and X1 will be equal to density times double partial of u1 which respect to t1.

So from the previous slide we simplified it we actually took the similar terms together and then we use this operator and that can be again simplified so we can see some similar terms here. So these terms this term can group them together so that will give us hmmm Mu partial with respect to X1 and then we can put it so what we did is so this term and this term we combined it and we are left with the one more term Rho 1 with respect to t. So as you know these are the definition.

So what we can do is again we want to simplify so basically what we are doing here is we want to simplify the expression and we want to get the expression in terms of the operator this operator and also the dilation delta. So these term you know is delta so from here again we can combine this term and this term so because this lambda and this is Mu and this part is similar with this you know this from here we took this and this entire thing also give us the same thing because that is delta this is equal to delta epsilon 11 + epsilon 22 + epsilon 33.

So this we have a second term Mu with this operator and finally we get this expression the t2 sorry I missed it u1 here. So similarly this is for only X1 direction similar we can get for X1 direction similarly we can get it for Y1 and sorry X2 and X3 as well.

(Refer Slide Time: 38:26)

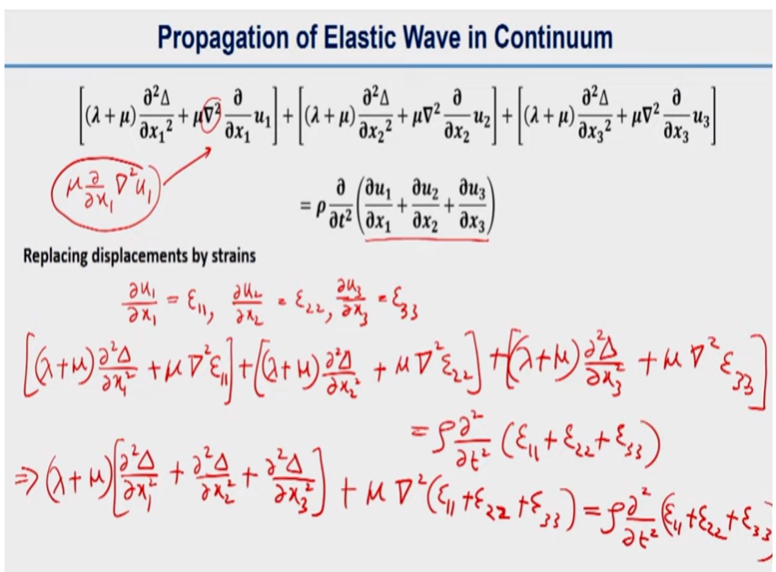


So we will write this equations again so whatever we found in the last slide is lambda + Mu partial of delta with respect to X1 + Mu with this operator e1 = Rho second partial of u1 which respect to t2 so this is for X1 direction for X2 direction we can have lambda Mu the same but which respect to X2 Mu u2 and Rho second partial of u2 with respect to t2. Similarly for X3 direction we can write here u3 and then second partial of u3 which respect to t.

Now these are called equation of motion these are equation of motion for an isotropic elastic body and the body forces are absent here body forces absent body forces means gravitation and even the moment so both are absent and the so what we will do is these equations so these three equations will be using to develop the equation of propagation of elastic wave in continuum.

So in continuum here we will be we are doing for infinite or unbounded media unbounded or infinite medium. So how we will do it we will the replace the replacement from these equations by strains and we can group some of the equations together. So what we will do is here is if for our earlier equation if we see these equations so let us say we call these equations as a, b and c.

(Refer Slide Time: 41:44)



So these equations what we will do here is to get the wave equation so the equation A will differentiate with respect to X1 partial derivative with respect to X1 similarly with the equation B with respect to X2 and equation C with respect to X3 so this will give us this expression. So this is the expression whatever we did it for this partial derivative with respect to X1 for equation

A and then similarly partial derivative with respect to X2 for equation b and then partial derivative with respective to X3 for equation 3.

