Plastic Working of Metallic Materials
Prof. Dr. P. S. Robi
Department of Mechanical Engineering
Indian Institute of Technology – Guwahati

Module 2
Lecture - 3
Slip Line Field Theory

So this lecture, we will be discussing about the slip line field theory. This slip line field theory is somewhat an old theory and basically in the mid 90s, this was of much prominence and this is based on the analysis of deformation, plastic deformation that is both geometrically, self-consistent, and statically admissible and this slip line is based on the assumption that this slip lines are planes of maximum shear stress and therefore generally they are oriented at 45 degree to the axis of the principals.

This is the basic, but make sure, you please note that this is different compared to the slip lines, which we discuss in material science or metallurgy courses, okay, these are entirely different okay. Here, we are based on the solid mechanics principles where the plane of maximum shear stresses are there, okay and mostly these are when you take it along the principal stress axis no, you will find that it is oriented at 45 degree to the principal stresses.

So, this slip line field theory was developed to analyze homogenous plane strain deformation and assumptions were there, the body is rigid, perfectly plastic isotropic material. So the body is rigid and perfectly plastic, so either you will just increase the stress and till it reaches a particular yield strength value, there your stress remains constant and you have large deformation, okay that is what is the rigid perfectly plastic material, and the material isotropic.

So, for this type of solids, the elastic strains are always assumed to be zero because it is not elastic, it is rigid okay, and the plastic flow occurs without work hardening. So, it remains almost a constant value, thus yield strength remains a constant value. So, when you look at the disadvantages of that, there are lots of disadvantages, but still it is very useful method to discus about few deformation studies, it is highly useful also.

The limitations are that this deals only with the non-strain hardening materials and this slip line field theory does not take into account the click which can take place during the deformation. It also does not take into account the strain rate effects during deformation and see it completely ignores, it ignores the heat generation during plastic deformation. During plastic deformation because you are giving in some energy into the material, a large amount of energy is being converted into heat.

So, here this is not considering that heat generation and during the deformation and when the heat is generated, there is going to be thermal stresses due to the heat generation in some localized area or anywhere. So, this is also not taking into consider the effect of thermal stresses and another is the shear, it also assumes that the shear stress at interfaces or surfaces remains constant and generally when you discuss about this slip line field theory, the friction is ignored, otherwise you consider the sticking friction.

So, friction is taken into account may be in some machinability studies and other thing, but in bulk plastic deformation studies, this friction is generally not considered for the case with slip line field theory. So, one assumption is that the materials deforms under plane strain condition. Plastic deformation takes place under plane strain condition. So, when it is plane strain condition, you assume that the plastic deformation is taking along a given particular plane, only one plane, and normally no, we can say that the plane is x, y plane.

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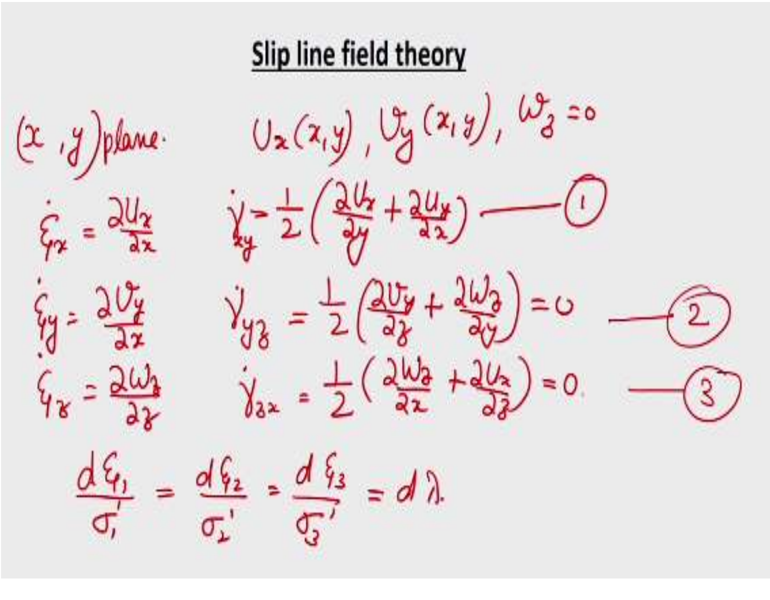

