Dynamic Behaviour of Materials
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Module No. #03
Lecture No. #08
General Solution of Elastic Wave Equation

Hello everyone, so far we have discussed about, the Wave Equation of Vibrating string, or a rope, and vibrating a helical spring. And, we also discussed about, Wave Equation of cylindrical bar, when a projectile hit the bar, from one end. And also, we discussed about the Wave Equation for, unbounded medium or infinite body. So, in this discussions, so either, we took a simple Wave shape, or we did not explicitly mention, the shape of the Wave. So, now in this lecture, we want to discuss, we want to derive, the General Solution of Wave Equation, for a general shape.

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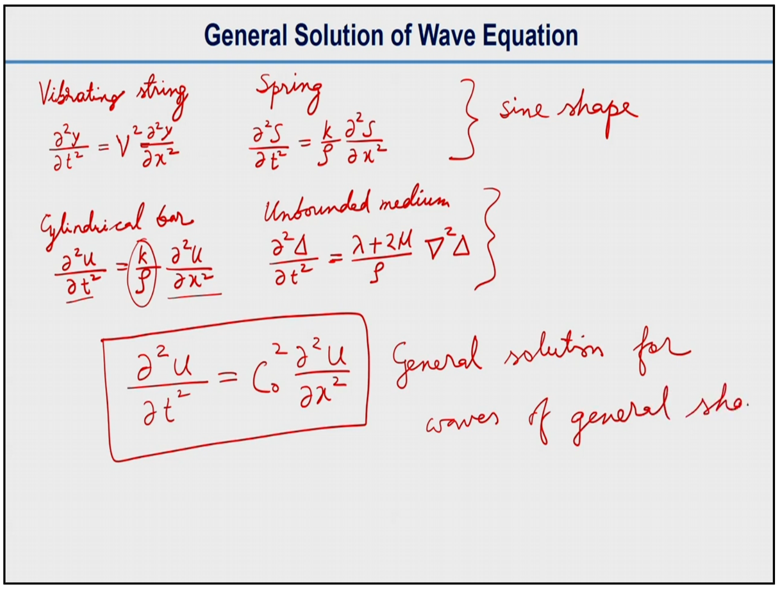

So, what we have learned so far is, for Vibrating string. So, the equation, we had is, second partial of Y, with respect to, Time is equal to V square, second partial of Y with respect to X. Similarly, for spring, we had these equations for cylindrical bar, hitting with a projectile, second partial of U, with respect to D, equal to 0. And, Longitudinal Wave Velocity for infinite body or unbounded media medium is, second partial of dilatation, with respect to T, Lame's constant Lambda plus twice of Lame's constant Mu, divided by Mass density.

So, these are the expressions, we had for, this different cases. So, for Vibrating string and spring, we assumed, Sine shape, and for other cases, we did not explicitly, defined the Wave shape. This is, just implicitly accepted, that the square of the velocity of the pulse, was given by the parameter, that relates, the two second partial derivatives. So, now in this lecture, so we will find, general solution, of this equation. So, general solution, for Waves of general shape.

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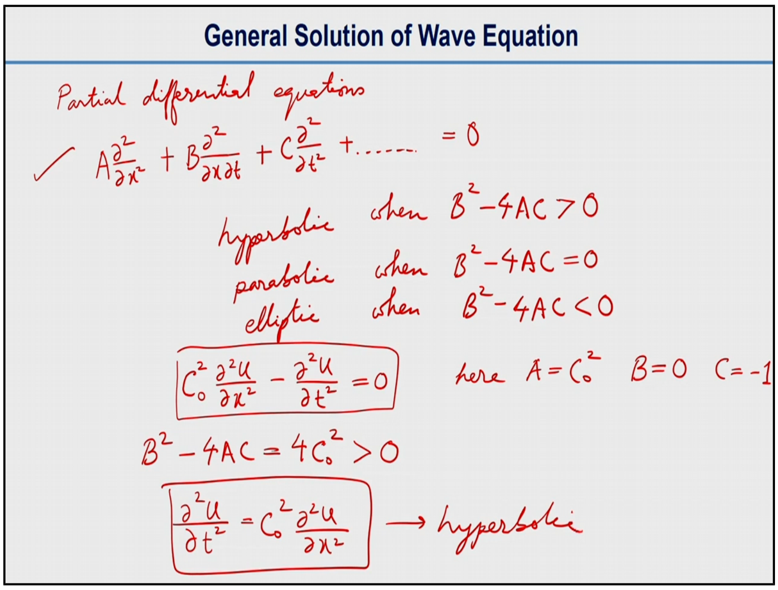

So, we know, that the partial differential equation equations of the form, A double-partial, with respect to X, B double-partial, with respect to X and T, plus C double-partial, with respect to T, plus …, equal to 0, are classified into Hyperbolic, when B square minus 4AC, is greater than 0. And, Parabolic, when B square minus 4AC, is equal to 0, and elliptic, when B square minus 4AC, is smaller than 0.