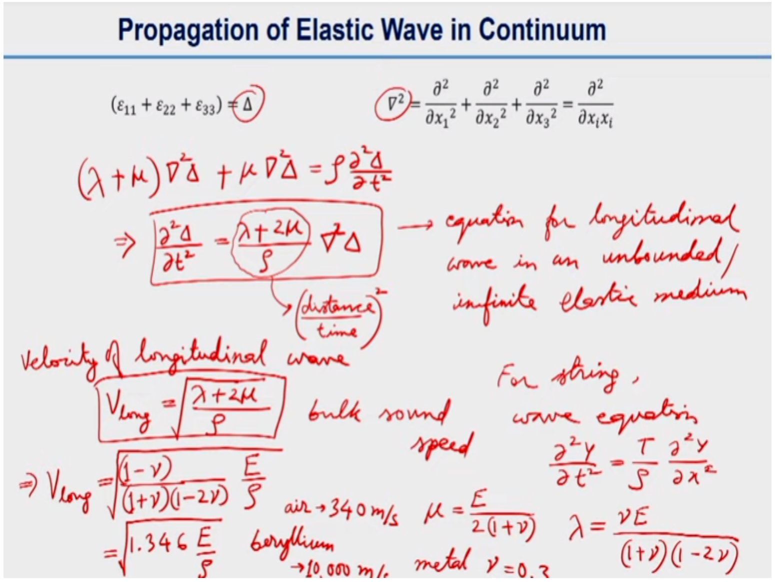
And similarly in the right hand side also we can have this bracket here so we have with respect to X1 with respect to X2 and with respect to X3. So if you want to simplify it more so we are actually operating with the partial derivative that means so that keeps us say second partial of delta which respect to X1 then here again second partial of delta with respect to X2 and similarly the second partial of delta with respect to X3.

So this expression we want to simplify more so you can see that we have did some more changes here so what we did is so Nabla is now outside here and then the earlier it was written as Mu partial with respect to X1 with the operator sorry this is u1 and then entire term we have rewritten like this and this terms are combined and so what we can do is now we can replacing the displacement by strains.

So as you know the displacement we can replace it like this partial of u1 with respect to X1 will give you epsilon 11 partial of u2 with respect to X2 will give epsilon 22 and then partial of u3 with respect to X3 will give epsilon 33. So if we replace if we substitute so what will happen is lambda + Mu double partial of delta which respect to X1 Mu with a Nabla operator we have epsilon 11 this is one term here and then Lambda + Mu double partial of delta with respect to X2 + Mu with a Nabla operator epsilon 22 + Lambda + Mu a this is delta with respect to X3 + Mu we need to be similar with 33 okay.

This will give you on the right hand side Rho double partial with respect to tt with epsilon 11 epsilon 22, epsilon 33 double partial sorry second partial of this with respect to time and then grouping similar terms we get that lambda + Mu this will delta sorry this is X1 + this X2 double partial of this X3 + Mu epsilon 11, 22 and 33 = Rho epsilon 11, 22 and 33.

(Refer Slide Time: 47:20)



So now if we use this delta and this Nabla operator so we can get from the previous equation from the previous equation what we can get is lambda + Mu with the Nabla operator and delta sorry + Nabla operator with delta Rho second partial of delta with respect to t so which can be again we can rearrange and that will give you double partial of delta with respect to t = lambda + twice Mu what will name is constant divided by your density to Nabla operator with delta.

So this is the equation for longitudinal wave in an unbounded or infinite elastic medium. So this is like elastic wave we are talking about longitudinal wave so this unit the dimension of this term is distance per times whole square. So you can check that term so dimension is distance divided by time the whole square and the velocity.

So you know that this analogous to the equation we derive for the wave equation of the string so for a string we derived earlier the wave equation is wave equation is what that it is something like that whatever we derived and so this also looks similar and from here we can tell that the velocity of longitudinal wave is V longitudinal is square root of this ratio lambda + twice Mu divided by Rho.

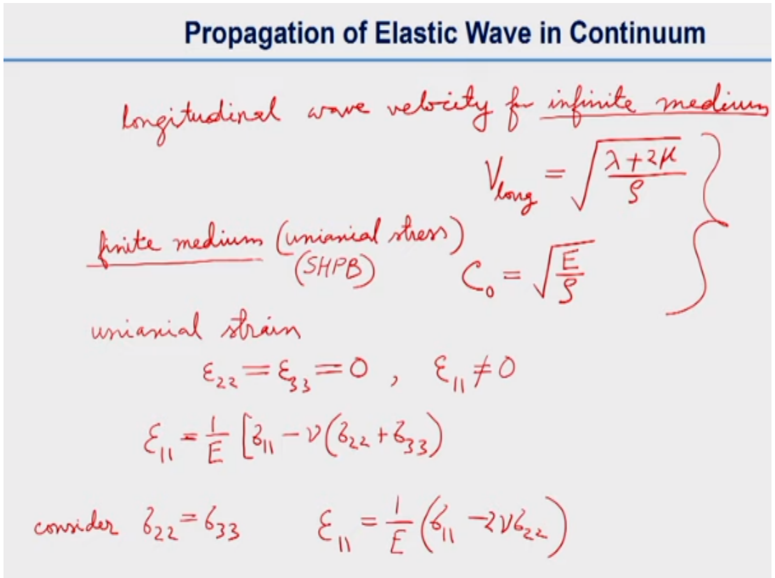
So this is the velocity and this also often called as the bulk sound speed velocity of longitudinal wave in the elective wave in an infinite media and the if we use on the relation between the Lame's constant and Young's modulus and Poisson ratio so Mu will be equal to e divided by 2 1