If you take a system of 3 mutually orthogonal plane, then we can say that and we can assume that this is taking place along the x, y plane, and in that case, the deformation is independent of z, wherever the z values you are taking, you are considering it is same okay. So, you have a one plane and above some along the z axis, there is no variation in the deformation which is taking place and elastic strains are neglected and plastic strain increments when you talk about plastic deformation, you talk about plastic strain increments.

These are written in terms of displacement or velocities, say like you say that Ux x, y and Vy and Wz = 0 okay. The strain rate or strain increment along the z direction is equal to 0 that is also there, the displacement it is equal to 0, and then we can write that the equations in this form say your strain increments = dou Ux by dou x and shear strain rate = 1 by 2 into dou Ux by dou y + so here it is xy + dou Uy by doux, let it as equation number 1. Similarly epsilon dot y is equal to = dou Vy by dou x and the gamma dot yz, z = 1 by 2 x dou Vy by dou z + dou Wz by dou y and that is equal to 0.

Similarly, you will find that this is equal to Wz by dou z and you will find that gamma dot zx = 1 by 2 into dou Wz by dou x + dou Ux by dou z, this is equal to 0. So these are the strain increments when you are writing. So this is equation 2 and this is equation 3. So, we are assuming that all the flow, plastic flow, occurs along the xy plane and if you recollect our earlier, we have mentioned in the previous lectures no, the Levy-Mises equation, so which is nothing but epsilon 1 by this sigma 1 dash, this is the deviatoric stress, this ratio always remains a constant d lambda okay. So in the tensor notation also, you can just write it.

$$(x,y)\ plane \quad U_x(x,y), V_y(x,y), W_z = 0$$

$$\dot{\varepsilon}_x = \frac{\partial U_x}{\partial x} \quad \gamma_{xy} = \frac{1}{2}\left(\frac{\partial U_x}{\partial y} + \frac{\partial U_y}{\partial x}\right)$$

$$\dot{\varepsilon}_y = \frac{\partial V_y}{\partial x} \quad \gamma_{yz} = \frac{1}{2}\left(\frac{\partial V_y}{\partial z} + \frac{\partial W_z}{\partial y}\right) = 0$$

$$\dot{\varepsilon}_z = \frac{\partial W_z}{\partial z} \quad \gamma_{zx} = \frac{1}{2}\left(\frac{\partial W_z}{\partial zx} + \frac{\partial U_x}{\partial z}\right) = 0$$

$$\frac{d\varepsilon_1}{\sigma_1'} = \frac{d\varepsilon_2}{\sigma_2'} = \frac{d\varepsilon_3}{\sigma_3'} = d\lambda$$

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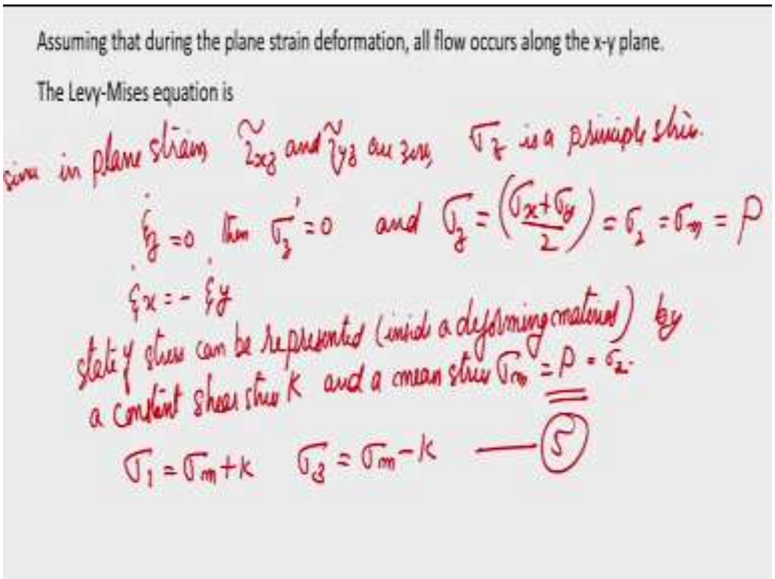

