**Plastic Working of Metallic Materials**
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**Module 2**
**Lecture - 5**
# Upper Bound Theorem

Today we will be discussing about the nest type of method of analysis that is by the upper bound theorem. This comes under the bound theorems. There are basically 2 bound theorems, one is upper bound theorem and another is the lower bound theorem and these are extensively used for analysis of determining the forces required for the deformation.

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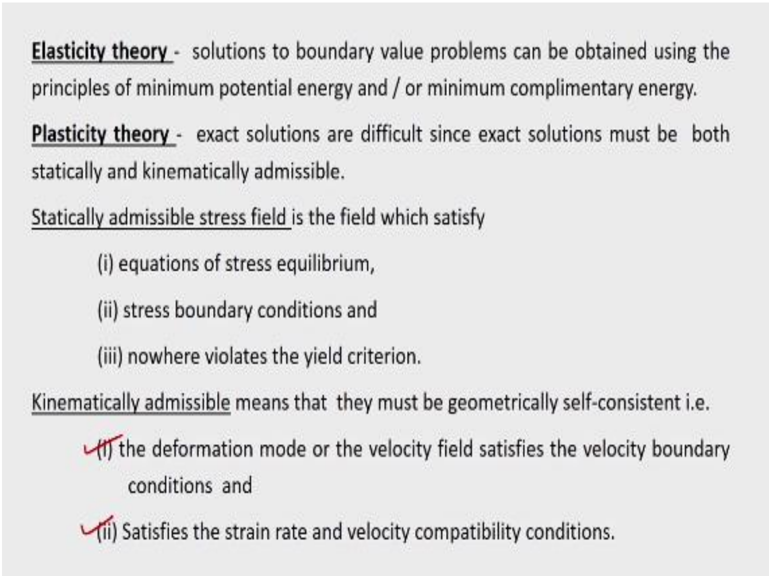


If you look at the solutions to the boundary value problems in elasticity theory, from elasticity theory point of view, the solutions to the boundary value problems can be obtained using the principles of minimum potential energy and/or the minimum complimentary energy, so that is how they and most of the case it is well established. You have good relationships by which we can determine the stresses or forces which are necessary for that, but when you come o the plasticity theory, the exact solutions are very difficult to determine.

Since its exact solution must be both statically and kinematically admissible, so this is the main problem, so in plasticity theory. So, exact solutions are very difficult when you look at these requirements. By statically admissible stress field, so when you talk about, it is the field where the stress field which satisfy the equations of stress equilibrium, it satisfy the stress boundary conditions, and nowhere it violates the yield criteria, so that is what is called a statically admissible stress field.

Now by kinematically admissible means that there must be geometrically self-consistent, that is, the deformation mode or the velocity field satisfies the velocity boundary conditions and it also should satisfy the strain rate and velocity compatibility condition. So these two conditions are if it is met then you say it is a kinematically admissible field.

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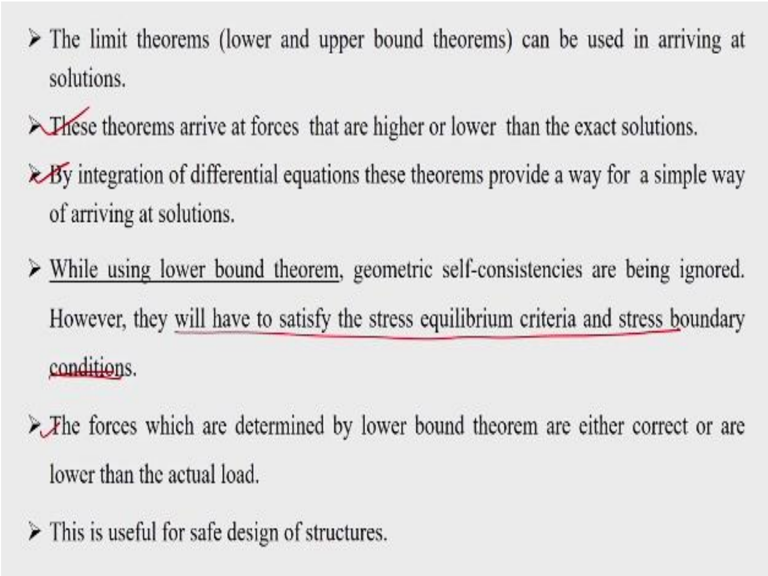


Now any of this whether it is lower bond or upper bound theorem, they can be used at arriving the solutions which are necessary for solutions of problems of practical interest, and these theorems arrive at forces that are higher or lower than the exact solutions. By integration of differential equations, these theorems provide a way for a simple way of arriving at solutions that is the biggest advantage of this.

So, when you look at though in this lecture, I am not going into depth, I am not going to discuss about lower bound theorem, but while using the lower bound theorem, the geometric self-consistencies are being ignored; however, they will have to satisfy the stress equilibrium criteria and stress boundary conditions, which is necessary for the lower bound theorem. The forces which are determined by lower bound theorem are either they are correct or correct to actual load or they are lower than the actual load.

So that way basically it is useful for safe design of structures because whatever you are going to get, it is lower than the actual load, so based on that if you do it, you will find that at no where it violates the yield criteria and your load is always lower, so that it is very safe for use.

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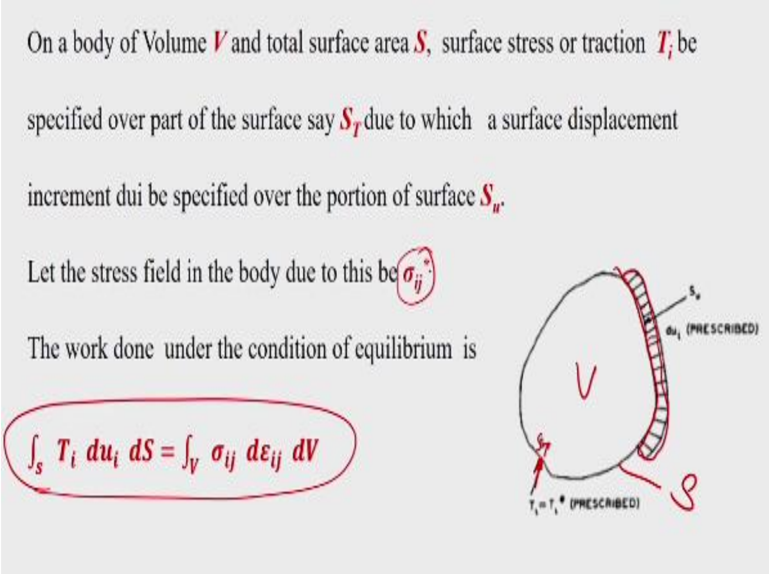


When you talk about that in a general way, I will just tell here suppose this is a body which is shown here is a body having a volume $V$, total surface area is S, and a traction Ti if you are applying at this point, traction $T$ on a specified part of the surface let us say some part $S_T$ if you are applying here a traction, when you are applying this traction here, a surface displacement increment is just taking place. So, we can say that a prescribed, what is that, displacement increment $du_i$ is coming into picture here on this path of that Su of the total surface $S$ okay.