So, in the previous slide, whatever equations, Wave Equation will be 2, so that can be, rearranged, to write in this way, C0 square, double-partial of U, with respect to X2, minus double-partial of U, with respect to T, is equal to 0.

So, here, A is C0 square, if we compare with this equation, at the top, and B equal to 0, then, C equal to minus 1. So, obviously, if we see, B square minus 4AC, is equal to 4C 0 square, because C0 is the way velocity. But anyway, this is square of the C0, and that should be positive. And, that means, this equation, or if we write, in the other form, C0 square double-partial of, which is this equation, is Hyperbolic.

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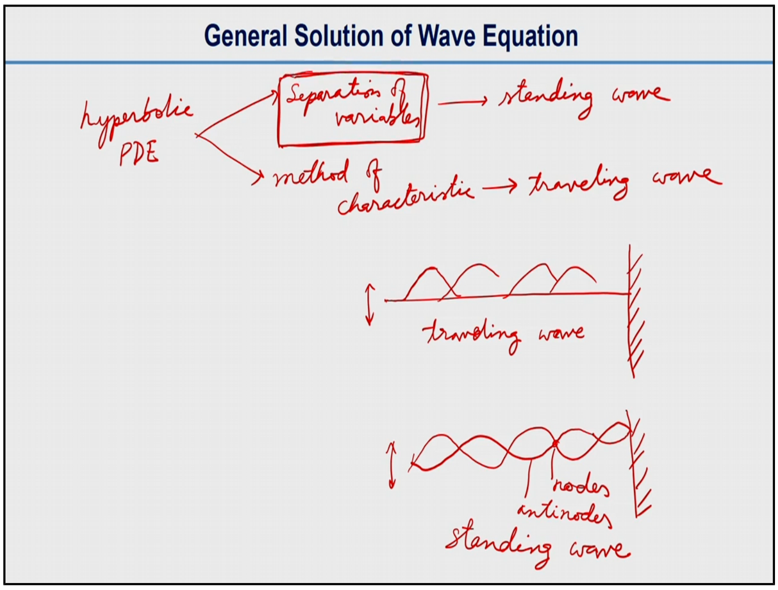

So, Hyperbolic equations, can be solved. Partial Differential equations can be solved, with two ways. One is, Separation of Variables. And, the other one is, Transformation or Method of Characteristics. So, this is appropriate for mostly for, Standing Wave, and this is appropriate for Traveling Wave. So, Standing Waves are combination of two Waves, that moves in opposite direction. Suppose, we have a fixed wall, and we have a string. And then, we are trying to, move the string, with a string like this.

And, if you want to, try to move the string, up and down, so the wave will go forward, like first this portion, will rise in the, then this portion will rise, and then this portion will rise. So, this is Traveling Wave. But, in case of Standing Wave, if we have a Wave like this, and then, if this Wave reflects back from the other Wave, and due to constructive and destructive interference, the Wave forms something like this, where these are called Nodes, and these portions are called, Antinodes. So basically, the Standing Wave, it does not look, appear like, it is moving.

But, this is, Traveling Wave. In case of Traveling Wave, this happens in early, when only one Wave travels and, we can see, the Wave traveling, from one side to other. But, in the Standing Wave, it happens, because of the constructive and destructive interference. And, this is, two Waves are involved. In this case, here it is, one is the incident Wave, and the other is the reflected Waves. And, we cannot see, the Waves moving, from one side to the other side. So, what we will do here is, we will use the separation of variables method, for this general solution of the Wave Equation.