In plane strain condition, since in plane strain condition, since tau xz and tau yz are 0, so we can consider that sigma z is a principal stress and also since epsilon dot z = 0, then from the Levy-Mises equation, you will find that this is equal to 0 and we can write that equation for this sigma z = sigma x + sigma y by 2, so that that will be like once if you are taking the principal stress no, you can just consider it as sigma 2 so because this is nothing but the mean stress, in that case it will be like a sigma m because these are mean stress of that.

So, we just generally in the slip line field theory because it is the mean stress, this is a hydrostatic stress, so when it is a hydrostatic stress, we always use it by this pressure okay, the term pressure P and the material is assumed to be incompressible. Since it is incompressible, you will find that the sigma dot x is equal to – epsilon dot y. So under this condition, each incremental distortion is in a state of pure shear. So whatever deformation is taking place, the distortion is taking place, it is basically due to the pure shear that is the thing.

$$Since\ in\ plain\ strain, \tau_{xy}\ and\ \tau_{yz}\ are\ zero, \sigma_z is\ a\ principle\ stress,$$

$$\dot{\varepsilon}_z = 0, then\ \sigma_z^{'} = 0\ and\ \sigma_z = \left(\frac{\sigma_x + \sigma_y}{2}\right) = \sigma_2 = \sigma_m = P$$

$$\dot{\varepsilon}_x = -\dot{\varepsilon}_y$$

So under such case, we can always, like earlier also we have mentioned that the state of stress where deforming material, a plastically deforming material can be represented as a constant shear stress k and superimposed on a mean stress or pressure p, so that is what. The state of stress can be represented inside a deforming material by a constant because you are assuming that it is rigid perfectly plastic and it is non--work hardening, so shear stress remains constant.

A constant shear stress K and a mean stress sigma m which is equal to P or you can say that is equal to sigma 2 also you can write, anything you can use it okay. So, we will we will be using this term P, sigma P. So that is, we can write it as sigma 1 = sigma m + K and sigma 3 = sigma m – K okay. So, this is equation number 5 when you write it.

$$\sigma_1 = \sigma_m + k \qquad \sigma_3 = \sigma_m - k$$

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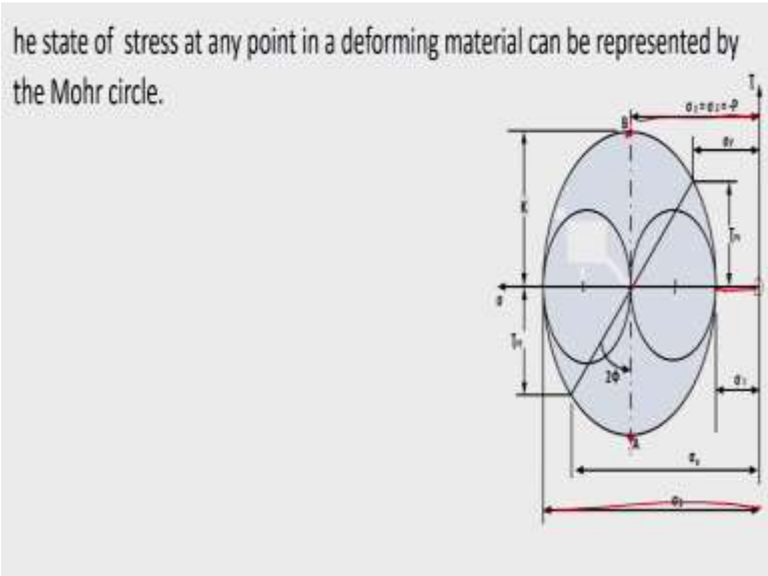

