Computational Continuum Mechanics

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Kinematics - 2
Lecture – 12-14
Linearized kinematics, Material time derivative, rate of deformation and spin tensor

Welcome. So, we are now coming to the 2nd part of Kinematics, ok. So, in next 3 lectures we will broadly cover following topics Linearize Kinematics ok, Material Time Derivative and the concept of Rate of Deformation and Spin Tensors, ok. And along the way, we will do a lot of examples. So, we will first.

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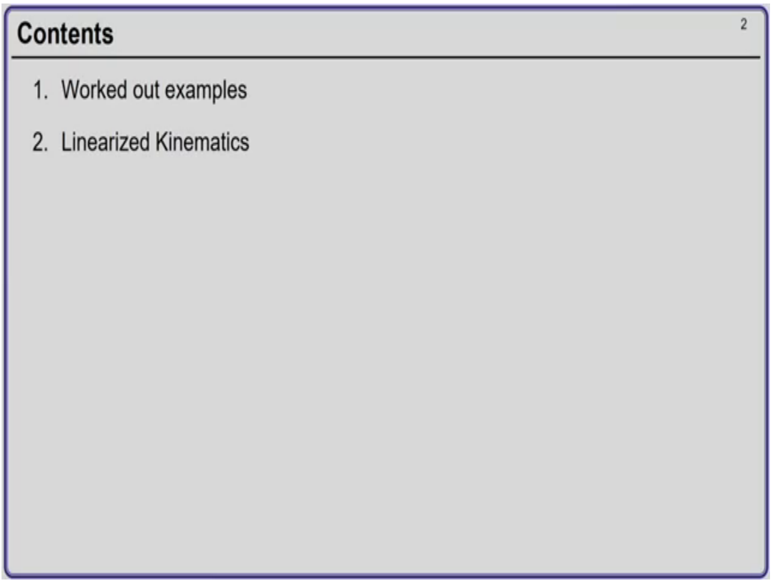

In today's lecture, we will do some worked out examples to have an understanding of the topics that we covered in the previous lectures on kinematics. And then we look into the concept of linearize kinematics. So, starting with our first example.

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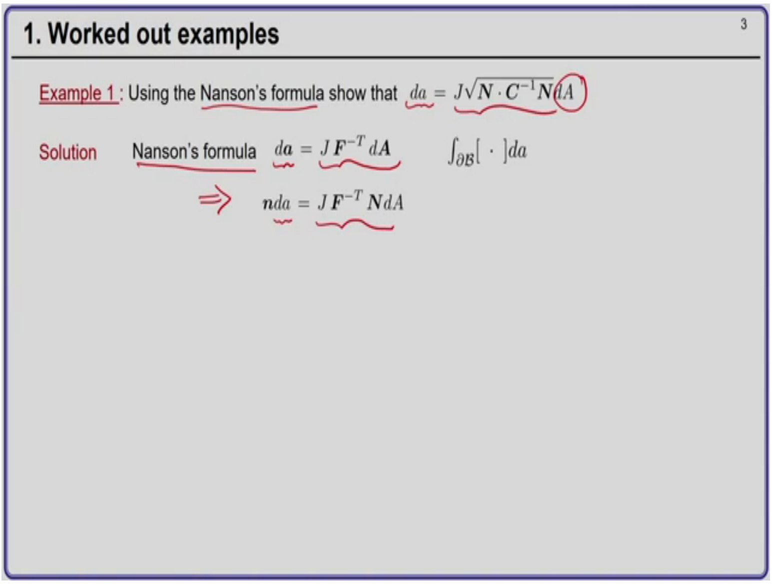

Say suppose you have been asked to show that the spatial area element d a is given by J under root N dot C inverse N into material area element d capital A and it is said you can use Nanson's formula, ok. Now from our Nanson's formula which is shown here, we know that the spatial area vector d a is J F inverse transpose into material area vector d capital A, ok.

So, this can be alternatively written in terms of n, where n is the normal to the area d a ok, spatial area d a equal to J F inverse transpose capital N; where capital N is the outward normal to the material area d capital A, ok. So, now why we need to show this is because many a times you will have these kind of integrals ok, integral over the current area and there is an integrand times d a, ok.

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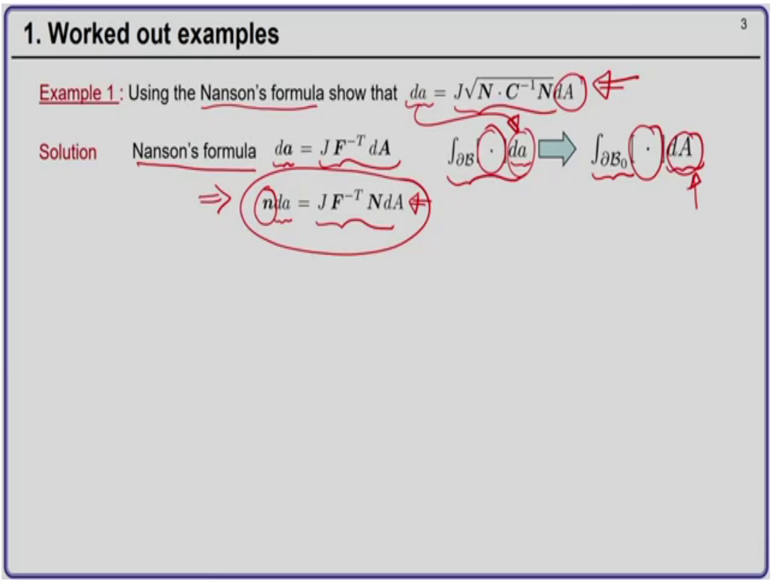

So, you need to convert it to integral over surface integral over the initial area times the integrand into d capital A. So, therefore, you need a relation for d a in terms of d capital A. While the Nanson's formula does not explicitly give us this relation over here ok, there is a normal which is well. So, if we can have this relation ok; so we can obviously substitute it here and then we will get d capital A, ok.

So, how do we show this? Ok. To show this what we do is, we start with this relation, ok. And then what we take is, we take the dot product of the left hand side with itself and the right hand side with itself; because both are vector quantities, therefore we can take a dot product.

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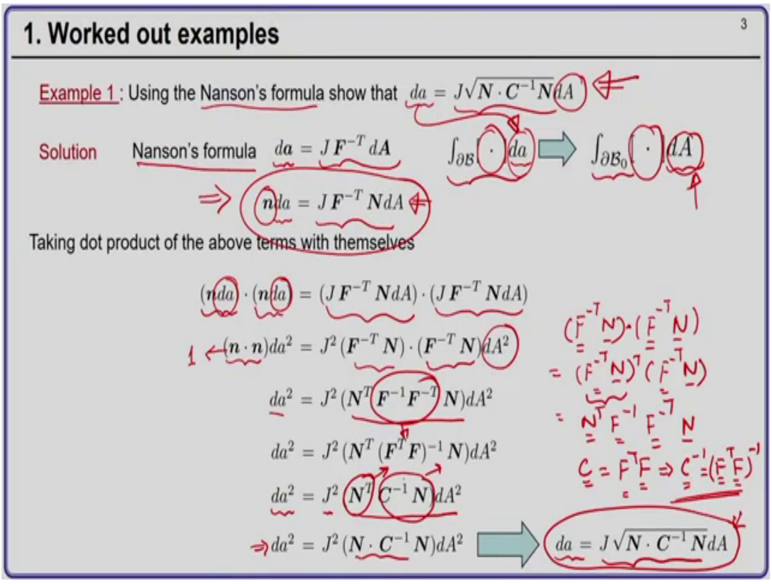

So, taking dot product of left hand side with itself and right hand side with itself ok, what we get? N dot n d a dot n d a is equal to J F inverse transpose N d A dot J F inverse transpose N d A.