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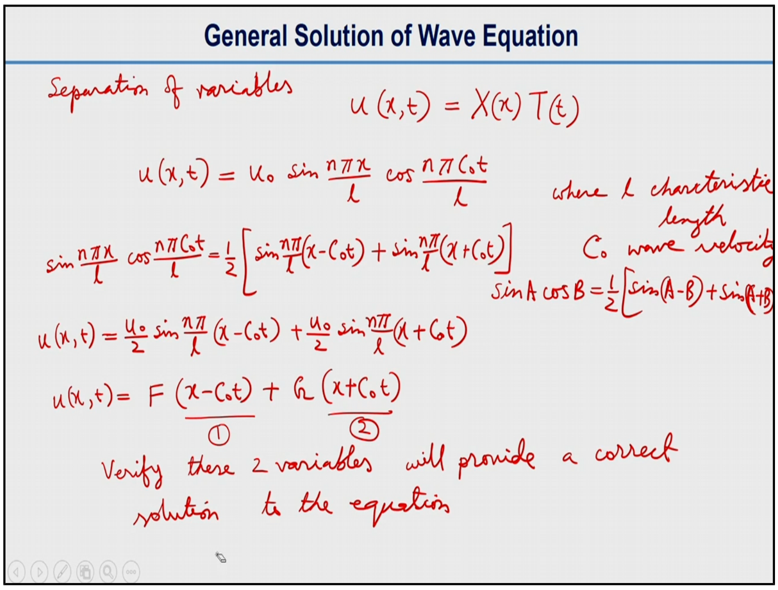

So, for separation of variable, so the solution, it generally, look like this. If u is a multivariable function of X and T, so we need to separate out, the two variables which, these two functions, capital X and capital T. So here, for Standing Wave, so we can have it like this, U0 Sine of N Pi X, divided by L, Cos of N Pi C0 by, t by L, where L is the characteristic length, and C0 is Wave velocity. So, we have separated the variables.

So now, from the Trigonometric relation, Sine E Cos B, is equal to, 1/2 of Sine A minus B, plus Sine A plus B. So, from this relation, we can write, the above expression, Sine N Pi X, L Cos N Pi C0 T, divided by L, is equal to is equal to, 1/2 of Sine N Pi by L, X minus C0 T, plus Sine N Pi by L, X plus C0 T. So, ultimately, if we express the displacement, U, X, T, this will be like, U0 by 2, Sine N Pi by L, X minus U0 T, plus U0 by 2, Sine N Pi by L, X plus C0 T. So, we can express, the displacement, in terms of two different functions, of the variables X and T.

We can generalize the above equation, by assuming two functions, which are actually nonharmonic functions, which are not harmonic. So, let us say, F and Z, X plus C0 D. So, now we have the two variables. So, variable of first one is, X minus C0 T, and the variable two is, X plus C0 T. So, we will verify, whether these two variables, will provide a correct solution to the equation.

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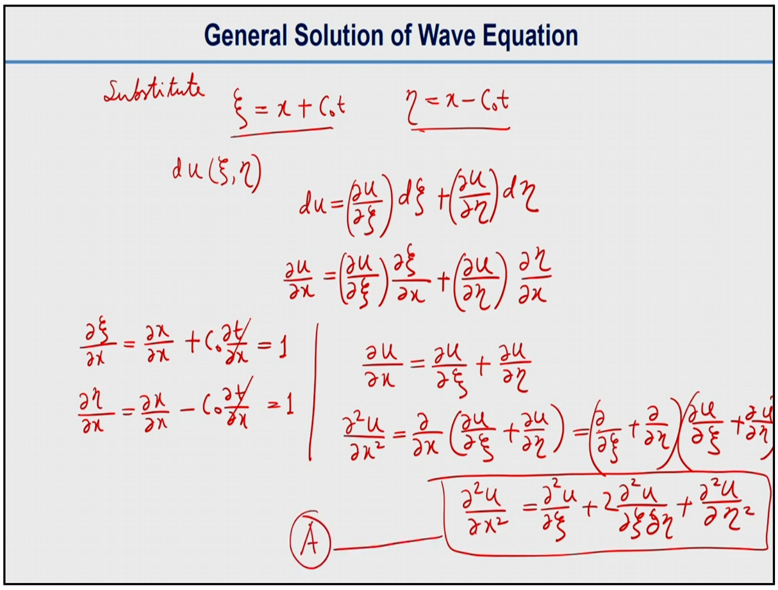

So, for that, we want to substitute, these two variables with, X plus C0 T, and eta X minus C0 T, by substituting these with, two variables. So, now we can express, the differential, of the multivariable function, d U, the differential, we can write it as, d U. And, from that, the partial of U, with respect to X will be, we can express it like that. From the above, from these relations, what we can get is, the partial of those variables, with respect to X, will give a value, equal to 1. This will be, 0. And, the second variable, will also be, 1.