+ Mu is the Young's Modulus Mu is the Poisson ratio and similarly lambda = Mu e Poisson ratio multiplied by Young's modulus divided by 1 + Mu and multiplied by 1 – twice Mu.

So now that V longitudinal waves velocity can be expressed in terms of Young's modulus and Poisson ratio this will be equal to 1- Mu divided by 1 + Mu 1 – twice Mu and here e / Rho the entire you need to take square root. So ultimately if let us say for metal we assume that Poisson ratio is 0.3 so which will give you 1.346 e / Rho square root of the whole thing. So that means longitudinal wave velocity depends on the Young's modulus and density and if you see for different material for air it is 340 meters per second.

And if you see some if you check for some material like beryllium so which has a low density but high modulus of elasticity that wave velocity will be very high 10,000 meter per second. So we will this is longitudinal wave velocity we will check that for other materials also later but this we can tell that for air it is very low it depends on the Young's model and the density but for the beryllium it is very high.

(Refer Slide Time: 53:47)



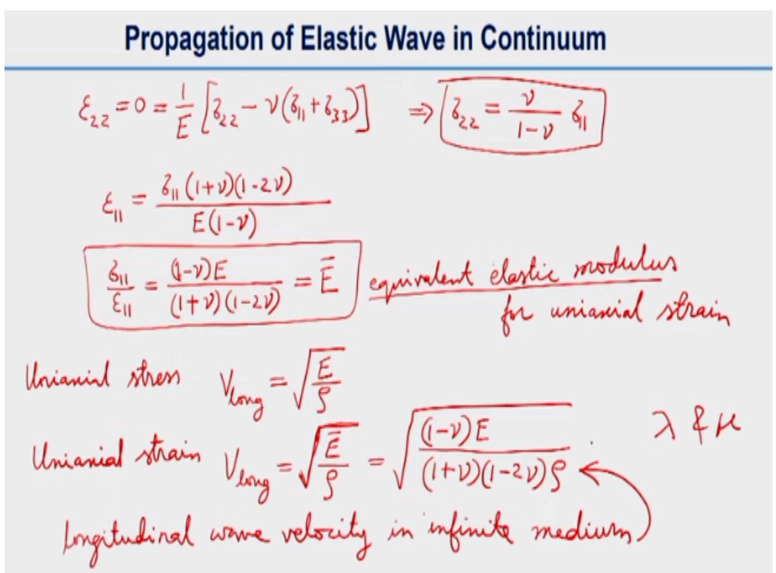
So what we have derived is the longitudinal wave velocity for infinite medium is square root of Lame's constant lambda + twice of Lame's constant Mu divided by mass density and we know for finite media finite medium for example the Split- Hopkinson pressure bar that is uniaxial case uniaxial stress case so the velocity which will (()) (55:08) C naught which is the longitudinal velocity is expressed as e / Rho.

Now we want to know this similarity between these two wave velocities the first one is the infinite and bounded medium and the second one is for finite medium which is the case of a striking bar hitting a long cylindrical bar that is uniaxial case stress case. So now we will see the similarity between these two so for that what will do is we will consider the case of uniaxial strain early interfinite media for let us say for Split- Hopkinson pressure bar we use we consider this as a uniaxial stress but now we are considering uniaxial strain which means epsilon 22 = epsilon 33 = 0 only epsilon 11 strain in the one direction is not equal to 0.

Now from stress strain relationship epsilon 11 will be equal to 1 / Young's modulus sigma 11 – Poisson's ratio multiplied by 22 + sigma 33. But here we know that we will consider here sigma 22 = sigma 33 we will consider this because it is only uniaxial strain and the other two direction stresses are equal so then the earlier expression will be epsilon 11 1 / e sigma 11 – twice sigma 22.

So we have removed sigma 33 and we are writing it is in terms of sorry this twice Mu / n multiply two times of Poisson's ratio multiplied by sigma 22 we have removed sigma 33 we are writing in terms of only sigma 11 and sigma 22.