The state of stress can always be represented using a Mohr circle by this. So we can say that in this particular case, so this is your σ_1 and this is your σ_3 okay. So, your mean stress, this is your mean stress σ_2 or σ_z which is equal to $-P$ okay, so and you are assuming that during plastic deformation studies no, basically bulk deformation studies, it is compressive, the nature of stresses are compressive in nature, that is one assumption you are taking.

So, at any plane, we can just assume the state of stress, at any point by this sigma x and other thing. So, by this Mohr circle representation, the state of stress can be represented. So if you look at it, see the largest principal stress in this case σ_3 is nothing but σ_m, from here to here is σ_m and + K, and the lowest one is σ_m – K. So σ_1 and σ_3 we can write like this. But if you look at this figure thing there, you will find that in this particular case, the maximum shear stresses are obtained at point B and A okay.

So, that is, you will find that the maximum shear strain rate, so strain increments it coincide with directions of the yield shear stress and it is represented by this point A and B. So these are directions of zero rate of extension or contraction okay, when you are deforming it when it is the maximum shear stress, the principal strain rate, maximum shear rates are there, the directions of the maximum shear strength rates, when they are coinciding the directions of yield strength, then we can say that okay along these directions no, there is no, because that is equal to your mean stress.

So when it is mean stress no, it is going to have any extension or contraction but shearing is taking place okay and so if you take that these directions loci of this maximum shear stress and shear strength rates, it should take, it forms a two orthogonal curves, they are called as slip lines okay. So slip lines at alpha, and generally the term which they use is alpha lines and beta lines okay. So, you can say that if you just take an element bounded by this alpha lines and beta lines because this, the locus of this B and locus of A along is the direction which you look at it no, you will find that it is called as alpha lines.

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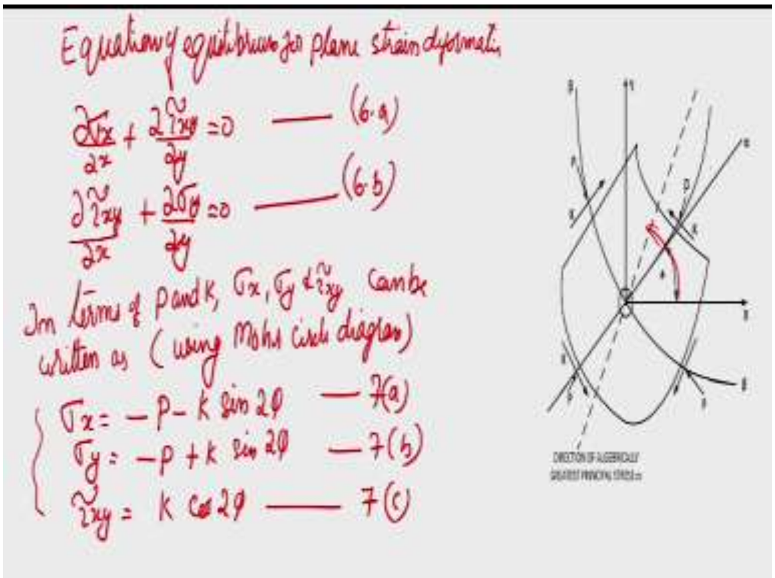

Because the locus when you look at it, one is this alpha line which is following like this okay and the another is the beta line which is going like this through this point. So these are two orthogonal lines and other things which are coming and if you take a Cartesian coordinate system x and y in this case no, this is the thing. This is the element under consideration, when you are taking an element under consideration this, then you can label this as alpha line and beta line, it is orthogonal directions okay and these are called as slip lines or alpha lines and beta lines.