If the stress field in the body due to this surface traction of the force which are applying on the surface if it induces a stress okay and the stress field we can represent it as $\sigma_{ij}$ *. Then, under the conditions of equilibrium, the work done can be in a general way written as integral over d Ti dui ds = integral over V sigma ij d epsilon ij dV where epsilon ij is the stress field inside the body due to which and V is the volume and sorry epsilon ij is the rate of velocity increment okay strain increment sorry.

$$\int_S T_i du_i dS = \int_V \sigma_{ij} d\epsilon_{ij} dV$$

It is the strain increment in the body and these are the force due to it, so it induces. So by equating these two things, we can find out the forces which are necessary. So let us come to this upper bound theorem.

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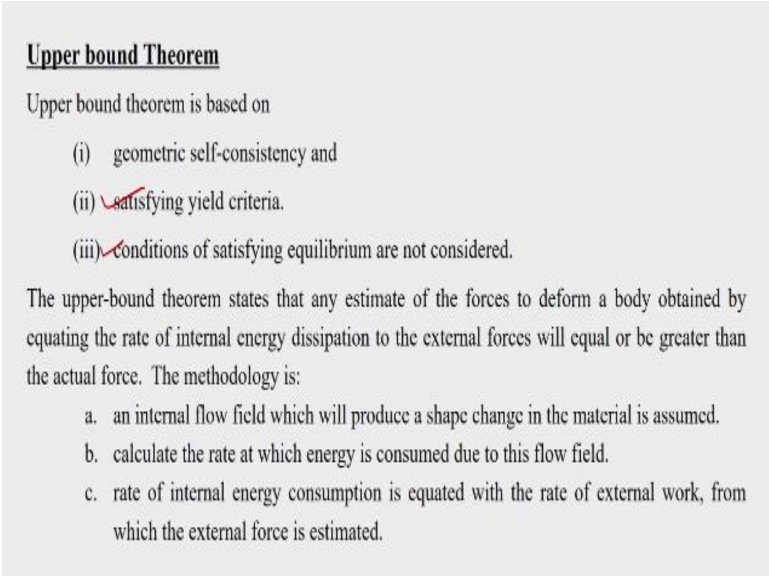

**Upper bound Theorem**

Upper bound theorem is based on

(i) geometric self-consistency and

(ii) satisfying yield criteria.

(iii) conditions of satisfying equilibrium are not considered.

The upper-bound theorem states that any estimate of the forces to deform a body obtained by equating the rate of internal energy dissipation to the external forces will equal or be greater than the actual force. The methodology is:

a. an internal flow field which will produce a shape change in the material is assumed.
b. calculate the rate at which energy is consumed due to this flow field.
c. rate of internal energy consumption is equated with the rate of external work, from which the external force is estimated.

The upper bound theorem as we discussed is based on the geometric self-consistency and it should satisfy the yield criteria, but in this case the conditions of satisfying equilibrium are not considered so that is one thing, so it is not necessary that equilibrium conditions has to be satisfied whereas this geometric self-consistency it should meet and it should satisfy the yield criteria, so these two things are necessary for this upper bound theorem.

The upper bound theorem states that at any estimate of the forces to deform a body obtained by equating the rate of internal energy dissipation to the external forces will be equal to or be greater than the actual force. This is the other way of opposite to the lower bound theorem. So if you are going to estimate the forces, which is required for the deformation of that body, there is a shape change and it has to undergo shape change, then if you are equating the internal energy dissipation to the external force and from that if you are calculating, that force will be equal to or greater than the actual force.

So whatever you are going to get, your estimation will always be higher than the actual case, that is what the upper bound theorem says. So for plasticity theory where you are going to plastically deform the material, this upper bound theory is of importance in determining the forces necessary for the deformation because by that if you determine the forces, it will always be higher than what is required, so naturally you can expect that material will plastically deform during this case. So that is what the upper bound theorem says.

The main process, the methodology of upper bound analysis is that you initially assume an internal flow field inside the material which will produce a shape change in the material, so that means we are assuming that internal flow field it produces a shape change and then when the shape change is taking place, you calculate the rate at which energy is consumed. So the internal energy rate or energy dissipation you calculate it due to the flow field okay and then the rate of internal energy consumption it is equated with the rate of external work done, when this external forces are applied.

So, you find out the internal energy consumption and then you equate it with the rate of external work done due to the imposed your say traction and then after equating it, from that you calculate this forces by equating these two.

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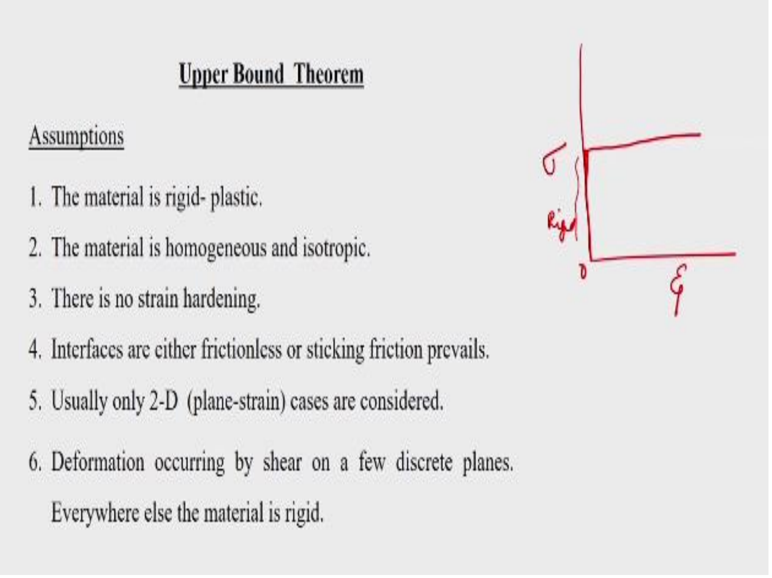


When you are going for this upper bound analysis, there are certain assumptions which have to be taken. One is that material is rigid-plastic, so that means the rigid-plastic is like so if you are just drawing $\sigma$ versus $\epsilon$ , you will find that from here, the material will remain like this and then it is plastic, so up to this region, it is rigid and then after that it is plastic, so that is one thing. Even though the material is rigid-plastic and the material is homogenous and isotropic, so anisotropy generally is not considered in upper bound theorem or upper bound analysis and there is no strain hardening.

So we are ignoring the strain hardening, so that is why we are assuming, this is just theoretical case which we are taking rigid-plastic, though it deviates from the actual case to a great extent, the analysis is fairly accurate by the upper bound analysis for plasticity studies and during this deformation because when you are trying to deform a material, you have to use some tools to apply the forces and other things okay and there may be a constraint also for material to flow, so all those things are coming.