Now, on the left hand side d a is a scalar ok, scalar quantity; therefore you can take it outside the bracket and n dot n is the unit outward normal vector, ok. So, you will have n dot n. And then here you can take d capital A square outside, ok. So, you have J square F inverse transpose N dot F inverse transpose N, ok. Now, because the spatial normal n is a unit vector, therefore n dot n will be equal to 1, ok.

Therefore we will have on the left hand side d a square equal to J square. And now F inverse transpose dot F inverse transpose N will be F inverse transpose N transpose F inverse transpose N, ok. So, a dot b is basically a transpose b, ok. And now this quantity is nothing, but N transpose F inverse F inverse transpose N, ok. Because A B transpose is B transpose A transpose, ok. So, that is what we have here. Now, we notice that, right Cauchy Green tensor is F transpose F. So, if we take the inverse, what we get? We get F transpose F inverse, ok. So, that is what we have here, ok.

So, I can write this as F transpose F inverse. And from here, I can realize that F transpose F inverse is nothing but C inverse. So, that is what I can substitute; I can write this as C inverse, the inverse of the right Cauchy Green tensor. So, once I do this, what I get? D a square is J square N transpose C inverse N d capital A square, ok. Now, it is very easy to see that this is a vector N transpose and this is another vector C inverse N, ok. So, a transpose b is a dot b. So, this is our a here and this is our b here. So, in the next step, what we can do? We can write d a square is J square N dot C inverse N d capital A square ok.

And then we can take square root on both the sides and we can get d a is J square root of N dot C inverse N d capital A. So, that is how you can relate the spatial area to the material area, infinitesimal spatial area to the infinitesimal material area, ok. So, that is what we wanted to show. So, using Nanson’s formula and using some identities, we have able to show following relation, ok. So, now moving to the next example. So, the next example gives you a deformation mapping, ok.

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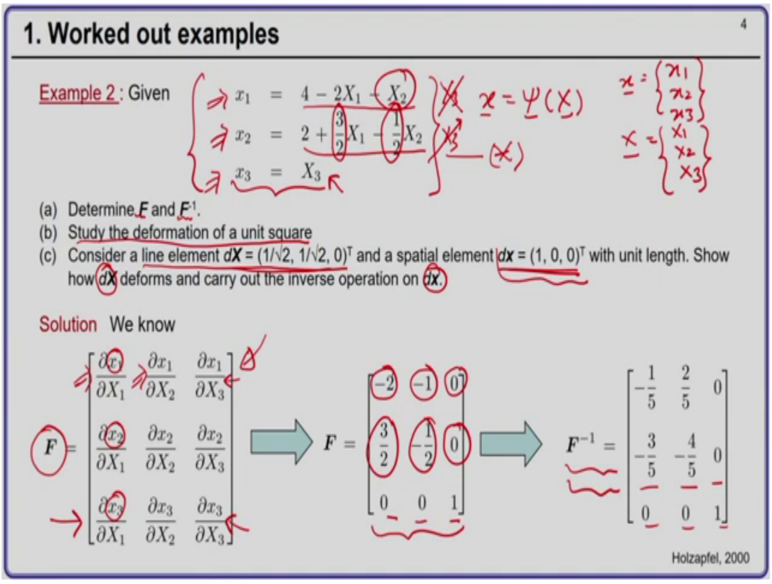

So, deformation mapping means, x equal to psi of X ok, where x is x 1, x 2, x 3 and x; capital X is capital X 1, capital X 2, capital X 3, ok. So, you have this deformation mapping that is the motion and this is given to you, ok. So, x 1 is 4 minus 2 X 1 minus X 2; x 2 is 2 plus 3 by 2 X 1 minus 1 by 2 X 2 and x 3 is equal to capital X 3, ok. So, last equation shows that, there is no motion in the third direction; all the motion is there in the x 1 x 2 plane, there is in one particular plane, ok.

So, this is the deformation mapping that you have been given. Now, in actual numerical computation, our objective is to find out this kind of mapping. This mapping will not be given to you; our objective will be to determine this mapping. But here for numerical example, we have just started with a known mapping and you want to just see how we can find the deformation gradient tensor F ok, find its inverse ok and then we want to study the deformation of a unit square whose one corner is at the origin ok; that means, x 1 equal to 0, x 2 equal to 0, x 3 equal to 0.

So, one corner of the unit square is at the origin. And finally, we want to consider a line element d capital X given by 1 by root 2, 1 by root 2, 0 and a spatial element d x equal to 1, 0, 0 of unit length. And you want to see, how these line element d capital X and d small x ok, the material element and the spatial element deform. So, let us start. So, we know that the deformation gradient tensor F can be put into a 3 by 3 matrix form ok and this is the relation that we discussed in our previous lectures, ok.

So, F 1 1 is del x 1 by del capital X 1; F 1 2 is del x 1 by del X 2. So, this is F 1 3, similarly you have all the way up to F 3 3, ok. And now these spatial coordinates x 1, x 2, x 3 which are here x 1, x 2, x 3; they have been provided to you in terms of the deformation mapping, ok. So, we can now substitute our deformation mapping which is here in the expression for the deformation gradient tensor, ok. Once we do this what we get? We get. So, take the first equation.

So, taking this equation del x 1 by del capital X 1 will be nothing but minus 2, del x 1 by del capital X 2 will be nothing but minus 1; because you have minus X 2 here. And del x 1 by del X 3 will be equal to 0; because there is no X 3 term present in the first equation. Similarly, you can take the second equation and you can compute del x 2 by del X 1 which will be nothing, but 3 by 2; because you have this term over here. Then del x 2 by del X 2 will be minus 1 by 2 which is here and del x 2 by del X 3 ok; there is no X 3 term again, therefore you will have a 0, ok.

And finally, you can find out the last row of the deformation gradient tensor by taking the derivative of the last equation with respect to material coordinates x 1, x 2, x 3, and you get 0, 0, 1, ok. So, in this way you have found out the deformation gradient tensor, ok. And this deformation gradient tensor you can see is independent of the material coordinates x 1, x 2, x 3; which means you have a homogeneous deformation, ok. Once we have found out the deformation gradient tensor; the next job is to found out find out the inverse of this deformation gradient tensor, ok.

So, you know how to find out the inverse of a 3 by 3 matrix. And once you find out the inverse you see that, F inverse is minus 1 by 5, 2 by 5, 0; minus 3 by 5, minus 4 by 5, 0; 0, 0, 1 that is the inverse, ok. So, your deformation mapping given by these equation has to be invertible ok; which means F inverse has to exist, which means you can go from the initial configuration to the current configuration and it is also possible to go from the current configuration to the initial configuration, and all the points will be mapped one to one, ok. So, this mapping will always be invertible, ok.

Next we were ask see. So, study the deformation of a unit square and also we wanted to study, how these line elements deform? Ok.