To distinguish between these two alpha lines, because you have to follow certain convention, for that the convention use is that this alpha and beta lines, they forma right hand coordinate system of axis okay. When this alpha and beta line form a right handed coordinated system of axis, then the line of action of the geometrically greatest principal stress, that is the σ_1, so if you do like this, this is the line of action of the geometrically stressed principal stress if you take it.

The greatest principal stress if you look, it should pass through the first and the third quadrant, this is the x axis, this is the y axis, so the largest principal stress, the line of action of the greatest principal stress should lie in the first quadrant. Of course if you extend it, it will go to the third quadrant and the clockwise rotation of alpha and beta lines, that means if you are just taking in this direction from the x axis, this is taken as the positive direction, so anticlockwise sorry, not clockwise, the anticlockwise rotation of alpha or beta lines is taken as positive, so that is the normal convention.

So the convention is that if there is, when the alpha and beta lines form a right handed coordinate system of axis, the line of action of the algebraically greatest stress may be the σ_1 or σ_3**,** whichever is the greatest, it pass through the first quadrant and then the anticlockwise rotation of the alpha and beta lines in this direction that is considered as the positive thing okay. Now, let us come to the slip line field analysis. For that particular case, let us write the stress equations or equations of equilibrium for plane strain condition.

So the equations of equilibrium for plane strain condition deformation, that is the equations of equilibrium are dou sigma x by dou x + dou tau xy by dou y = 0 we can write equation 6.a. Similarly dou tau xy which is the shear stress by dou x + dau sigma y by dau y = 0 you can write 6.b. So if use the Mohr circle diagram, the stress component sigma x, sigma y, and tau xy expressed in terms of P and K, we can express these in terms of P and K.

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} = 0$$

$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial_y} = 0$$

If you write that the previous figure which was there for the Mohr circle diagram, so that we can write it and where P is your mean stress which is given by this, this is the P, mean stress and it is a normal hydrostatic pressure on the two planes of the yield stress okay. So, that if you write by this equation, by this Mohr circle, σ_x and at any point if you write it between this, you can write it like this, in terms of, it can be written as may be using Mohr circle as sigma x = because it is compressive –P – K sin phi and sigma y = – P + K sin, sorry it is 2 phi, sin 2 phi, so this is 7a and this is equation 7b and tau xy = K cos 2 phi, 7c.

So this is the general equation which we can write for sigma x, sigma y, and tau xy using the Mohr circle equation. Now, this equation 7 you just differentiate and substituting in equation 6.

$$In\ terms\ of\ P\ and\ K, \sigma_x, \sigma_y and\ \tau_{xy} can\ be\ written\ as\ (using\ Mohr\ circle\ diagram)$$

$$\sigma_x = -P - K\ Sin\ 2\varphi$$

$$\sigma_y = -P + K\ Sin\ 2\varphi$$

$$\tau_{xy} = k\ Cos\ 2\varphi$$

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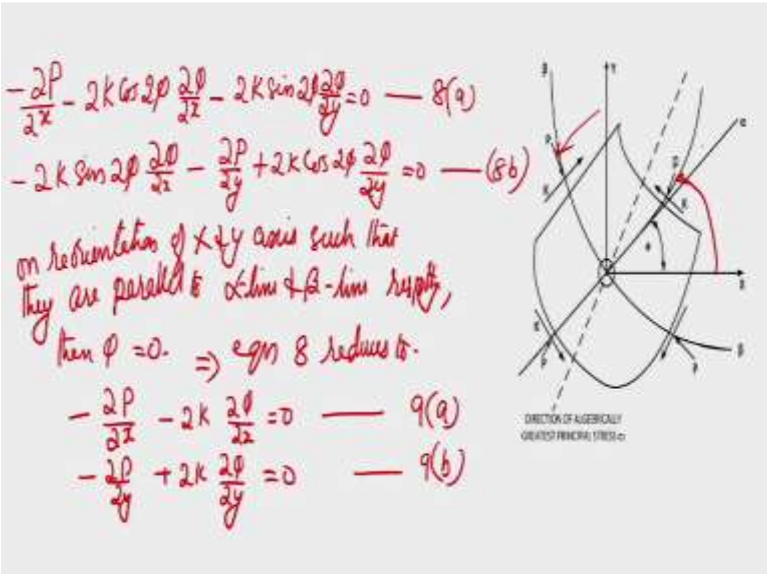