So, naturally what happens is that there it is going to be in contact with some sort of die okay, so at the die work this material at the interface region, there is no friction, this is major thing we people assume is that friction is neglected or if friction is assumed, then it is sticking friction which prevails, rather than the column friction okay. So, interfaces are either frictionless or sticking friction prevails.

In most of the case, upper bound analysis is in the simplest form it uses only the three dimensional cases, that is the plane-strain condition that is what it is generally assuming and in this upper bound analysis because the material is rigid-plastic, the deformation is occurring by a shear on a few discrete plane, not entire planes. So along certain planes only this is happening. So everywhere else or other places, you are assuming that material is a rigid block and the block as such it is moving along certain discrete interfaces that is what it is, along certain discrete planes, so that is another assumption which is there.

In spite of these assumptions which are moving away from the actual case, but still upper bound analysis gives fairly good result for practical application, especially when you are dealing with plasticity theory.

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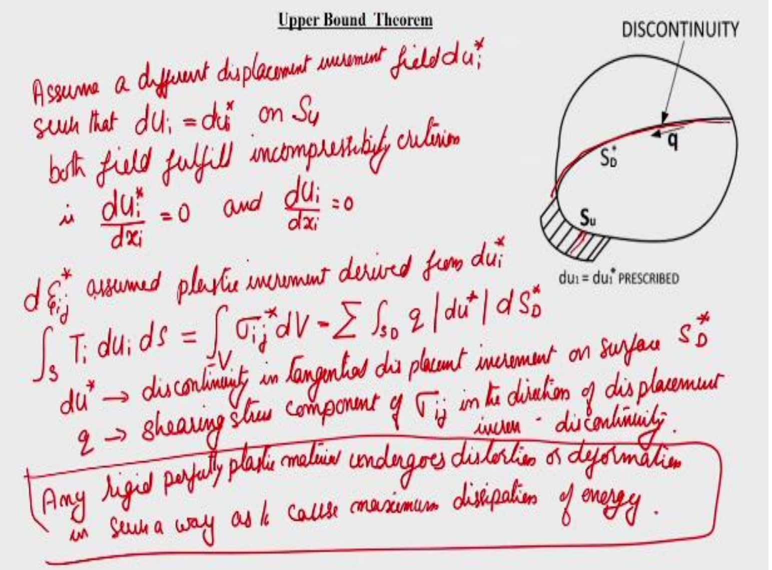


So when you just come in the upper bound analysis, you are assuming that okay say in this body suppose actual displacement increment is $du_i$ which his there and if you assume a different displacement increment $du_i$which is prescribed on some area on the surface Su if applied on the surface so such that assume a different displacement increment dui star such that $du_i$= $du_i^*$ on surface Su, a part of the surface Su, and these both fields are required to fulfill the incompressibility criteria.

When you are assuming that there is a plastic deformation, one of the major assumptions which you assume is that the material is incompressible. So the volume remains constant, so constant volume relationship is mentioned that is for both the fields fulfill the incompressibility criteria, that is dui star by dxi = 0 and dui by dxi = 0. So if this is not met, what will happen is the spherical component will start doing some work, which is not correct actually.

$$\frac{du_i^*}{dx_i} = 0 \; and \; \frac{du_i}{dx_i} = 0$$

So spherical component will always be doing either expansion or dilation, that is what is coming, but the actual shape change which we have earlier discussed is mainly due to the deviatoric component of the stress okay or strain. So, now a kinematically admissible displacement increment field will have discontinuities so when you are applying some say as I mentioned earlier, you are applying some traction and due to okay there is a displacement increment field $du_i$.

If you are assuming another increment displacement field $du_i *$ , so then we are trying to do that, but when you are doing it when the material is deforming, see there will be say discontinuities okay in the tangential component along certain directions along certain surface. Let us say along the surface SD star there is some tangential discontinuity which is taking place, discontinuities in tangential component is taking place okay.

Though tangential components are there, discontinuities are there along certain direction, you can find out the tangential component, but the normal component must be the same on other side, in the region where it is deforming and in the region which is not deforming, so there this normal component of the displacement increment should be the same okay. So that is what, when you are doing the kinematic studies and other things that is velocity of flow when you are discussing with that, that should be the same.

Now let us denote that $d\epsilon_{id} *$ is the plastic increment is the assumed plastic increment derived from your prescribed displacement increment field dui star okay and if you apply the principle of virtual work to the kinematically admissible displacement increment field and the actual stress field sigma ij, we get that integral over s Ti dui ds = integral over V sigma ij star dV = the sum of sigma over integral SD into q into du star dSD star okay where $du *$ denotes the discontinuity in the tangential displacement increment on the surface SD.

$$\int_s T_i du_i ds = \int_V \sigma_{ij}^* dV = \sum \int_{SD} q|du^*|dS_D^*$$

So du star denotes the discontinuity in the velocity field in tangential displacement increment on surface SD star for the kinematically admissible displacement increment field and $q$ is the shearing stress component of $\sigma_{ij}$ in the direction of displacement, the direction of dui star, increment discontinuity. Now if you look at the principle of maximum work dissipation, the principle of maximum work dissipation states that any rigid perfectly plastic material undergoes distortion or deformation in such a way as to cause maximum dissipation of energy. So this is the principle of maximum work dissipation.

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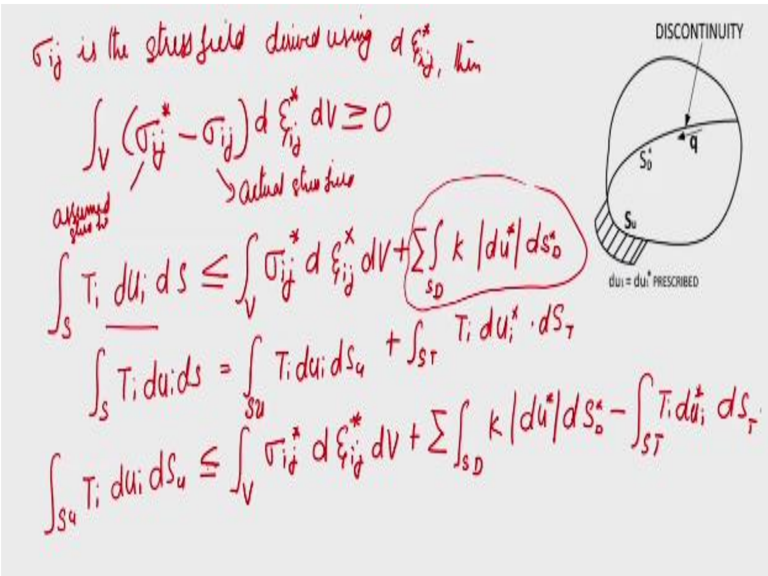


So if you say that $\sigma_{ij}$ is the stress field derived using the strain increment field then we can write V into sigma ij star – sigma ij into d epsilon ij star dV is always greater than or equal to 0 where this is the actual stress field and this is the assumed or we can say that always the assumed stress field if you just expand it and bring this term onto the right side, you will find that always the work done by the assumed stress field is always higher than that of the actual stress field that is one thing.