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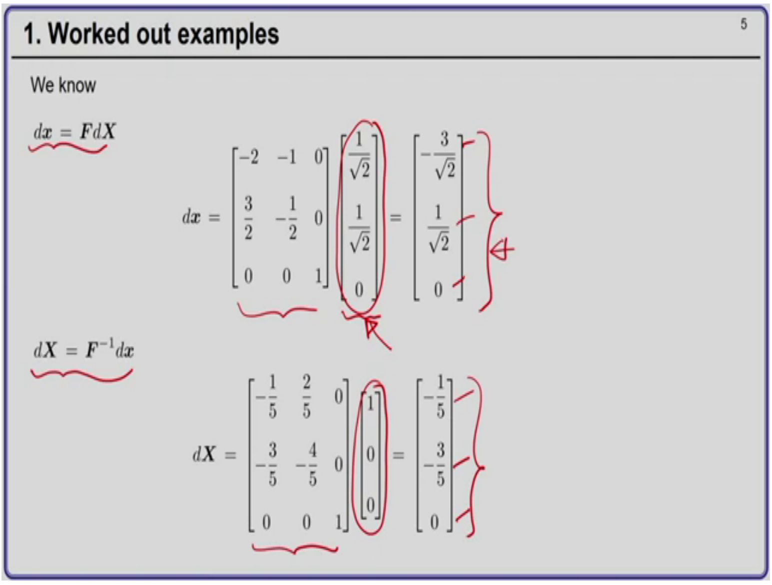

So, first say how the, will see how the line elements deform. So, we know that spatial element d x is F times material element d capital X; and d capital X is nothing, but 1 by root 2, 1 by root 2, 0, ok. So, all we need to do is take the matrix vector multiplication and then you will get the spatial element d small x corresponding to material element d capital X given by following vector.

And what you will see, you get minus 3 by root 2, 1 by root 2, 0 that is what you get. So, your material element deforms to following spatial element, ok. Now you also have been given, you can see here you also have been given this spatial element 1, 0, 0 and you have been asked how it deformed, ok. So, from which actually we want to see; what was the material element which deform to this spatial element d x, ok? So, d capital X will be nothing, but F inverse d x, ok. So, we already have computed F inverse which is here and d small x is 1, 0, 0, ok.

So, you can carry out the matrix vector multiplication and then you will get the answer as minus 1 by 5, minus 3 by 5, 0. So, that is here material vector which had deform to 1, 0, 0 this particular spatial vector, ok. Now, let us see how a unit square deforms, ok.

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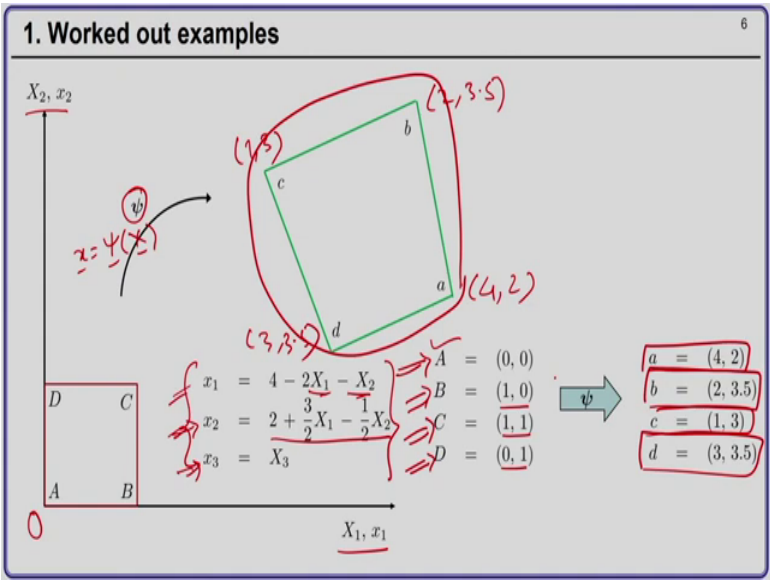

So, because everything goes on in one plane; there is nothing in the X 3 direction, so we just consider x 1, x 2 plane, ok. And let us position our unit square at the origin; one corner is at the origin and let the four corners the a, b, c, d capital A, B, C, D.

So, this is your deformation mapping, ok. So, that is your deformation mapping and you know the coordinates of each corner point, ok. For example, A is at the origin, so the coordinates of a are 0, 0, ok. B is at along the x 1 direction, so it is coordinate is 1, 0; because the square is of unit length, ok. So, you have 1, 0; C will be 1, 1; and D will be 0, 1, ok. And now using our deformation mapping ok; so this will be your capital X 1, capital X 2 ok, and for all these points x 3 is 0, the material coordinate x 3 is 0. So, that is why we have not written.

Once you have these X 1, X 2, you can substitute here and then you can obtain the current position of each corner point A, B, C, D ok, once you do this, ok. For example, if you want to see where did your point A go, ok. So, the x coordinate, current x coordinate of point A will be the substitute 0, 0 here, ok. So, you will have 4 minus 2 times 0 minus 0. So, the current x coordinate of point A will be 4; and then if we substitute 0, 0 in the second expression, you will get 2. So, the current y coordinate of point a will be 2, and x 3 is anyway 0, ok.

Similarly, you can do it for all other points and you can derive, ok. So, for example, the current location of point a is 4, 2; the current location of point b is 2 comma 3.5, ok. So, you can substitute 1, 0 in these two equations and you can get the current positions,. Similarly, point c will go to 1, 3 and point d will go to 3 comma 3.5, ok. So, your deformation mapping this x is psi of capital X takes. So, your deformation mapping takes this unit square and it deforms it to following shape, ok.

This is the shape ok, this I have drawn only approximately ok; you just see that this coordinate is 2, 3.5; coordinate of a is 4, 2 ok; coordinate of c is 1, 3 and coordinate of d is 3, 3.5. Now, let us see what happened to our material vector d X given by 1 by root 2, 1 by root 2, 0, ok.

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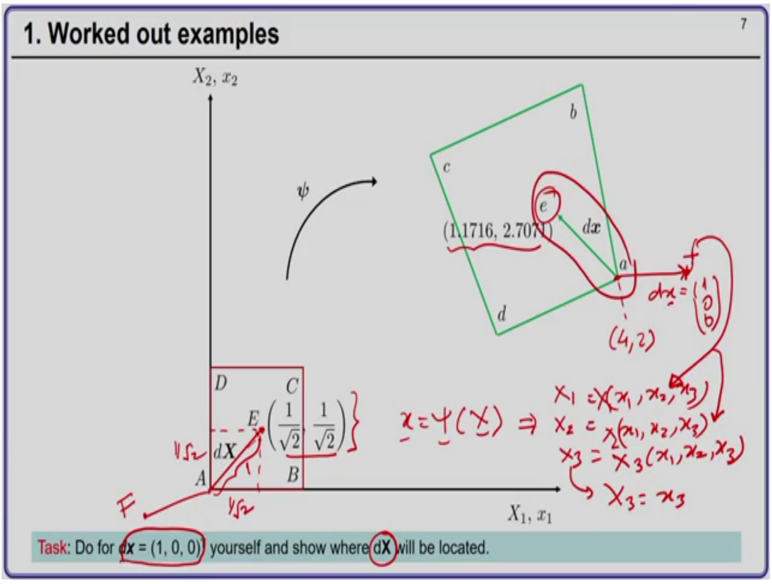

Let us see is the unit square which has mapped to this particular shape and now this is our unit vector d capital X given by 1 by root 2, 1 by root 2, 0 the tip, ok. So, the length of this is E equal to 1, ok.

So, if the length is 1, then its coordinates will be 1 by root 2 and 1 by root 2. So, that from Pythagoras theorem, these side third side will be 1 by root 2 square plus 1 by root 2 square which is nothing, but 1. Now, you want to see where does this vector go, ok. So, the coordinate of point is nothing, but 1 by root 2, 1 by root 2, ok. And now using our deformation mapping psi, we can see what will be the location of point E, ok. So, point A we already know has gone here. So, one end of the vector A E has gone here.

Now, we want to see where is the other position. So, from our deformation mapping given by x equal to psi of capital X; we can know the location of point E and that is what you will get 1.1716 comma 2.7071, ok. So, your A E vector in the material configuration will go to vector a's e in the spatial configuration, ok. So, let the task of finding out d capital X for a given d x 1, 1, 0; 1, 0, 0 transpose yourself, ok. You can take a vector d x given by 1, 0, 0 transpose.