So differentiate this part and then write it in this equation 6 you can get the following equation. So that is–dou P by dou x – 2 K cos 2 phi into dou phi by dou x – 2 K sin 2 phi into dou phi by dou y = 0, that is when you differentiate this equation, substitute here, differentiate this equation and substitute it here okay, so that is how you are getting 8a. Similarly when you the second equation is 2K sin 2 phi dou phi by dou x – dau P by dau y +

2K cos 2 phi into dou phi by dou y = 0, this is equation 8b. So, these two equations you are getting.

$$-\frac{\partial P}{\partial x} - 2K\ Cos2\phi \frac{\partial \phi}{\partial x} - 2K\ Sin2\phi \frac{\partial \phi}{\partial y} = 0$$

$$-2K\ Sin2\phi \frac{\partial \phi}{\partial x} - \frac{\partial P}{\partial y} + 2K\ Cos\ 2\phi \frac{\partial \phi}{\partial y} = 0$$

So now Px and Py, sorry sigma x and sigma y, so if you just orient this x and y axis so that this, see x axis is parallel to your alpha, sorry it is made parallel to this alpha line and y axis is made parallel to the beta line, so if you are just rotating your coordinate axis in such a way, then you will find that when it is made like you on reorientation of x and y axis such that they are parallel to alpha line and beta line respectively, so in that case what happens is that, this you just rotate like this so and this will come like this and it coincide with alpha and beta axis.

Then what happens is that then phi is equal to 0. When phi is 0, so equation 8 reduces to, you will say –dou P by dou x – 2 K dou phi by dou x = 0. Similarly, – dou P by dou y + 2 K dou phi by dou y = 0. So that is 9a and this is 9b. So, these two equations we are getting.

$$\phi = 0,$$

$$-\frac{\partial P}{\partial x} - 2K \frac{\partial \phi}{\partial x} = 0$$

$$-\frac{\partial P}{\partial y} + 2K \frac{\partial \phi}{\partial y} = 0$$

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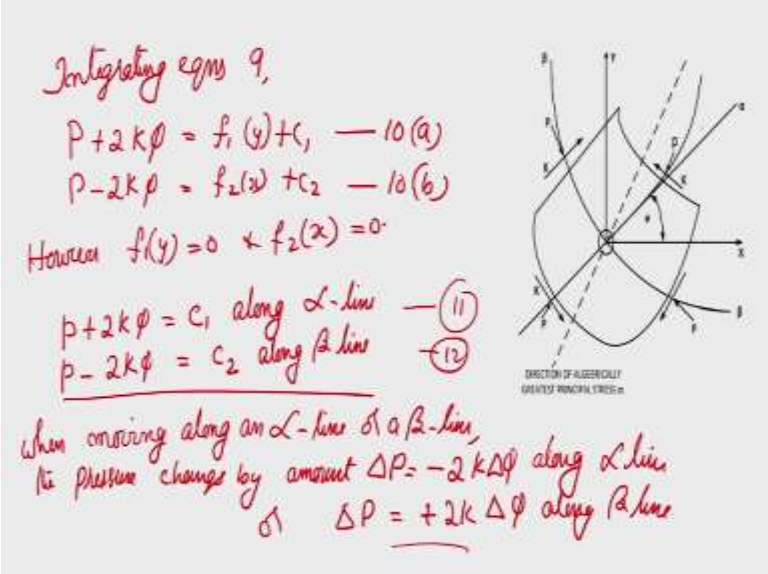