$$\int_v (\sigma_{ij}^* - \sigma_{ij}) d\, \epsilon_{ij}^* dV \geq 0$$

This is the concept of plastic potential from the strain increment field d epsilon ij star if you do. So if you apply this in the first equation, so you can write it that integral over s Ti dui ds is less than or equal to integral over v sigma ij star d epsilon ij star dV + summation over integral SD into K where is the shear yield stress K into du star dSD star okay, we can write this equation based on this okay. So in this case, where K is your shear yield stress of the material okay and since normal case K is greater than q, so that is why we are getting this relationship.

$$\int_s T_i du_i ds \leq \int_V \sigma_{ij}^* d\epsilon_{ij}^* dV + \sum \int_{SD} k|du^*|dSD^*|$$

This term, we can write it in this form, this is the total surface Ti dui ds can be written in terms of two component over SU Ti dui dSU + integral over the remaining part total ST that is Ti into dui star into dST, so that we can write so that if you just substitute into this one you will end up with this integral over SU Ti dui dSU is less than or equal to integral over the volume sigma ij star d epsilon ij star dV + sigma or integral over SD K du star dSD star – integral over ST surface T Ti dui star dST, so you are getting this relationship.

$$\int_s T_i du_i ds = \int_{su} T_i du_i ds_u + \int_{ST} T_i du_i^* \, dS_T$$

$$\int_{su} T_i du_i ds_u \leq \int_V \sigma_{ij}^* d\epsilon_{ij}^* dV + \sum \int_{SD} k|du^*|dS_D^* - \int_{ST} T_i du_i^* \, dS_T$$

So in this, you should understand that this term which is coming, so here the du star denotes the as we mentioned discontinuity in tangential displacement increment on a surface $dS_D^*$ which can be the internal part, inside the material where it is going to, the deformation is taking place along some discrete planes that is what we have mentioned. Now for achieving the final shape and the final size, the material may have to undergo a shape change along more than one directions okay or more than one planes, that is why the summation is taking place.

If multiple planes are there along with deformation is taking place or tangential velocity discontinuity is there in place, so you have to add the energy, that is the internal energy generation or dissipation through all those things, so that is why it is the summation is written.

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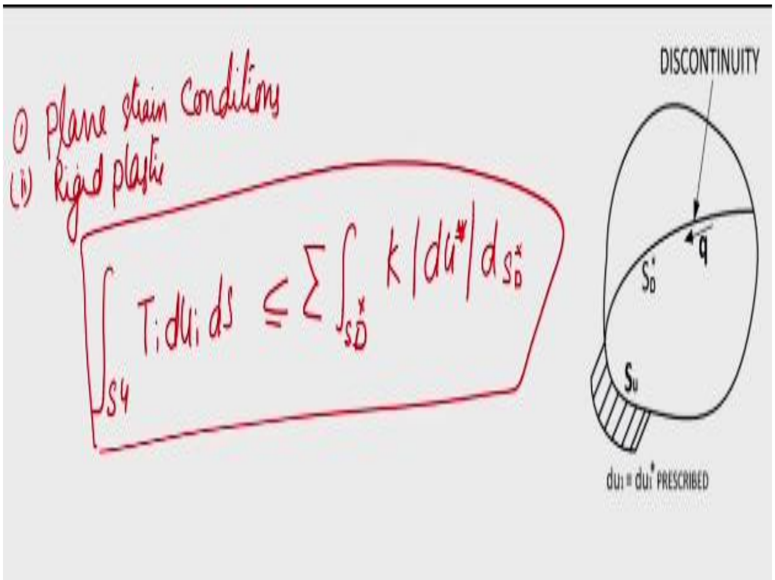


Now in the same thing assuming that plane strain condition, so I am not going to too much depth in that planes because we will with the practical situation, we will just discuss with that case following this, so that is why. Assuming the plane strain conditions and material is rigid plastic, that is where the material consists of rigid blocks of material separated by lines of tangential displacement discontinuity.

Then the same equation we can write it as integral over Su finally it will end up with Ti dui ds is always less than or equal to say sigma over integral SD star K du star into dSD star. So we can arrive at this relationship and then do that okay. So here, this term on the right hand side, you will find that that multiple tangential discontinuity if it is taking place, then we have to add it, if it is only one, only one will come.

$$\int_{su} T_i du_i ds \leq \sum \int_{SD^*} k|du^*|d_{SD}^*$$

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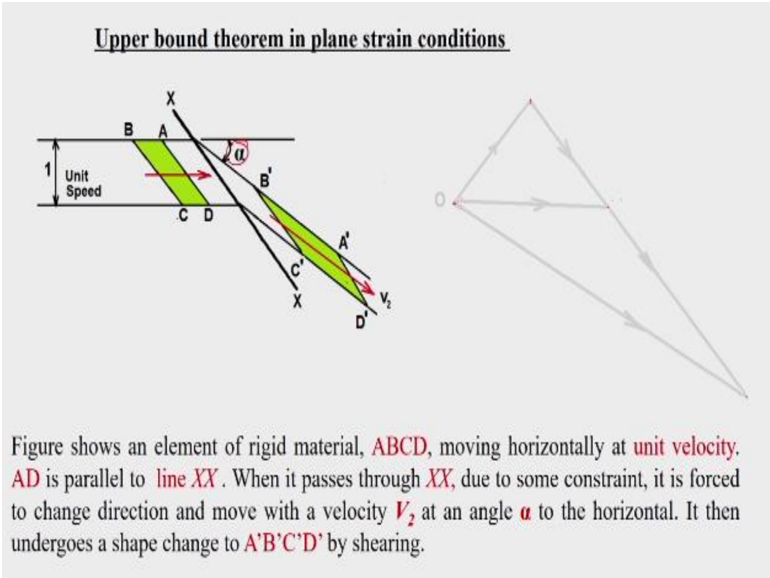


So let us just see that upper bound theorem under plane strain condition, let us just discuss one case. Say in this the figure, it shows that a material during the plastic deformation, material is rigid and it is represented by ABCD which is shown here ABCD is there moving, it is moving horizontally with a unit speed okay or unit velocity towards the right, and AD, you assume a line XX such that AD is parallel to XX.

When it passes through this element of the rigid material, when it passes through this plane XX because this is the plane XX, though here it is showing as a line, when it passes through the plane XX, due to some constraint, may be constraint may be your die material or some other things, due to some constraint, there is a change in the direction of movement. So metal when it is flowing in the plastic region, when it is flowing from left to right and when it cross a certain plane, you will find that there is a change in the direction with which.

So finally, the material will be moving with a different velocity V2 and which is inclined at an angle of $\alpha$ to the initial horizontal direction. This is the case we are going to assume okay. So in this case, what happens when it crosses this plane XX, the shape of this ABCD, the element ABCD, there is a shape change in this region and it ends up with a shape A dash B dash C dash D dash and this happens by internal shearing of the material. So, shear yield strength is very important in this case, so it happens by internal shearing.