You know the location of this 4 comma 2; let us you take a point f such that a f is this vector d x 1, 0, 0 and then you see where does this vector go, ok. So, you have to map. So, first what you have to do using deformation mapping; you have to find out X 1, X 2, X 3 in terms of small x 1, x 2, x 3,. Let us say X 1. For example, the last one will be X 3 equal to small x 3 ok that is very easy; the other two you can do it yourself. There are two equations in two unknown; I mean two unknowns, you can solve and get X 1, X 2 in terms of spatial coordinates small x 1, x 2.

And then you can substitute the coordinates of f here ok, and you can get the coordinate of f; it will come out somewhere here, ok. So, this is the task that I leave it to you,.

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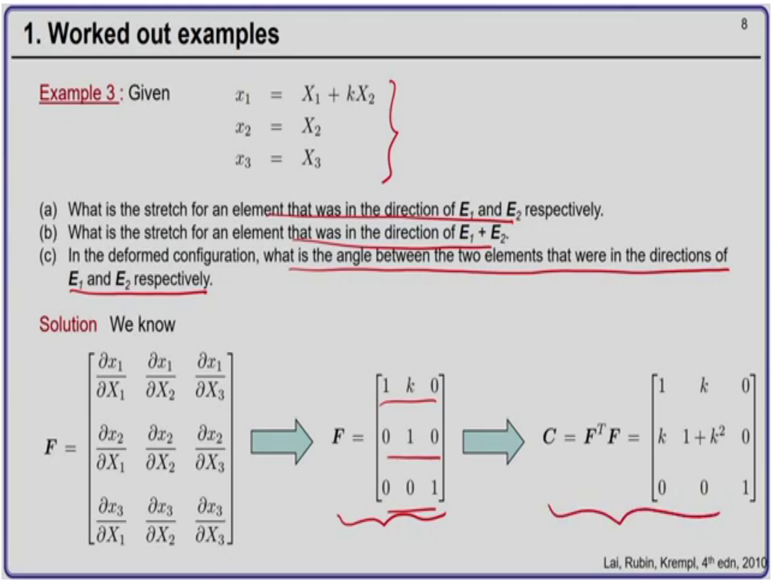

So, moving to our next example, which is again a deformation mapping which is given by x 1 is X 1 plus k X 2; x 2 is X 3 and x 3 is, x 2 is X 2 and x 3 is X 3. Now we have to find out the stretch of an element that was in the direction capital E 1 and capital E 2 respectively. Remember capital E 1 capital E 2 are the basis vectors in the material coordinates of the initial configuration,

And we also have to find out the stretch of an element that was in the direction E one plus E 2, ok. Now in the deformed configuration, we also have to find out what will be the angle between two elements that were in the directions of the E 1, E 2 respectively. Which means, initially if there were two elements, one in the x 1 direction which is E 1 and the other in the E 2 direction; then after the deformation, what will be the angle? So, initially they were like this and after deformation what will be the angle between these two elements, ok?

So, starting we can in first find out our deformation gradient tensor which will be nothing, but 1, k, 0; 0, 1, 0; 0, 0, 1, ok. And then we can find out the C ok, which is F transpose F, ok.

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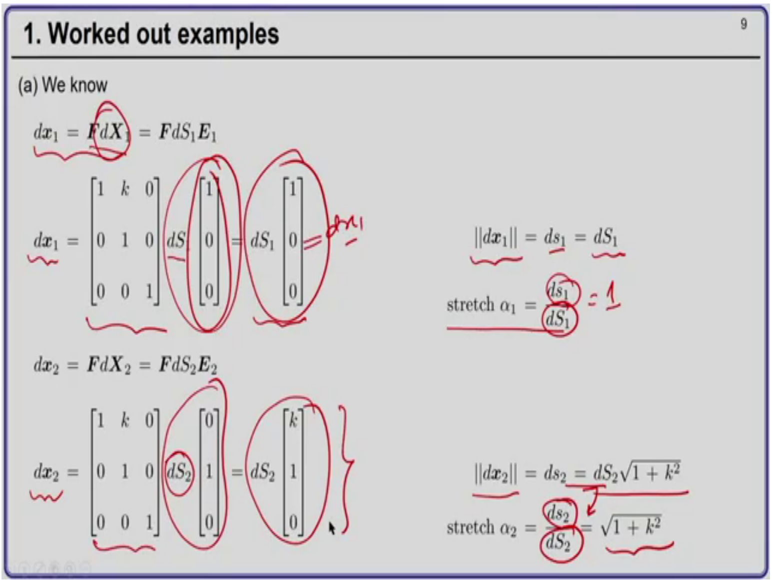

Now, using. So, our first objective is to find out what happens to a material vector which was along the E 1 direction, ok. So, why, for simplicity just take the length of the vector be d S 1, ok. So, the vector d capital X one will be nothing, but d S 1 into the unit vector E 1, which is nothing, but 1, 0, 0. And to calculate the spatial element d small x 1, we just have to multiply by F.

So, F d capital X 1 gives you d S 1 1, 0, 0, ok. So, what this deformation mapping does? It does not change the length of any vector which is along the E 1 direction, ok. So, you see your initial vector was this and your final vector also comes out to be the same ok; which means, there is no length change in the E 1 direction, ok. So, the stretch in the E 1 direction will be alpha one equal to d s 1 by d capital S 1, ok. So, d s 1 will be nothing, but the norm of the material l 1 d x 1, ok. So, this is d x 1. So, you can compute the norm will be d capital S 1; the stretch will be equal to 1, which means there is no change in the length, ok

Now, in the E 2 direction. So, in E 2 direction any material vector will be length d S 2 times the basis vector in E 2 direction which is nothing, but 0, 1, 0, ok. So, this is here vector in the E 2 direction. And to get these spatial vector corresponding to this material vector, you have to multiply by F and you get the current position which is d S 2 k, 1, 0, ok. So, the length of the vector becomes the norm which is nothing, but d S 2 root over 1 plus k square.

So, the stretch will be the length of the vector in the spatial configuration divided by the length in the material configuration; which is nothing, but if you see from here, you can get the stretch will be root over 1 plus k square, ok. So, in the E 1 direction, there is no change in the length of the vector and the E 2 direction there is following change in the vector, ok.

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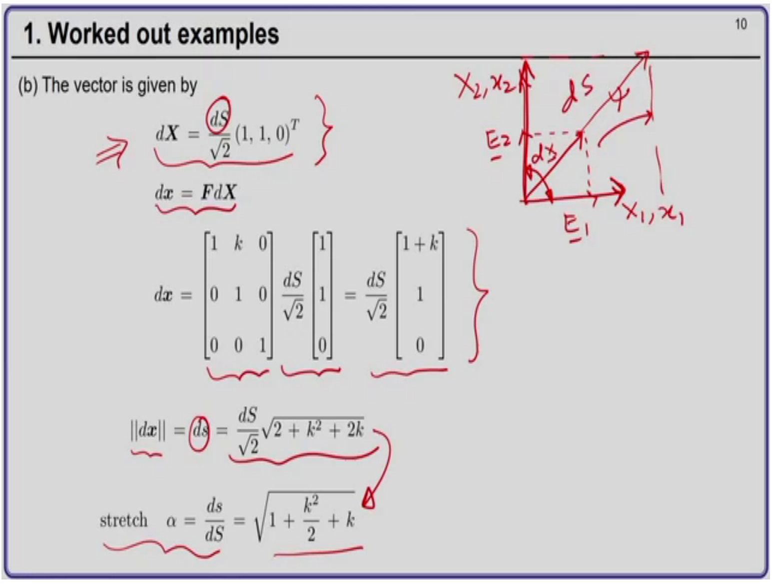

So, now we have to find out what happens to a vector which is E 1 plus E 2, ok. So, you have capital X 1, x 1; we have capital X 2, x 2, ok. So, this is your E 1 vector, this is your E 2 vector.