So in this equation if you just integrate it, if you integrate these equations, we will just get two equations, that is integrating equations 9, we will get P + 2 K phi = a function of y + C1 and P – 2 K phi = f2 into x + C2 okay. So you are getting 10a, equation 10b and then if you look at this f1y and f2y, they are equal to 0. However, f1y = 0 and f2x = 0, so you will find that P + 2 K phi = C1 along an alpha line and P – 2 K phi = C2, this is C1, C2 along a beta line.

$$P + 2K\phi = f_1(y) + c_1$$
$$P - 2K\phi = f_2(x) + c_2$$
$$However\ f_1(y) = 0\ and\ f_2(x) = 0$$
$$P + 2K\phi = c_1\ along\ \alpha\ line$$
$$P - 2K\phi = c_2\ along\ \beta\ line$$

So, the significance of this equation, these two equations, let me just say that it is 11 and 12 is that when we are moving along an alpha line, suppose we are moving along an alpha line in this direction or may be along a beta line if we are moving along this direction, the pressure, the hydrostatic stress or the pressure P changes by a small amount, that is the delta P. When moving along an alpha line or a beta line, the pressure changes by an amount delta P = -2 K delta phi along alpha line or delta P = + 2 K delta phi along a beta line.

$$\Delta P = -2K\Delta\phi\ along\ \alpha\ line$$
$$\Delta P = +2K\Delta\phi\ along\ \beta\ line$$

This constant C1 and C2 no, they vary from one slip line to the other, so only thing is that the difference when you are moving along this either an alpha line or a beta line, there is a change in the pressure and along these lines okay, so that is given by these equations and. Now if you just this ox, the coordinate axis ox and oy, they are a general set of Cartesian axis and once the alpha and beta lines have been correctly designed, they may be taken into account along any chosen direction, so that is one advantage.

So normally what happens is, the most convenient thing is that this coordinate axis you adjust it may be in a such a way that the tangent to the alpha line is made parallel to your x axis okay, so that x direction no, that is tangent to the point, any point on the alpha line and then phi is then measured positively when it is rotated anticlockwise as it moves from one point to the other, so that is what it is done okay along the given alpha or beta line when it is moving, that is the normal practice for the convenience sake.

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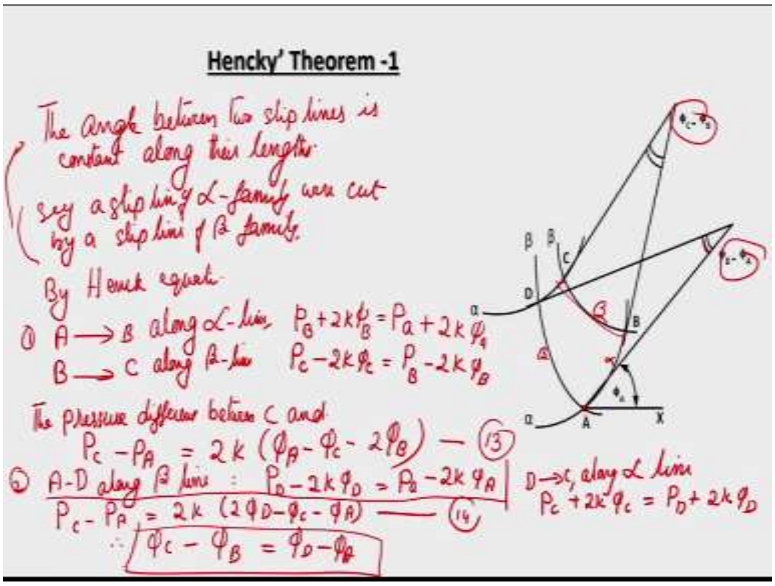

Now let us come to this Hencky’s theorem. So we will see because initially we were talking about this, what is its implication in this? Then Hencky’s theorem, it states that angle between two slip lines is constant along its length. The angle between two slip lines is constant along their length, that means if you say that a slip line of an alpha family if it was cut by a slip line of a beta family, then the angle between these two remains a constant.