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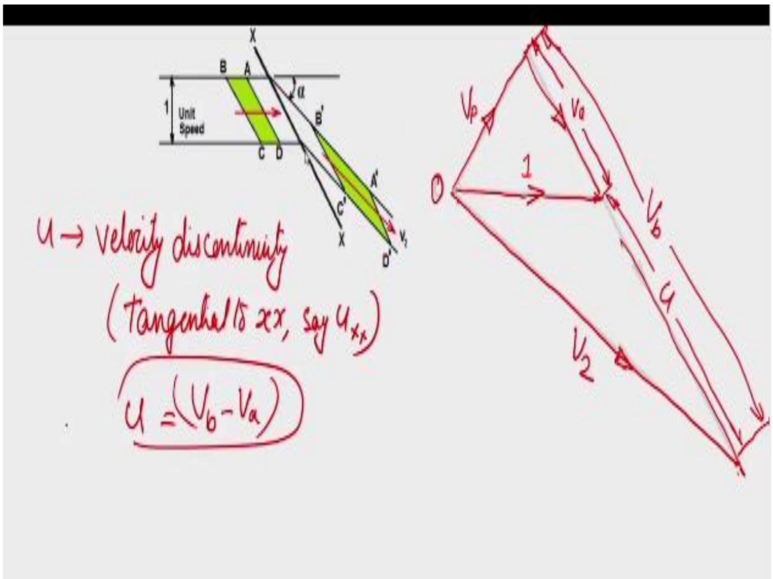


Now for this particular case, let us draw the velocity diagram. So because for this to be kinematically satisfied admissible, then we have to draw the velocity diagram which is called as holograph, so let us see that. The absolute velocity, the initial velocity it is moving with a unit velocity, so you can just star with origin O and then let us give to some scale of the velocity you draw a unit velocity with 1 okay. So, this is that value.

Now, this horizontal velocity with which it is moving before it reaches the plane XX, it will have two components, one is a component which is parallel to the XX that is the tangential component which is parallel to our XX and another which is normal to the XX. It is the tangential component which causes the shape changes because that is the discontinuity, the velocity discontinuity which is taking place. So, we can say that the normal component is $Vp$ and other case it is $Va$.

So let me just say this is $Va$, so that means $Va$ is parallel to XX and $Vp$ is perpendicular to XX. Now the thing is that after this, let us see what is happening at the exit side. So, the velocity changes **so** both in magnitude as well as in direction. So in this case, the direction is along this direction parallel to this what we show is as $V2$. So from O, let us draw a line which is parallel to this $V_2$, we don't know what is that value okay.

Since then condition of constant volume is to adhered, the $Vp$ should be, that means normal component of the velocity before it passes the plane XX and after it passes the plane XX, these should be the same. If there is a difference, it will do work, so that is why. So since for the volume constancy relationship to be maintained or the constant volume relationship is to be maintained, then this $Vp$ should be the same for both these case.

So in the second case at the XX, you will find that, okay this is the thing, so tangential component at the exit also will be there parallel to that, so that will be from here to here, so you just draw a line parallel to $V_2$ parallel to the tangential component, tangential to your $xi$. So both these, so the $Va$ produced, it will reach at what you call it as $Vb$, so you get the $Vb$. When these two meet, $Vb$ you are going to get it. Now the intersection of this line of $Va$ produced with this $V_2$, this is your $V_2$, will be the terminating point of the final velocity $V_2$ okay.

So, the velocity discontinuity what you are going to get? Tangential to a line say $Vb$, say you can tell that as u, this is your, so that is there, u is the velocity discontinuity so that is tangential to to XX. So we can say that say $u_{xx}$ we can put it like that okay. So, this is the so when the material closes this plane XX, there is a change in the flow direction and so there is going to be a tangential velocity discontinuity which is parallel to your this plane XX, so that is represented by u and which is nothing but say u = the difference between $Vb - Va$. So that is what we can get it, it is magnitude is equal to the difference between this $Vb$ and $Va$, so that is what the discontinuity you are going to get it.

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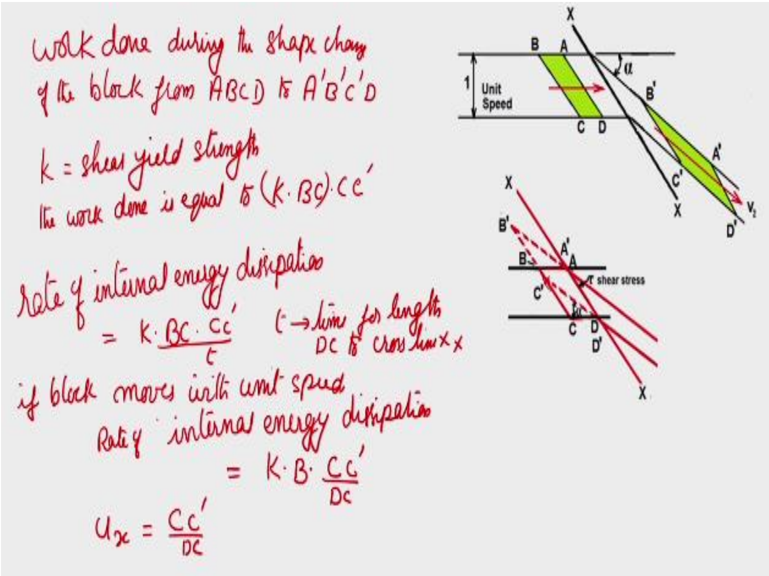


Now, when this is happening, let us consider the work done because in this case there is an internal energy dissipation because the shearing has taken place, so we have to find out what is the work done. So when the shape of the particle has changed from ABCD to $A'B'C'D'$ when it has changed the shape, sowe have to find out the work done, ABCD to $A'B'C'D'$.

So in the initial case, let us say that AD that is parallel to xy line and that line AD and all lines parallel to this line which is parallel to XX will after traveling through, after undergoing the shape change, all those lines which are parallel to AD will always remain parallel to XX itself, so that is one thing. So if you in this below figure you say that this is the ABCD which you have drawn, ABCD, and then this shape also you superimposed so that no $A'$ and $D'$ they are coinciding with A and D.