And then we want to find out what happens to this vector; this vector is nothing, but E 1 plus E 2. So, after the deformation, what happens to this particular vector ok? So, our vector is d S by root 2 1, 1, 0, ok. So, d s means, I just let take any vector along this direction, ok. So, unit vector and let d capital S, I can take any vector along this direction, ok. So, let it be the length be d S, ok. So, what happened to this particular vector? I just have to multiply by F to get its current position d small x, ok. So, F times d X and it gives you d S by root 2 1 plus k, 1, 0, ok.

So, this is these spatial vector corresponding to this material vector. Now I can compute the length of the spatial vector which is norm is nothing, but d small s, and if I compute this will be d S by root 2 into root 2 plus k square plus 2 k. So, the stretch will be alpha equal to d s by d capital S which from here you can see its root over 1 plus k square by 2 plus k, ok. Now coming to the angle change, ok. So, we also have been asked what happens to this angle change, ok.

Initially if you have a material element along the x 1 direction and you have another material element along the x 2 direction ok; then what will be the change in the angle? So, initially you can see the angle is 90 degree and we want to find out, what will be this angle ok?

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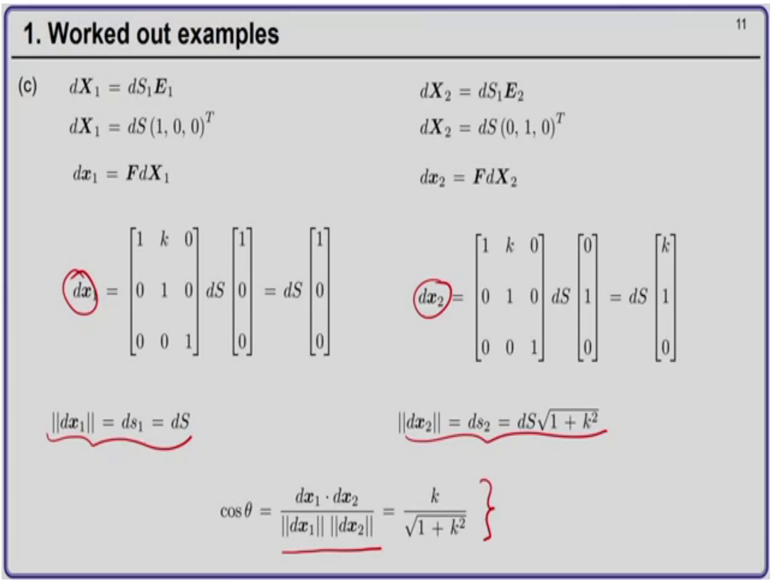

So, this we already had derived in the first case, there is no stretch in the x 1 direction and this is the change in the x 2 direction, ok. So, now, you know vector d x 1 and vector d x 2. So, the angle between these two vectors, ok.

So, if you have two vectors a and b; so the angle between the two vector will be cos theta equal to a dot b divided by norm of a into norm of b. A here is d x 1 and b here is d x 2, ok. So, if you do that, cos theta will be d x 1 dot d x 2 divided by norm of d x 1 into norm of d x 2. And if you simplify you will get k divided by root of 1 plus k square, ok. Now, with these examples, it should be clear to you how to find out the deformation mapping, ok. I mean using deformation mapping, how to find out the deformation gradient and then visualize how the shape of the body changes, ok.

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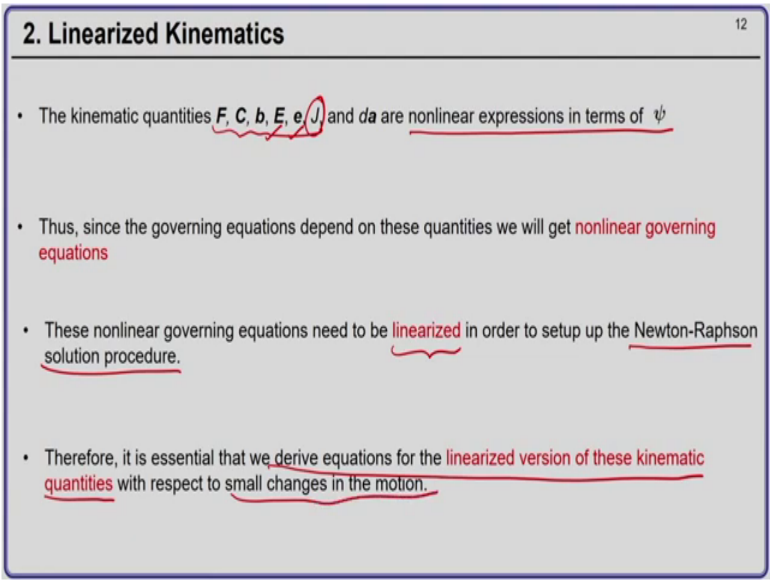

So, coming to our next topic which is linearize kinematics, ok. So, these kinematic quantities which we discussed in our previous lectures like deformation gradient tensor, the green right Cauchy Green tensor, the left Cauchy Green tensor, the Green Lagrange strain tensor, the Euler Almansi strain tensor, the Jacobian J and the area element d a all these quantities are non-linear quantities ok, they are non-linear expression in terms of the deformation mapping, ok. So, each quantity is derived from deformation mapping and all these quantities are non-linear quantities, ok.

And therefore, the governing equations that we derived later on in terms of these kinematic quantities will be non-linear, ok. There and to solve these non-linear governing equations; we have to linearize the governing equations ok, in order to set up the Newton Raphson solution procedure, ok. So, the, there are non-linear governing equations and then we have to linearize, so that we can set up the Newton Raphson solution procedure to solve for the unknown displacements, ok. So, therefore, linearization of the governing equation means, you have to do the linearization of the kinematic quantities like deformation gradient tensor, the right Cauchy Green tensor, left Cauchy Green tensor and so on ok.

So, you have to derive the linearized version of these kinematic quantities, because of small changes in the motion, ok.

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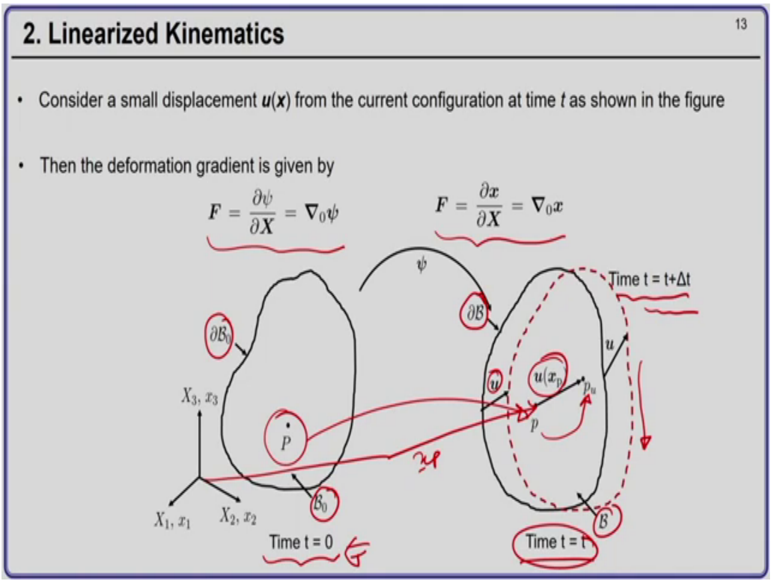

So, consider again you have body occupying the current position B 0 bounded by surface del B 0 at time t equal to 0, ok. And after deformation the body occupies position B bounded by surface del B at time t, ok. Now you consider a small displacement ok; a small change from it is current position t ok, consider displacement u from the current position at time t, ok.

