Computational Continuum Mechanics

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Kinematics - 2
Lecture - 12-14
Linearized Kinematics, Material time derivative, rate of deformation and spin tensor

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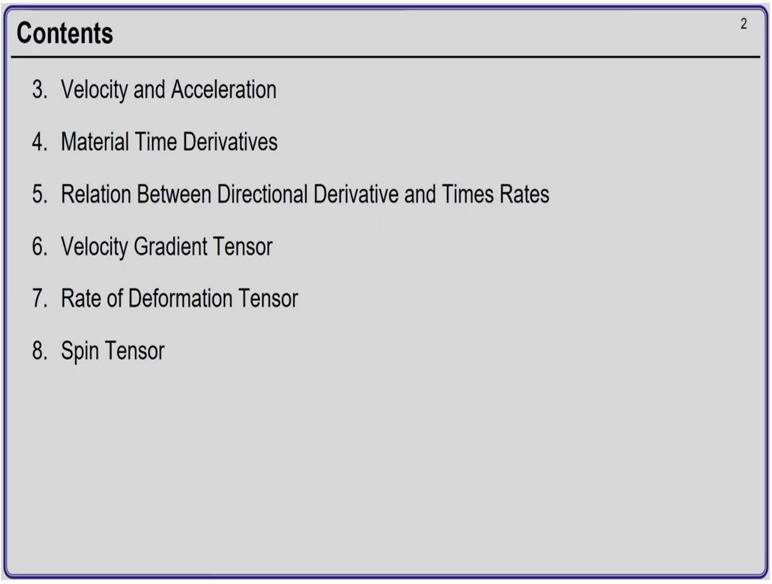

So, today we are going to start some new topics. So, they are listed here; first, we will look into how to define velocity and acceleration for a deformable body, then we are look into a very important concept of material time derivative.

Finally, we will see there is a relation between directional derivative and the material time derivative ok. So, we will look into this important connection. And towards the end we will look into the concepts of velocity gradient tensor, rate of deformation tensor and the spin tensor ok.

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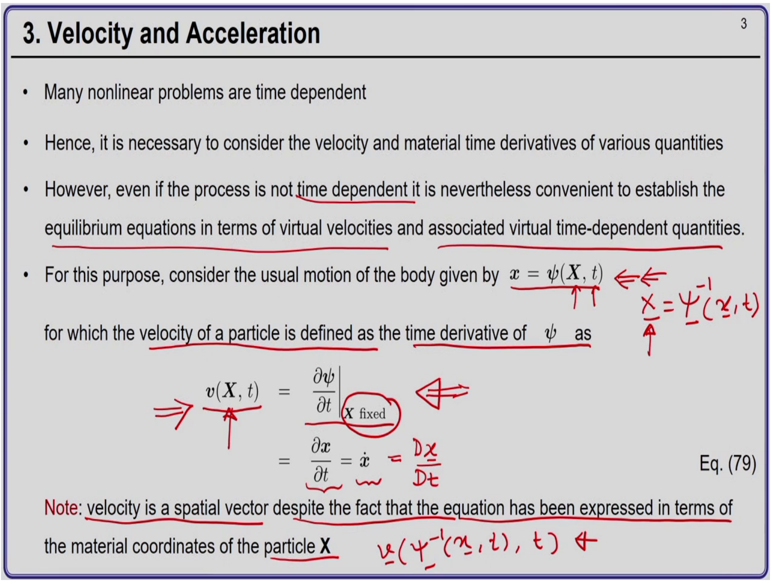


So, let us start. So, many non-linear problems you would know are time dependent. So, static problems are just a idealization of dynamic problem; all problems are dynamic is only when the inertia effects can be neglected, then we get the static problem. But there are certain problems like crash impact and all these kind of problems where time effect cannot be taken out ok; in that case we have to consider velocity and material time derivative of various quantities.