(Refer Slide Time: 58:21)



And similarly for epsilon 22 which is equal to 0 but the expression for epsilon 22 is 1 / E sigma 22 Poisson's ratio times 11 sigma 33 and this gives us sigma 22 = Mu 1 – Mu sigma 11 so if we replace this sigma 22 in the earlier expression of epsilon 11 with whatever we got here. So then the expression will look like sigma 11 1 + Mu 1 – twice Mu Young's modulus times 1 – Mu so if you want to know the ratio of normal stress and normal strain in the X1 direction then that will look like 1 – Mu 1+ Mu twice Mu which will be.

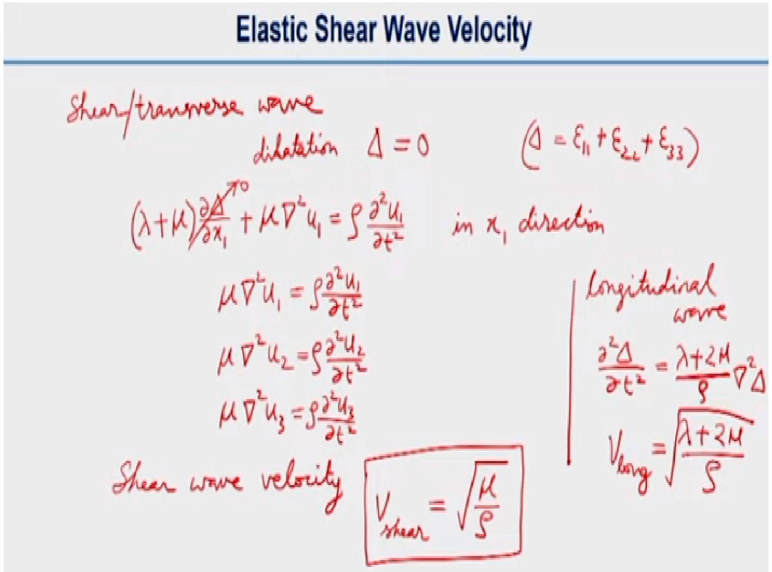
So the ratio of normal stress and normal strain which is the equivalent elastic modulus for uniaxial strain case uniaxial strain so now as we know that for a finite medium that is what for a uniaxial case uniaxial stress case. So the V longitudinal is expressed by E / Rho now in this case for uniaxial strain if we want to express in a similar fashion so E bar which is earlier expression regard / Rho and that will be equal to 1 – Mu multiplied by elastic modulus 1 + Mu 1 – twice Mu multiplied by Rho.

So if you if we check the earlier expression in the case of earlier expression what we derived in case of infinite medium is exactly the same expression and also we expressed it in terms of lambda and Mu as well Lame's constant but there we actually derive the same expression earlier. So that means for infinite medium for infinite medium or we can write longitudinal wave velocity in infinite or unbounded medium takes the same expression.

So that means now we have a connection and we can say that this is can be considered as the uniaxial strain case because we have derived it that the equivalent elastic modulus for uniaxial strain has this expression and if we incorporate this expression into square root of ratio between elastic modulus divided by density then we will end up with same expression of longitudinal wave velocity. And that means that we can consider it as the uniaxial strain case.

So next we will discuss about shear wave velocity what we discussed earlier is the longitudinal wave velocity.

(Refer Slide Time: 1:03:26)



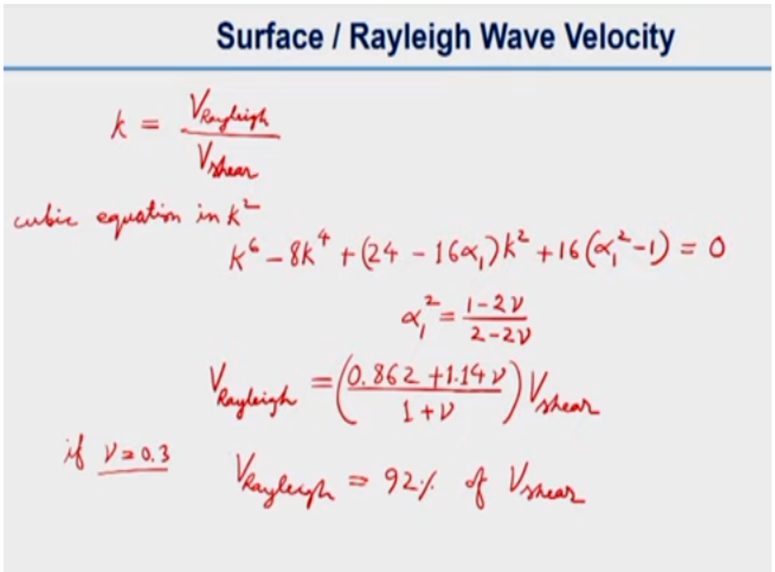
So shear wave velocity or transverse wave velocity is shear wave or transverse wave as the dilation = 0 the dilation term dilatation = 0 which means dilatation is epsilon 11, epsilon 22, epsilon 33 so that is equal to 0 and earlier we had the expression for that equation of motion which is means constant lambda + Lame constant Mu partial derivative of dilation with respect to X1 Mu which the operator Nabla this u1 and = mass density second partial of u1 which respect to t.