So, let the red, dotted red lines show the position at time t plus delta t ok; delta t correspond to small increment in the time, ok. In static problems time is a virtual quantity, so a small change ok; in dynamic delta t the physical quantity. So, that the point P in the material configuration which are deformed to small point p in the spatial configuration. Now goes to P u ok, where the displacement of point p is a function of its spatial coordinate x p ok, x p is the spatial coordinate of point p, ok.

Now because there is a small change in the position of the body at time t ok; therefore and this change is given by u the displacement, there will be changes in the kinematic quantities ok, because of this small change in u, ok. So, recall that our deformation gradient was del psi by del X equal to del 0 psi or F is del x by del X equal to del 0 x, ok.

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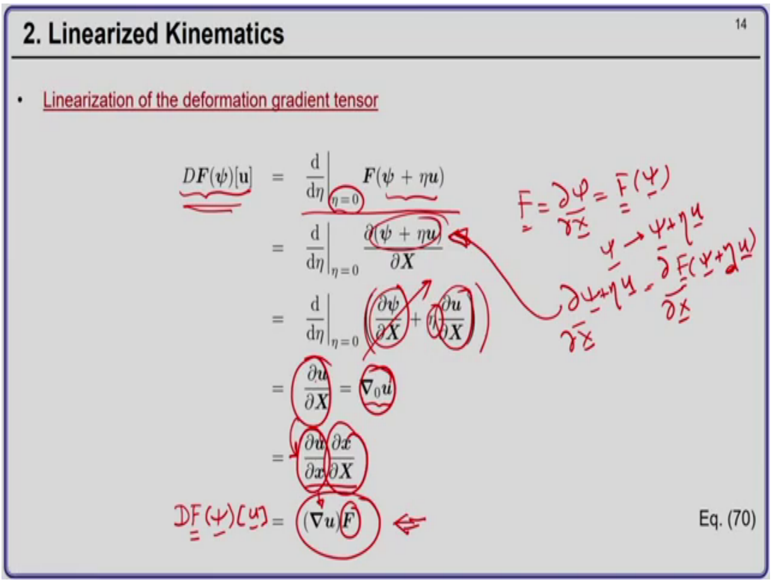

Therefore first kinematic quantity that has to be linearized ok; we want to understand what will be the small change in the deformation gradient tensor because of this small change u in the deformation, ok. So, the current position of the body changes by u; so we want to ask what will be the change in the value of the deformation gradient tensor F, ok.

So, basically we want to compute the directional derivative of the deformation gradient tensor at psi in the direction of u, ok. So, from our definition of computing the directional derivative, we can write this expression as d by d eta of F evaluated at psi plus eta u and then taking eta equal to 0, ok. Now F evaluated at psi plus eta u is nothing, but, ok. So, F is del psi by del X, ok. So, it actually means F of psi, ok. So, changing psi to psi plus eta u means, you want to compute del psi plus eta u by del X ok, that is your del F by del X of psi plus eta u, ok. So, that is what we have written here.

So, now we can open up the brackets ok, I can open up the bracket and you can write this quantity as del psi by del X plus eta times del u by del X, ok. Now I can take the derivative of the term inside this bracket with respect to eta, ok. So, the first term does not have any eta, so this will go away; and there is only eta in the second expression, so del by del eta will give you 1. So, all you are left with is del u by del X, ok. So, in short I can write this as del 0 u, ok. Now I can simplify it further to get relation in terms of the gradient of u in terms of spatial coordinate. So, I can write del u by del X as del u by del small x into del x by del capital X, ok.

So, del u by del x is nothing, but spatial gradient of u, ok. So, this was material gradient, this is called the material gradient of u. So, the spatial gradient of u times and del x by del X is nothing, but F, ok. So, the directional derivative of F evaluated at psi in the direction u ok; it is nothing, but the gradient the spatial gradient of u times the deformation gradient tensor F, ok. So, this means this is the change in deformation gradient tensor at point x, because of a small change u in x, ok. So, if x changes by u, following will be the change following will be the change in the deformation gradient at that point ok that is what it means.

So, this is the, equation 70 is the linearized deformation gradient tensor, ok. So, once we have done the linearization of the deformation gradient tensor, we can now linearize the Green Lagrange strain tensor, ok.

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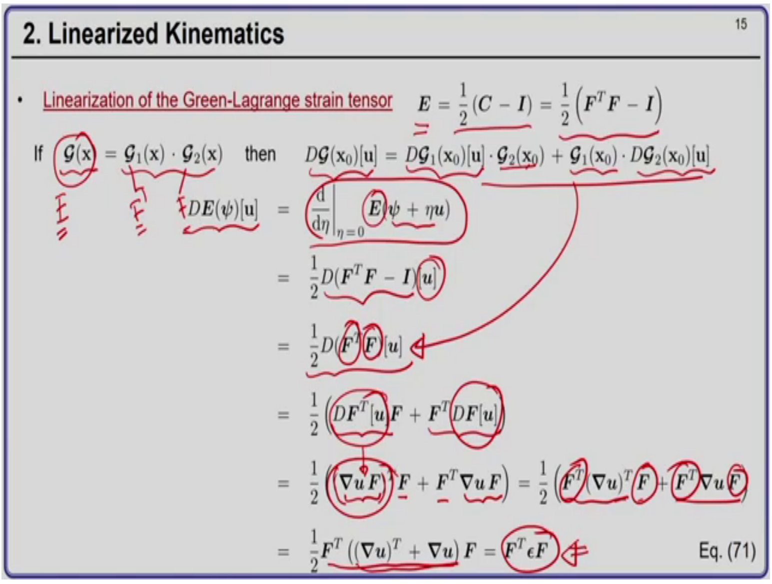

So, Green Lagrange strain tensor E is nothing, but 1 by 2 C minus I 1 by 2; right Cauchy Green tensor minus the identity tensor, and the right Cauchy Green tensor is nothing, but F transpose F, ok. So, we can use these second property that we discuss of linearization which is; if non-linear quantity is some kind of product of two non-linear quantities, then the directional derivative of this non-linear quantity type X 0 in the direction u is nothing, but the directional derivative of G 1 at X 0 in the direction u times G 2 plus G 1 times the directional derivative of G 2 at X 0 in the direction u, ok.

So, our G here is E and our G 1 and G 2 are F, F transpose n F, ok. So, now, if we do this, so the directional derivative of the Green Lagrange strain tensor at psi evaluated in the direction u is d by d eta of the Green Lagrange strain tensor evaluated at psi plus eta u; which means you have to compute.

And if I substitute for E here, which means I have to compute 1 by 2 times the directional derivative of F transpose F minus I at psi in the direction u, ok. Now I is a identity tensor which is constant, so its directional derivative in the direction u will not change, so that is 0, ok. So, that takes away. So, we have 1 by 2 directional derivative of F transpose F in the direction u, ok. Now here I can use this property ok, I can use this property and I can write. So, my G 1 is F transpose; this is my G 1, this is my G 2, ok.

So, I can write directional derivative of F transpose in the direction u into F plus F transpose directional derivative of F in the direction of u, ok. Now directional of derivative of F transpose in direction u is nothing, but the directional derivative of F transpose, ok. Now directional derivative of F is nothing, but the spatial gradient of u times F.

So, this is directional derivative of F transpose ok, this is the external delta F transpose u into F plus F transpose and directional derivative of F in the direction u; we have derived in previous slide is nothing, but spatial gradient of u times F, ok. Now I can. So, this I can open up the bracket A B transpose is B transpose A transpose, so this is B transpose A transpose, ok.

So, F transpose del u transpose F plus F transpose del u F. Now you can take out F transpose from the left hand side and F from the right hand side, ok. If I do this, so I can write 1 by 2 F transpose del u transpose plus del u into F. And now I can recognize from our knowledge in solid mechanics that del u transpose plus del u by 2 is nothing, but the small strain tensor epsilon, ok. So, del u plus del u transpose by 2 is nothing, but small strain tensor epsilon.