So, even though a process may not be time dependent ok; it is nevertheless convenient to establish equilibrium equation in terms of virtual velocities. That is what we are going to do later on when we are going to derive the virtual work we are going to use virtual work. So, there instead of virtual displacement we are going to use virtual velocities ok. So, equilibrium equations need to be established in terms of virtual velocities and their associated virtual time dependent quantities.

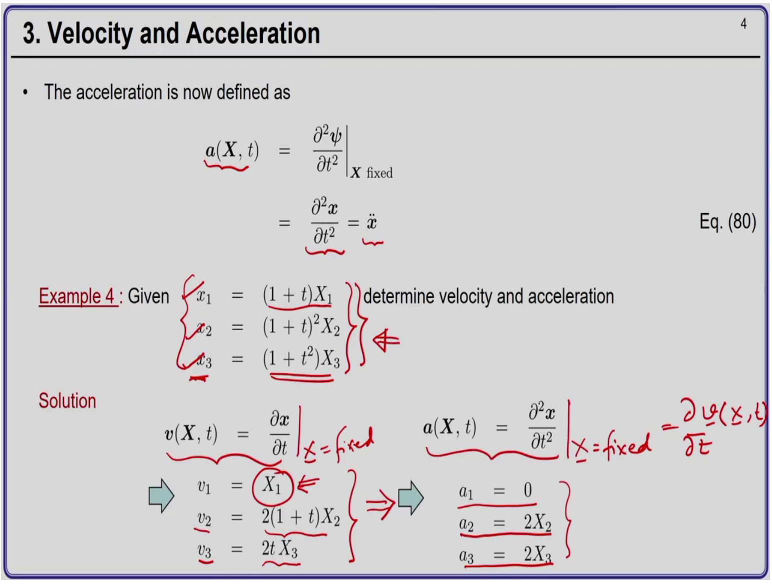
So, for this purpose now; consider the motion of the body which is given by x equal to psi and psi is a function of the material coordinates X and current time t. So, what this tells you is the current location of a material particle which was originally at coordinate capital X ok. So, now, the velocity of the particle is now defined as the time derivative of psi and the velocity is written as v equal to del psi by del t for a fixed material particle.

So, this definition ok gives you the velocity of a particular material particle x at time t. So, during the deformation the particle is moving over space ok. So, as this changes its position in space how does its velocity change? Ok. So, this relation gives you that particular information ok. So, this velocity is also written as del x by del t or equivalently as x dot ok.

And in some books you will also find dx by dt ok you will find dx by dt capital D actually is used for derivative time derivative where X is fixed the material coordinate is fixed ok. So, equation 79 gives you the velocity at time t of a particle which was originally located at material coordinates x. So, this will always give you the velocity of a particular particle ok.

Now, you note that velocity is a spatial vector ok its not a material vector even though you see there is a material coordinates, but it tells you that velocity is a spatial vector despite the fact that the equation has been expressed in terms of material coordinates of the particle that is capital X ok. You can always invert this relation over here you can always get X as psi inverse x comma t ok.

And then you can substitute for this capital X in equation number 79 to get velocity v as psi inverse x comma t comma t ok. So, this is velocity at the spatial position x. (Refer Slide Time: 06:28)



Now, moving on to acceleration. So, acceleration is now defined as the double derivative of the motion psi and its given by a equal to del square x by del t square or equal to x double dot ok. So, this equation number 80 gives you the current acceleration of a material particle which was located at material coordinate x. So, with this now let us take one example ok.

So, consider the motion remember now we have a time dependent motion, so time now comes into the picture. So, some of the previous examples that we did there was no component of time there was no t involved, but here in this motion you see the spatial coordinates are related to their material counterparts ok and also there is a factor of time.

Now, you have been asked to determine the velocity and acceleration ok. So, velocity is nothing, but del x by del t and acceleration is nothing but del square x by del t square. And both of these have to be computed for fixed value of x which means; you have to keep the material coordinates X 1, X 2, X 3 fixed during the differentiation.

So, v 1 what will be v 1? v 1 will be del x 1 by del t ok. So, from first relation if you take del x 1 by delta t you will get material coordinate capital X 1. Then v 2 will be del x 2 by del t ok. So, you will get twice of 1 plus t material coordinate x 2 ok.