So this U shape change you are doing that that is $A'B'$ and $D'C'$ you are getting it. So, this is the figure which is coming. So in this case if K is the shear yield strength on both the sides, the work done is equal to due to shearing, we can y $K\ into\ BC\ into\ CC'$, this is the work done. If you just look at the rate of internal energy dissipation, so that means per unit time if you have to move that = K into BC into CC dash divided by t where t is the time for the length DC to cross line XX, that means it is the velocity.

$$the\ work\ done\ is\ equal\ to\ (K \cdot BC) \cdot CC'$$

$$rate\ of\ internal\ energy\ dissipation = \frac{K \cdot BC \cdot CC'}{t}$$

So if you are assuming unit velocity no, then it becomes much easier okay. So since you are using this unit speed no, so that means, the rate at which the CD crosses this line, so that is the time which is coming. So if it is a unit time, then it will be much easier okay. So if the block is moving with a unit speed, then the internal energy for a unit speed, if block moves with a unit speed, then internal energy dissipation = K into BC into CC dash by DC that is what you are getting because that is the time by which DC just takes to cross the line XX, so that is what we are getting.

$$rate\ of\ internal\ energy\ dissipation = K \cdot B \cdot \frac{CC'}{DC}$$

So now comparing these CC C dash CD and the holograph, you will find that they are similar to say U xx = CC dash by DC, so that you will get the relationship.

$$u_x = \frac{CC'}{DC}$$

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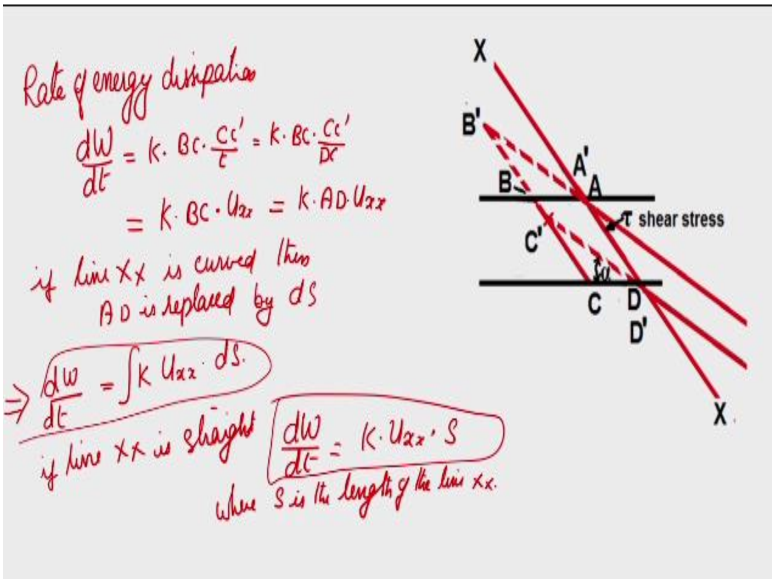


So in that case, the rate of energy dissipation which also can be written as dW by dt where w is the work done = K into BC into CC dash by t = K into BC into CC dash by DC. So that you can write it as K into BC into Uxx or BC and AD are same, so that is equal to K into AD into Uxx where this is the distance CD which is going to change the shape okay. See the line over which this discontinuity occurs is curve, this is for straightened, in this case we have considered to a straight line, XX is a straight line, but if you are just writing that if the line is curved okay.

$$Rate\ of\ energy\ dissipation$$

$$\frac{dW}{dt} = K \cdot BC \cdot \frac{CC'}{t} = K \cdot BC \cdot \frac{CC'}{DC}$$
$$= K \cdot BC \cdot u_{xx} = K \cdot AD \cdot u_{xx}$$

So we can say if XX is curved, then AD is replaced ds, the term ds so that dW by dt, it implied dW by dt, then you have to write it as integral K Uxx x ds. So if it is, if the line XX is straight, the dW by dt = K into Uxx into S where S is the length of the line XX, but you will find that under in all these relationship whether it is this or this, at no point the condition of stress equilibrium is fulfilled, so that is one thing we have to see that.

$$\frac{dW}{dt} = \int K\, u_{xx} dS$$

$$if\ line\ xx\ is\ straight\ \frac{dW}{dt} = K \cdot u_{xx} \cdot S$$

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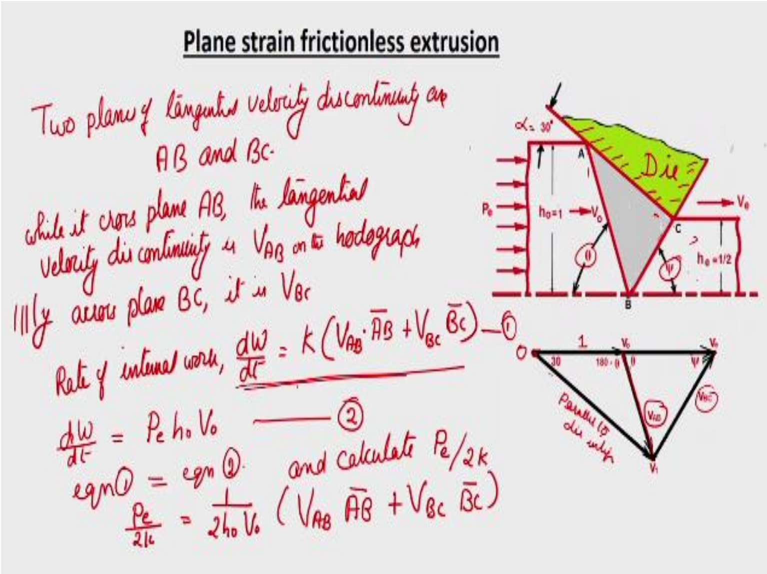


Now, let us just consider, take a typical example and discuss how this is applicable. So in this particular case if you look at it, you see that with a unit velocity it is moving and then when it crosses certain plane so it undergoes a shape change and due to the shape change, there is going to be say velocity discontinuity along certain line and that is how you calculated. So, for that shape change to take place, what is the internal energy generated and you have to equate it with your external applied force or traction force, and what is the rate or power which you have applied into that and from that only you are going to calculate the forces.

So let us take a case of a frictionless extrusion. So here since it is a slab method, if you just slab extrusion, if you assume a slab extrusion, a rectangular block which you are assuming for the theoretical purpose. So here, this is your die and this is the work piece material. So since it is symmetric, we have taken only half of the extrusion part. So, this is your initial or you are applying a pressure so through and then the material is allowed to deform plastically, so initially the material just moves along this direction with a velocity $V_0$ and then it enters into this, this is the deformation zone.

It enters into this region, so in that case there is a change in the flow direction because of the constraint imposed by the die, the metal cannot move along that, so there is going to be a change in the plastic flow of the material. So then after that, it comes out and again the metal is moving along the horizontal direction. So, once it comes out of that, there are no stresses here, so and you will find that okay it is coming out and then it is moving out with a different velocity depending upon what is the reduction, the initial volume and the initial cross sectional area and final cross sectional area.

Under plane strain condition, if you take it and we can just calculate it for a unit width if you just calculate it. So, let us say that this part the initial height is h0 and the final height is he at the exit side, it is half, if this is one, this is half, that means your reduction is 50% in this case and the die angle $\alpha$, this is the die angle $\alpha$ which you are getting it okay, so that is 30 degree. So suppose this is 30 degree if you assume, should not be read as 300, it is 30 degree, and when you are applying a force, external force $P_e$ from the left hand towards the right, the metal is shrinking.