And then we finally, arrived at our expression for the directional derivative of the Green Lagrange strain tensor E evaluated at psi and the current position in the direction u as F transpose epsilon F, ok. So, that is the directional derivative of E ok, the Green Lagrange strain tensor.

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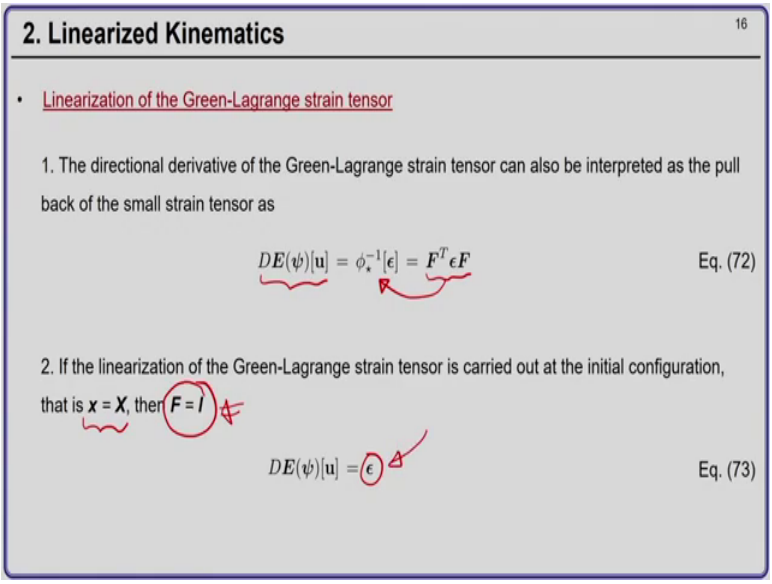

Now, I can interpret the directional derivative of the Green Lagrange strain tensor as the pull back of the small strain tensor, ok. So, you see the directional derivative of psi in the direction u is F transpose epsilon u ok, or this is nothing, but phi star inverse epsilon, ok. So, the directional derivative of the Green Lagrange strain tensor in the direction u is nothing, but the pull back off small strain tensor epsilon, ok.

Now, if I do this linearization of the Green Lagrange strain tensor about the initial configuration; which means, say if my current configuration and my initial configuration do not differ by much ok, which is for example, the case in elasticity, the current and the final configuration are not differing by much. In that case my, the spatial coordinates nearly equal to the material coordinates or my the deformation gradient tensor F is nothing, but nearly identity, ok. So, in equation number 72, if I substitute F is identity. So, F transpose epsilon u F will be I transpose epsilon I which will be nothing, but epsilon, ok.

So, when I do my linearization of the Green Lagrange strain tensor about the initial configuration, I get nothing but the small strain tensor. So, in elasticity, linear elasticity your initial and the final configuration do not change by much. So, your Green Lagrange strain tensor boils down to nothing, but the small strain tensor, ok.

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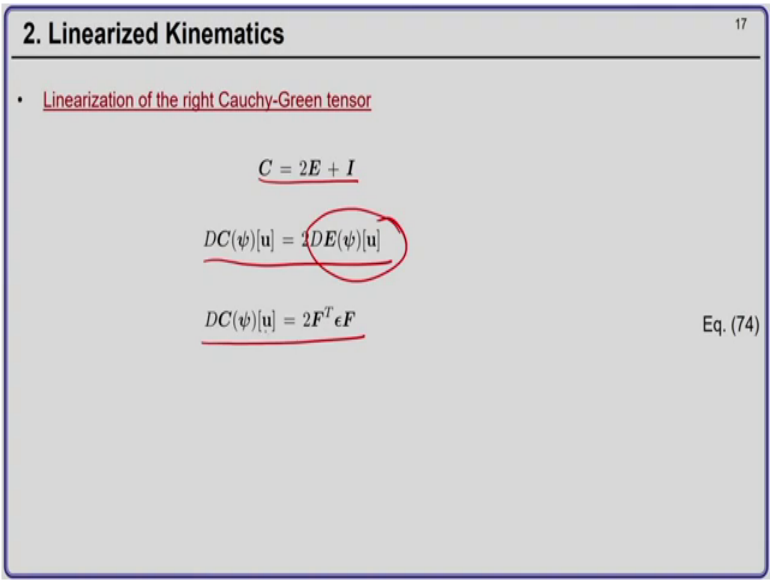

So, moving to our next linearization, which is the linearization of the right Cauchy Green tensor. Now C is nothing, but 2 E plus I. So, the directional derivative of C will be twice of the directional derivative of E, ok.

Now, we have already computed the directional derivative of E which is F transpose epsilon F. So, I can substitute that here, ok. And I can write the directional derivative of the right Cauchy Green tensor at psi evaluated in the direction u is nothing, but 2 times of F transpose epsilon F, ok.

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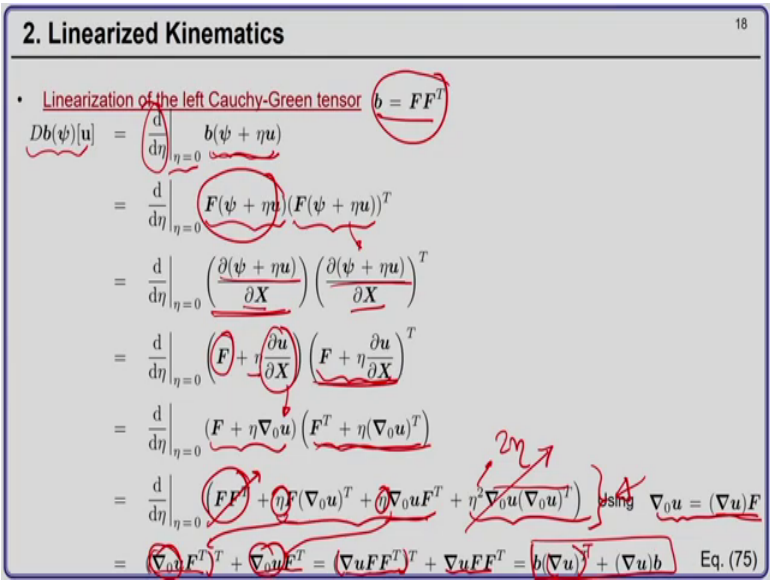

Now, coming to the linearization of the left Cauchy Green tensor, which is b equal to F F transpose, ok. So, to see this, we write the directional derivative of b in the direction u as b evaluated at psi plus eta u derivative of d by d eta and eta equal to 0, evaluated at eta equal to 0.

Now, b evaluated at psi plus eta u will be nothing, but F evaluated at psi plus eta u times F evaluated at psi plus eta u transpose, ok. Now this we have already seen during linearization of deformation gradient tensor; this is nothing, but del psi plus eta u divided by del X and the second quantity will be del psi eta u divided by del X transpose, ok. So, opening up the brackets here, we will have del psi by del X which is nothing, but F and eta times del u by del x. In the second bracket we have again F plus eta del u by del x transpose, ok.

So, the second bracket, if I take transpose inside that will be F transpose plus eta del u the material derivative gradient of u transpose, ok. So, I can write del u by del X as material derivative of u, del 0 u. So, I have F plus eta del 0 u into f transpose eta del u 0 transpose. I can open up the bracket, therefore I will have a constant term and have two terms which have eta and have one terms which has eta square, ok. Once I have this term in eta, I can take the derivative of this quantity in the bracket in terms of eta. If I do this, the first quantity does not have any eta. So, it goes away.