So, both of them are, if we take partial derivative, with respect to X, that will give us, 1. And then, we want to have the second partial of displacement, with respect to X. So, what we will do is, now, differentiating u with respect partial of U, with respect to X. That is, nothing but the partial of U, with respect to the two variables, we define. And, if we differentiate once again, so this will be, partial with respect to X, and which can be again, written as, so this will give us, plus 2. So, this is one expression, for second partial of U, with respect to X.

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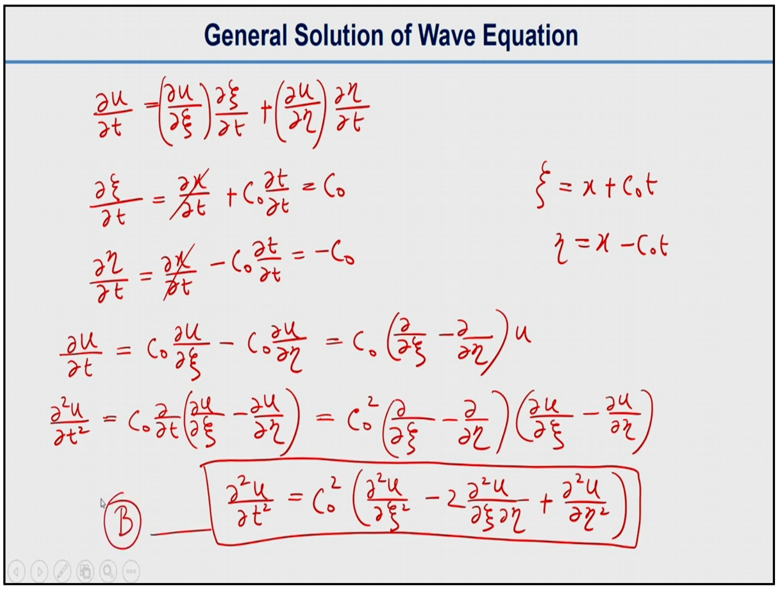

Similarly, we will do, for second partial of U, with respect to T. So, for that, we have partial of U, with respect to T, will be, so from our definition of the variable, X minus CT, so the partial derivative of this variable with respect to T, will be, so this will give C0. And, partial of the second variable, with respect to T, will give us, minus this is, minus C0. It should give us, minus C0. So, this will be, 0.

And then, we want to have our, partial of U, with respect to T. That is, C0, minus C0. Partial of, with respect to the second variable. Or, we can write it, this way. U, so second partial of U, with respect to T2, will be equal to, C0, which will be equal to, C0 square. Partial of U, with respect to the first variable, partial of U, with respect to the second variable.

And, this will finally give us an, expression like, C0 square, minus twice second partial of U, with respect to the second variable, plus second partial of U, with respect to the, second variable. So, this is what we got, for second partial of U, with respect to T. So, now we will use, both these relations, like first relations, we got this, let us say, it is A, second partial of U, with respect to X2. And, the second one, let us assume, this is, B.

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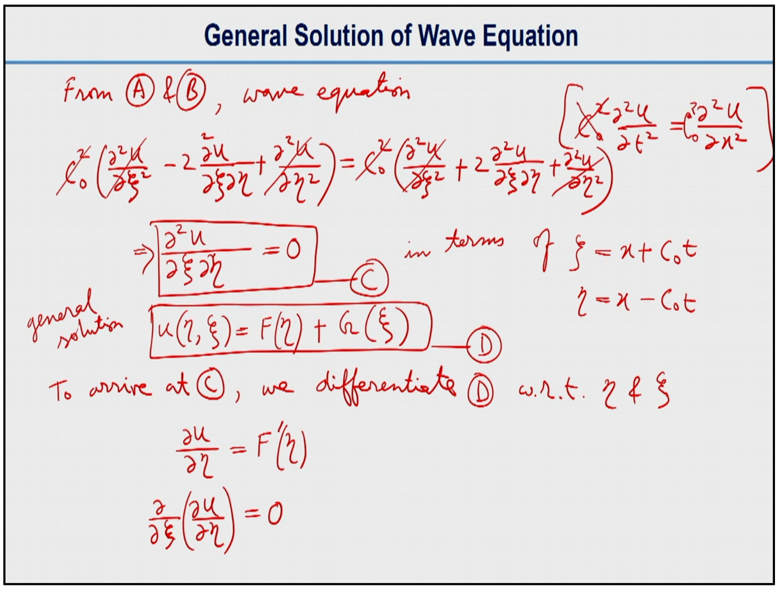

So, from A and B, if we substitute in the Wave Equation, and the Wave Equation will be, C0 square. So, the Wave Equation, we have is, C0 square. This, is equal to, second partial of U, with respect to X. So now, we will write the Wave Equation, from A and B, using the relation, we got an A and B. So, that will be, something like this, twice, plus, this is second partial, is equal to C0 square, plus second partial of U, we just picked the second variable.