So let us see how the metal flows inside this cavity, inside the extrusion chamber and inside the deformation zone. What is the velocity at which the metal is flowing, so that is what we have to find out. So in this particular case, we can do that. See you start with a point O here, so this is the velocity diagram which we have written, so what is done is that since you are assuming this $V_0$ as the initial velocity with a unit velocity V, $V_0$ or any to scale you draw it, but when it enters the deformation zone, though initially it is moving, you may find that the metal is flowing parallel to this die metal interface.

So this is how it comes, metal will flow in this we are assuming. Because we are assuming that it is rigid plastic and it moves along certain discrete planes, otherwise the material is remaining as rigid blocks, so we have to assume that it is moving in this direction which is parallel to the die and this is mainly due to the imposed constraint of the die, but once it reaches here, again it will just move in this direction okay, in the horizontal direction. So, initial velocity and the final velocity are the same.

So when it comes out, you will find that this velocity it is here, this is your $V_e$, this is initial your $V_0$ and this final is your $V_e$ which we are getting okay. So, both are horizontal, but here this is moving with an absolute velocity $V_1$ inside the deformation zone and it is parallel to the die metal interface, so from the origin itself you draw a line parallel to the die surface, die interface you can say, interface okay. So from O you draw a line. Now the thing is that when it is flowing, there is going to be a tangential discontinuity, so that we have to find out.

So, this is the absolute velocity of the material which is taking place. The initial velocity was $V_0$, so the tangential velocity discontinuity will be like if you just draw this, so this is the tangential velocity discontinuity you are getting, so $V_{AB}$, we can put $V_{AB}$ or $V_{AB}^{*}$whatever you can put it, so that is the velocity discontinuity which is coming, so that way you get it. So when the material is flowing through this, this is the line corresponding to that XX which we derived earlier.

So when the metal is coming here and it now after crossing this plane AB, so there are two planes, the two planes are, two planes of tangential velocity discontinuity are AB and BC. So when it crosses plane AB, discontinuity is $V_{AB}$ on the hodograph. Similarly, across plane BC, it is $V_{BC}$ which you can do that because the outlet velocity is much higher, so in this case it will be we can get the exit velocity Ve and from here to here if you cross that, you will get the $V_{BC}$, so these two things you can determine from this velocity discontinuity diagram okay.

Now, the rate of internal work or internal energy dissipation rate dW by dt is equal to as per our earlier relation K into $V_{AB}$ into AB + $V_{BC}$ into BC, so where AB and BC are the lengths AB and BC here, this is the AB so you have to. So if you draw this, you can get it. You can also either you draw this hodograph and from this you to some scale you draw it and from this figure, you get this AB and BC as well as $V_{AB}$ and $V_{BC}$ or you can once you get the hodograph you can just either physically you measure it, the other way is that it is easier to, by analytical method you can get it, so that is another thing.

$$Rate\ of\ internal\ work, \frac{dW}{dt} = K(V_{AB} \cdot \overline{AB} + V_{BC}BC)$$

So in this case, you know if you look at this is H0 = 1. So you can get the $P_e$ by 2K is what you have to calculate, $P_e$ by 2K. So we can evaluate it analytically or graphically, which way is, so like we can say that h0 by AB = sin theta, so this is that, this is your sin theta if you draw like this, h0 by AB, and similarly he by BC = sin phi, so this is your sin phi and sin theta. So from this relationship and for a 50% reduction, that means he = 1 by 2, so your velocity V will be very high.

$$\frac{P_e}{2K}\ evaluate\ analytically\ or\ graphically$$

$$\frac{h_0}{AB} = sin\theta \qquad \frac{h_e}{BC} = \sin\psi$$

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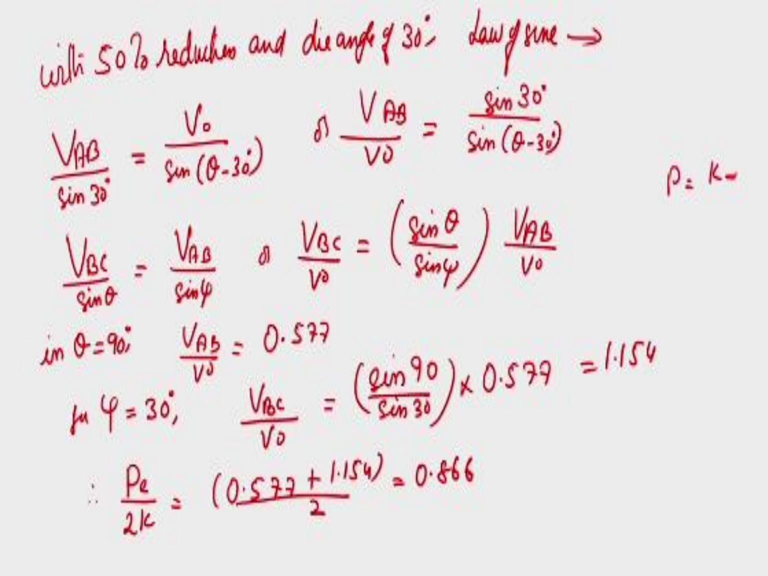

So for that purpose what we can do is that 50% reduction and die angle of 30 degree using the law of sins will lead you to say V star AB by sin 30 from the geometry you can get it V0 by sin theta – 30 or V star AB by V0 = sin 30 by sin theta – 30 and similarly VBC by sin theta = VAB by sin phi or VBC by V0 = sin theta by sin psi into VAB by V0. So th is way, you can calculate it and, if theta = 90 degree, V star AB by V0 = say 0.577 and for psi = 30 degree, VBC by V0 = sin 90 by sin 30 into VAB by V0 okay. VAB by V0 into that is equal to 0.577.

$$\frac{V_{AB}}{sin30°} = \frac{V_0}{\sin(\theta - 30°)} \quad or \quad \frac{V_{AB}}{V_0} = \frac{sin30°}{\sin(\theta - 30°)}$$

$$\frac{V_{BC}}{sin\theta} = \frac{V_{AB}}{sin\psi} \quad or \quad \frac{V_{BC}}{V_0} = \left(\frac{sin\theta}{sin\psi}\right)\frac{V_{AB}}{V_0}$$

$$in\ \theta = 90°, \frac{V_{AB}}{V_0} = 0.577$$

So, that is equal to 1.154 your are getting. So therefore, P exit by 2K is equal to summation, so that means you will get it at it is 0.577+1.154 by 2 = 0.866 because this 2 is coming here okay so because you are getting P is equal to say K into okay. Rate of in internal work is there. Rate of external work dW by dt = Peh0V0 h not v not, this should come here okay, so h0V0, so that from that we can say Pe by 2, K we can calculate, so this is 1 and this is 2. So, equate 1 and 2, 1 and 2 and determine Pe by 2K. So Pe by 2K = 1 by 2 h0V0 into V Star AB into AB + V Star BC into BC. So something has gone wrong, I will just.