The second and the third quantity, ok. So, the second and third terms have eta. So, del eta by del eta is 1. So, we will have this here and then last term will be twice of eta times this quantity. And now when I substitute eta equal to 0, this term also goes away. So, what I am left with is, this material gradient of u F transpose plus the material gradient of u F transpose, ok. Now I can use this relation which we already have derived the, at material gradient of u is nothing, but the spatial gradient of u times F that if I substitute here, ok.

I will get the spatial gradient of u F F transpose whole transpose plus spatial gradient of u F F transpose. Now recognizing that F F transpose is nothing, but the left Cauchy Green tensor b, I arrive at the final expression b times the spatial coordinate of u transpose time plus spatial gradient of u times b, ok. So, that is the equation 75 gives you the change in the left Cauchy Green tensor b, when the current configuration goes a small change u, ok.

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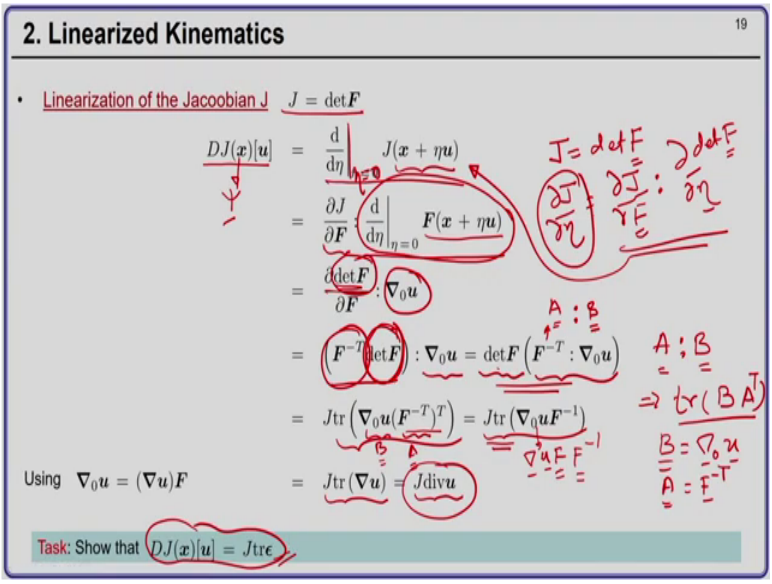

Now, you come to the linearization of the Jacobian, which is nothing, but the determinant of F, ok. And you can also recognize that Jacobian is nothing, but d v the spatial volume element divided by d capital V which is the material volume element, ok. So, now, J is nothing, but determinant of F, ok. So, the directional derivative of J evaluated at x, I can also write this as psi no problem. In the direction u is nothing, but d by d eta of J evaluated at x plus eta u and then putting eta equal to 0, ok.

Now, see J is determinant of F, ok. So, if I have to calculate del J by del eta, I can use a chain rule del J by del F. Now, what happens, del J by del eta is a scalar quantity; therefore when I do del J by del F, F is a second order tensor, I get a second order tensor. So, to get a scalar I have to do what is called the double contraction and then del by determinant of F times del F. So, that is the chain rule, ok. So, if I do this. So, if I use this here, I get del J by del F double contraction d by d eta, eta equal to 0 ok, sorry this is. So, this should be eta, ok. So, d by d eta of F plus F at x plus eta u, ok.

Now, I have already have derived. So, this is nothing, but the directional derivative of F which is nothing, but the material gradient of u; and J is nothing, but determinant of F, ok. So, the del J by del F is nothing, but del of the determinant of F divided by del F,. Now you know, we have already computed the gradient of the third invariant of a tensor with respect to the tensor itself, ok. What was the third invariant of a tensor second order tensor? It was the determinant of a tensor, ok. So, here also we have determinant of F which is nothing, but like the third invariant of the deformation gradient tensor F.

So, now we want to compute the derivative of determinant of F divided by del F, it is determinant of the third invariant of F divided by del F. And if you can recall that relation was F inverse transpose, that was the third invariant of the second order tensor times the second order tensor inverse transpose ok; which is nothing, but if we see here is determinant of F, the third invariant times the tensor inverse transpose, ok. So, we have determinant of F times F inverse transpose, double contracted with the material derivate material gradient of u, ok.

So, because determinant of F is a scalar, I can take it outside the bracket and I can write F inverse transpose double contracted with del 0 u, ok. Now I can write. So, A double contraction B is if you can recall that identity that we discuss in the, our mathematical preliminaries was B transpose A or is B into A transpose, ok. So, our B here is del 0 u and our A here is F inverse transpose, ok. So, this is our B and this is our A ok, this is our, A, ok.

So, this is our A double contraction with B. So, A double contraction with B is nothing, but trace of B A transpose. So, I can write this relation as J, J is determinant of F times trace of this quantity, ok. Now F inverse transpose is nothing, but F inverse. So, I can write J trace of del 0 u into F inverse, ok. Now, del 0 u is nothing, but; see del 0 u is nothing, but del u into F, ok. So, you have del del u into F into F inverse, and F F inverse is nothing, but identity. So, finally, our relation here boils down to J times trace of spatial gradient of u.

And now you also know that trace of gradient of a tensor is nothing, but divergence of a tensor. So, finally, you get J divergence of u. So, this relation shows you the directional derivative of the Jacobian evaluated at x in the direction u; that is J divergence of u is the change in the Jacobian for a small change u in the current configuration, ok. So, the task for you is you can also show that the directional derivative of the Jacobian in the direction u is nothing, but J trace of epsilon ok, that is a where epsilon is the small strain tensor. So, this job is left to you, ok.

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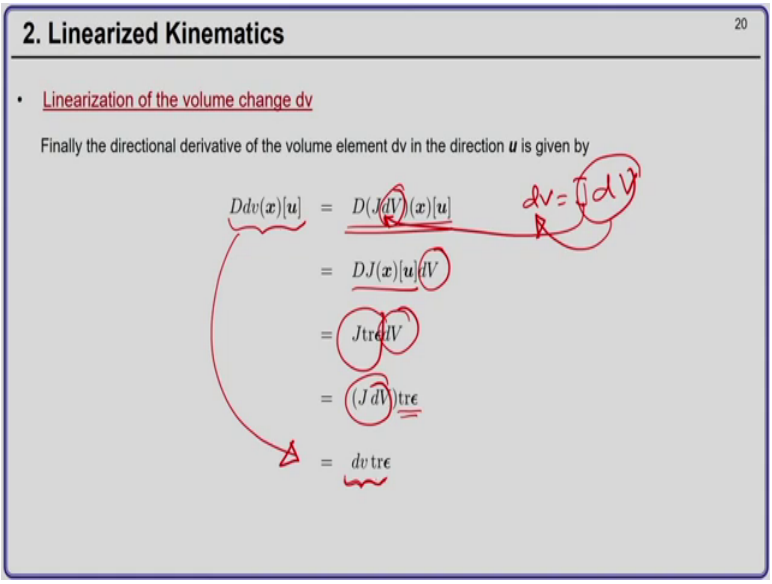

Finally we come to linearization of the volume element d v, ok. So, the directional derivative of the volume element d v at x in the direction u is nothing, but see d v is nothing, but J d capital V that is what we substitute, ok. So, now, d capital V is a constant quantity, I can take it outside the bracket. So, this reduces to directional derivative of the Jacobian in the direction

u. And just now we have computed that the directional derivative of J in the direction u is nothing, but J trace of epsilon, which was a task that you were to do, ok.

So, J trace of small strain tensor epsilon times d V, ok. So, J d V times trace of epsilon. Now J d V here is nothing, but our spatial volume element d V. So, we finally, we get the directional derivative of the volume element, spatial volume element in the direction u is nothing, but spatial volume element d v times trace of epsilon. So, with this we end today's lecture.

Thank you.