So, this was the original Wave Equation. So, now here, this all terms will cancel out, except the middle terms. So, this will give us, second partial of U, with respect to, both the variable, is equal to 0. So, this is the Wave Equation, in terms of, these two variables, which is equal to, we assumed as, X plus C0 T, and X minus C0 T. Now, what we assumed is the, general solution.

Solution is, F function of the second variable, and the first variable. So, to arrive at this Wave Equation, let us say, we write C, to arrive at C, we will just verify, we differentiate this. That is, we are writing D, with respect to our two variables. Okay. And, I will see, whether we end up, with that equation.

So first, we are differentiating with, was one variable. And, which will have a, we can write it as, F prime. And then, if you differentiate, with the other variable as well, which will be equal to, 0. That means, the general solution is, we will write it in the other place. Oh sorry, so I made a mistake here. This C0 square, should be on the right hand side. C0 square, that you must have noticed, that this was a mistake.

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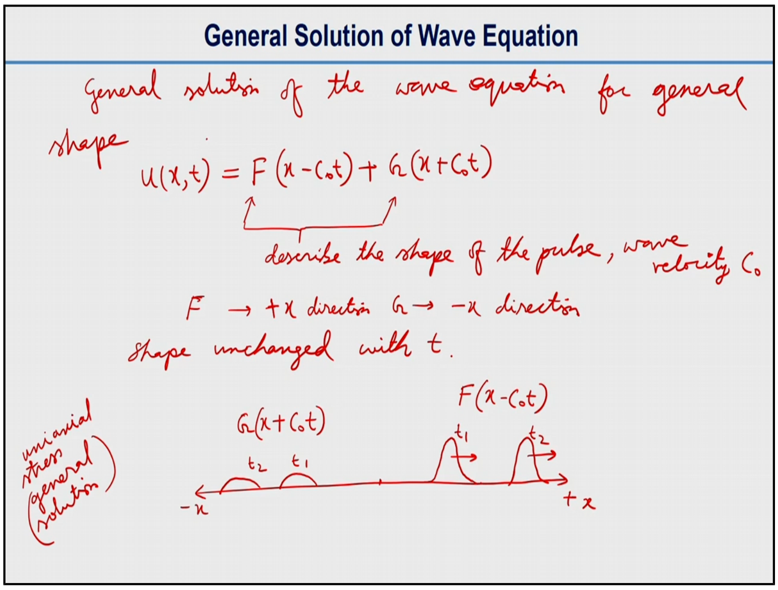

So, the general solution of the Wave Equation, for the general shape, can be given by, u function of X and T, can be given by, X minus C0 T, G X plus C0 T. So, physical meaning of this equation is, function F, and function G, describe the shape of the pulse, and F in the positive X-direction, and G in the negative X-direction. So, wave that is, pulse, that is Wave velocity, is C0. In this case, Wave Velocity is C0. And, F and G, these functions, describe the shape of the Wave.

And, the shape of the Wave is unchanged with, Time T, if we draw a general solution, for a uniaxial stress. So, for a uniaxial stress, if we draw the shapes, let us say, this is positive X, this is negative X. So, for uniaxial stress, general solution of Wave, will give, that is in the positive X direction, the Wave shape is something like this. And, let us say, this is traveling, towards this direction. And, this is, let us say, Time t equal to 1, this is a Time t equal to 2, and Time t equal to 1.

So, this is actually represented by, F of X minus C0 T. And, in the negative direction, that Wave is shape will, let us say, it will be different with, this is Time equal to T1, this is Time equal to T2. And, the function G will define here, to find a Wave shape, C0 T. So, this is the general solution of the Wave Equation, for general shape. So next, we will discuss, about the Elastic Wave in cylindrical bar, which we have already discussed. But, there are some additional considerations, that we will be discussing in the, next lecture. Thank you.