$$for\ \psi = 30°, \frac{V_{BC}}{V_0} = \left(\frac{sin90}{sin30}\right) \times 0.577 = 1.154$$

$$\frac{P_e}{2K} = \frac{0.577 + 1.154}{2} = 0.866$$

$$Rate\ of\ external\ work \quad \frac{dW}{dt} = P_e h_0 V_0$$

$$\frac{P_e}{2K} = \frac{1}{2h_0 v_0}(V_{AB}^{*}\ \overline{AB} + V_{BC}^{*}\ \overline{BC})$$

So this rate of internal work, internal energy dissipation due to this shearing process that we got this relationship as equation number 1. Now this you have to equate it because by the very first assumption itself, we said first you calculate the rate of internal energy dissipation then you find out the rate of external work, so that is that way by due to the external work done due to this force P dW by dt, we can write that it as Pe into h0 into V0 where V0 is your initial velocity, h0 is the height we are taking it here as unity, but width also you are taking unity under plane strain conditions or per unit volume you are calculating it.

$$\frac{dW}{dt} = P_e h_0 V_0$$

So from this, so this is equation number 2. So now equation 1 you equate it to equation 2 and calculate the external force Pe which is required for the deformation. So that means, we can say by non-dimensionless value we can say Pe by 2K. So that is P or pressure you can say, Pe by 2K, not force it is the pressure. Pe by 2K is equal to from this you will get it as 1 by 2 h0 into that is initial height into V0 into VAB into AB, the length AB here in the figure, + VBC into the length BC of your physical diagram, upper figure.

$$\frac{P_e}{2K} = \frac{1}{2h_0 V_0}(V_{AB}\ \overline{AB} + V_{BC}\ \overline{BC})$$

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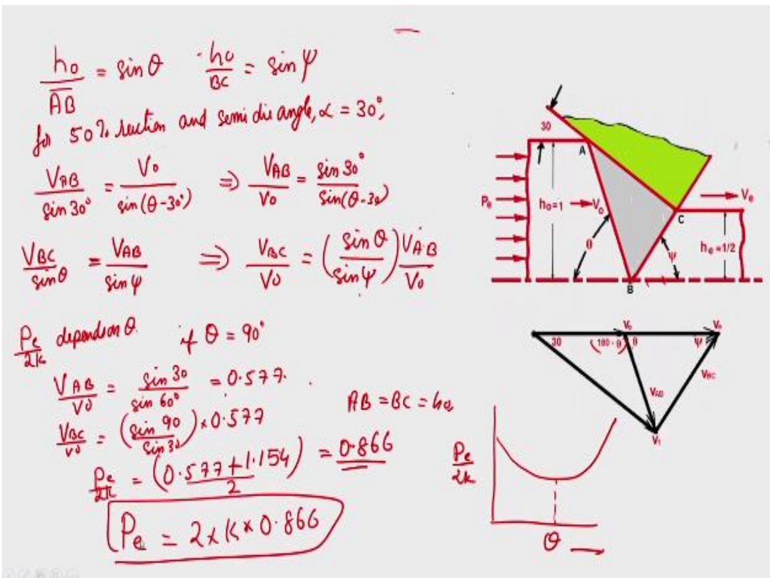

Now this you can calculate either by measuring from this velocity diagram, from the hodograph, or you can do it by analytical method also or from physical method also. However, it is easier to evaluate it by analytical method because you can just take h0 by AB = sin theta and then h0 by BC = your sin psi okay. So from that if you do it for a 50% reduction and semi die angle alpha = 30 degree. Applying law of sin you can say V Star AB or VAB by sin 30 degree = V0 by sin

So from here V0 by sin theta – 30 degree or you can say that VAB by V0 = sin 30 degree by sin theta – 30 and VBC by sin theta = VAB by sin phi because it is a tangential velocity discontinuity, we have to always put V star AB and V star BC, but I am not using it here for simplicity in this one, but actual case it should be there okay. So that means, here now you will get V star VBC by V0 = sin theta by sin psi into VAB by V0. Now the magnitude of the value of Pe by 2K, it depends upon theta value, so that is there actually.

$$\frac{h_0}{\overline{AB}} = sin\theta \quad \frac{h_0}{BC} = sin\psi$$

$$for\ 50\%\ reduction\ and\ semi\ die\ angle\ , \alpha = 30°,$$

$$\frac{V_{AB}}{sin30°} = \frac{V_0}{\sin(\theta - 30°)} \qquad \frac{V_{AB}}{V_0} = \frac{sin30°}{\sin(\theta - 30°)}$$

$$\frac{V_{BC}}{sin\theta} = \frac{V_{AB}}{sin\psi} \qquad \frac{V_{BC}}{V_0} = \left(\frac{sin\theta}{sin\psi}\right)\frac{V_{AB}}{V_0}$$

$$\frac{P_e}{2K}\ depends\ upon\ \theta$$

So, you are arbitrarily choosing this inclination okay, it can may be from here to here, it can be there, it can be here, so like that you can have, you can assume different inclination $\theta$ value. So for each value, you will get a different value, maybe now you may have to assume that at which the case it is the minimum value which gives because as per the law of nature itself no, it will always take that path that which energy is minimum, so that is also there.

So in that case no, so you may have to determine for different values of theta and then determinant, so you can, it is a question of optimization also or minimization problem also, but maybe let us assume one particular case if theta is equal to 90 degree. So, assume if $\theta$ =90 degree, so that means it is vertical this one there will be say what you call it as a dead metal zone and other things may come. So $\theta$ is equal to 90 degree means, it will be like this. The die will be like this, straight, and metal which is coming it will just flow like this, internal shearing is taking place okay.

This will be the dead metal zone which you will find, here also, that in actual case, theoretically okay you can always take it as theta is equal to 90 degree okay. So if theta = 90 degree, then VAB by V0 = sin 30 by sin 60 and that is equal to 0.577. **S**imilarly VBC by V0 = sin 90 by sin 30 thirty into 0.577, and if AB = B C = h0 if you take it, that means it will be vertical okay. Therefore P by Pe by 2K = 0.577+1.154 by 2 because that equation we are getting, that distance also you have to take it okay, equation 2, so that you are getting.

$$\frac{V_{AB}}{V_0} = \frac{sin30°}{sin60°} = 0.577$$

$$\frac{V_{BC}}{V_0} = \left(\frac{sin90°}{sin30°}\right) \times 0.577$$

$$AB = BC = h_0$$

So that you will end up with the 0.866, but with the different values of theta if you are just plot it no, you will get the value of something like this. It will reach a minimum value and so the sum minimum value you will say if it is Pe by 2K with different theta value if you just plot it, some minimum value get, that you can minimize it and optimize the problem and take it. So this is how you calculate it. So once you know that P2 by 2K, you can calculate Pe = 2 into shear stress of the material into 0.866, so that way you can calculate it okay. Thank you very much.

$$\frac{P_e}{2K} = \frac{(0.577 + 1.154)}{2} = 0.866$$

$$P_e = 2 \times K \times 0.866$$