So like if you say that angle between A and D, say A and D is equal to the angle between D and C in this equation, D and C, if you assume that this is your alpha line, these are your alpha lines and similarly these are your beta lines, which are also constant shear stress value. These are the two alpha and beta lines, let us say that okay, one is this alpha and another is this beta if I just say that.

The angle between the tangent at A and D, the tangent at A and D goes along any alpha line, so this angle should be equal to, say the angle between the tangent is this one, that is $\phi_D - \phi_A$, that should be equal to the angle between the tangents at C and B, so that should be equal to $\phi_c - \phi_b$, so that is what the Hencky's theorem says, so that is what angle between two, so like it says a slip line of alpha family were cut by a slip line of beta family, then we can say this is that, the angle between the two slip lines is constant along.

So that means $\phi_D - \phi_A$ is equal to $\phi_c - \phi_B$ because if you move along this alpha line or if you move along the beta line, at different points you will find the different in the angle between these two are remaining constant. So this can be proved by arriving at the pressure differences between C and A. So let us take one point here, one point here. If you take along this line and then along this line of if you take along this line and this line, what is the difference between A and C and that should be equal to difference between B and D, okay A and D should be equal to B and C.

So let us just see that. If the axis direction is taken along the A, along A as shown, then remembering that the positive phi axis, phi is defined as anticlockwise, rotation of the alpha line from the x axis we are using this Hencky equation. If you use the Hencky equation, we can write this as, by Hencky equation, say may be one from A to B, if you move along A to B along an alpha line, so A to B along an alpha line, we can write this pressure equation that is PB + 2 K phi B = PA + 2 K phi A okay.

$$P_B + 2K\phi_B = P_a + 2K\phi_a$$

From there B to C if you are moving, moving along B from B to C so along and that movement this is beta, this is alpha, this is beta, so that is that. So along the beta line, the equation for beta line is say like if you just write that say PC along beta line – 2 K phi C, PB – 2 K phi B, this is along the beta line. The beta line you will find it is negative, along the alpha line it is positive, that is what we are writing**.** Hence the pressure difference between C and A, so that PC - PA = 2 K you can write, phi A – phi C – 2 phi B, this is what we are getting.

$$P_c - 2K\phi_c = P_B - 2K\phi_B$$

$$P_c - P_A = 2K(\phi_A - \phi_c - 2\phi_B)$$

So let me just write how much was which was the equation number 13. Now second approach if you do with that if you move from A to D along the beta line and D to C along the alpha line okay. From A to D along beta line, that is PB, sorry A to D, because it is beta line, so PD – 2 K phi D = PA – 2 K phi A. Similarly, if you say from D to C along alpha line, you can write it as PC + 2 K phi C = PD + 2 K phi D because alpha line you have to write it as plus.

So hence from this equation, we can say that PC – PA = 2 K into 2 phi D – phi C – phi A, so this is equation number 14. So therefore no, finally we can get from 13 and14 if you write it, you see that phi C – phi B = phi D – phi A, so that is what we are getting, this and this is same, so that means the angle between these two slip lines is constant along any length if you take it, so whether that is along here or here if you take it, you will find that it is same. So, this is what we are finding. So Hencky's first law is this one, theorem 1. So now, next part we will discuss.

$$A - D \ along \ \beta \ line \quad P_D - 2K\phi_D = P_a - 2K\phi_A$$

$$D - C \ along \ \alpha \ line \quad P_C + 2K\phi_C = P_D + 2K\phi_D$$

$$P_C - P_A = 2K(2\phi_D - \phi_C - \phi_A)$$

$$\phi_C - \phi_B = \phi_D - \phi_A$$