So this is X1 direction and X1 direction so if we take the dilation delta = 0 so if it is 0 then we will end up with Mu it is expression will end up with this expression and similarly for other two directions will have and for third direction. So now if we compare with the our earlier derivation so we will find that velocity shear wave velocity can be written as V shear = Mu / Rho. So earlier for longitudinal wave what we found is for longitudinal wave our equation was looks something like this it look something like this and from there we evaluated the our longitudinal wave velocity is this.

And similarly in this case for shear wave velocity we can say that this is the square root of ratio of Lame constant Mu / mass density Rho. So this is the shear wave velocity and we will see different wave velocity is of different materials in a short while. So for shear wave velocity longitudinal velocity for different materials but before going to that we will also see about surface or Rayleigh wave velocity sorry this is I wrote the Rho but it is somehow erased off so this is Rho lambda / twice Mu divided by Rho and that next lined is correct.

So we have Rho here so the derivation for Rayleigh wave velocity is not included here however ahh we will just show the ratio of Rayleigh wave velocity to shear wave velocity.

(Refer Slide Time: 1:08:16)



So K the ratio is Rayleigh wave velocity V Rayleigh by V shear is a K and then K can be found is a cubic equation in equation in K square. So it is a complicated equation it looks little complicated here. So this is the equation and where alpha 1 square is 1 – twice Mu divided by 2 – twice Mu and this will be this will come out to be V Rayleigh = 0.862 + 1.14 Mu divided by 1 + Mu times shear wave velocity.

And if we assume Poisson’s ratio is 0.3 which is almost metals then V Rayleigh is 92% of shear velocity shear wave velocity. So that is the relationship for Mu = 0.3 V Rayleigh is just 8% lower than the shear wave velocity.

(Refer Slide Time: 1:10:48)

Wave Velocities of Different Materials

Material	Young Modulus E (GPa)	Density ρ (kg/m^3)	Longitudinal Wave Velocity (m/s)		Shear Wave Velocity (m/s)
			Thin Bar	Infinite Body	
Copper	130	8930	3812	4758	2325
Aluminum	70	2700	5103	6394	3109
Iron	211	7850	5189	5961	3224
Alumina (Al_2O_3)	365	3900	9674	11225	6000
Diamond	1000	3510	16879		

So next we will see of different materials so here are some common materials like copper, metal, aluminum, iron they are very commonly used materials and then alumina is like a ceramic Al2O3 and then very hard material diamond. So if you see the Young's modulus for the copper it is 130 and the density is 8930 if we talk about different wave velocity we have not included a Rayleigh wave velocity here but there are two types of longitudinal wave velocity we included here one is the tin bar that is for a finite medium and then second one is the infinite body.

If you can see the tin bar has 3812 meter per second that means almost kind of 4 kilometers per second 3812 meter per second so infinite body it has 4.7 kilometer per second and then shear wave velocity is 2325 meter per second. Similarly for aluminum it has a lower young's modulus and lower density so it will result however a higher longitudinal and shear wave velocity for iron it has higher Young's modulus and higher density.

However this longitudinal wave velocities are very much similar with the aluminum if we can see the very much similar with aluminum. Now we come to the ceramic which is quite high strength and low density this will give as a very high longitudinal and shear wave velocity. So this is very high 11 kilometer for a infinite body 11 kilometer per second and for diamond it is even higher because the Young's modulus says unusually high and then density is quite low and then this gives us for a tin bar it is 16.8 kilometers per second for a tin bar others we have not keep the values here.

So that is all about wave velocity we derived expression for longitudinal wave velocities in continuum that is infinite medium or unbounded medium we also mentioned about shear wave velocity expression and we talk little about Rayleigh wave velocity as well we also listed some of the materials wave velocities with their Young's modulus and density values. So that is all for this lecture thank you.