Similarly, the third component of the velocity v 3 will be del x 3 by del t which is equal to 2 t times material coordinate X 3 ok. So, these are the so you will see you have material coordinates on the right hand side which means these correspond to the current velocity of the material particle which was originally located at x.

Now, let us see how to compute the acceleration. Acceleration you can take the time derivative of velocity or you can take doubled time derivative of the motion itself ok. So, this can also be del v by del t, but remember here it has to be in terms of material coordinates ok. So, if you do this you will get a 1 equal to 0 because you see there is no time derivative I mean there is no time component here a 2 and the second component will be twice of x 2 and third component of acceleration will be twice of x 3 ok.

So, in this way you can get the acceleration and velocity of a material particle x at time t. So, you can get the current position of that material particle by this equation of motion and you can get its velocity and acceleration by taking the derivative of the motion with respect to time keeping the material coordinate x fixed ok; with this we now move to our very important topic of material time derivative.

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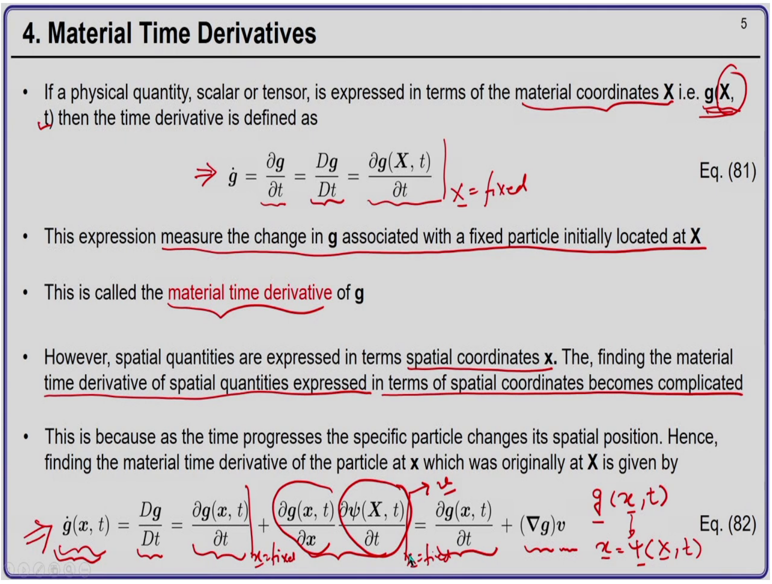

So, when a physical quantity ok; so any physical quantity for example, scalar or a tensor like temperature, velocity, stress, when they are expressed in terms of material coordinates X; that is g equal to X comma t ok. So, g can be scalar, g can be vector, g can be tensor, or any other higher order tensor.

Then the time derivative of this physical quantity g is written as g dot is equal to del g by del t or I said in the previous slide that material time derivative is conventionally in many books you will find they also use this capital D. So, del g by del t Dg by Dt equal to del g by del t ok. So, this is for fixed value of material coordinate.

So, if you are given a physical quantity in terms of material coordinates and time its very easy to compute the time derivative because the material coordinates are fixed you known their location, so its fix it will not going to change over time. So, this expression what it does, is it measures the change in the physical quantity g associated with a fixed particle which is initially located at material coordinate X?

Now, this equation number 81 is called the material time derivative of the physical quantity g. Now, what happens? There are spatial quantities which are expressed in terms of spatial coordinates x. Now, you know as a motion progresses the spatial coordinate of a material particle x changes over time.

So, then finding the material time derivative of the spatial quantities expressed in terms of spatial coordinates becomes complicated; why? If you have g in terms of spatial coordinate x and you have in terms of time. Now, if have to take the derivative of that spatial quantity in terms of time, then not only the time changes, but also the spatial coordinate x is changing with time ok.

So, in our previous expressions the material coordinates X here they were not changing with time you know knew their position the initial position. So, they were not changing with time; what it means is now we have to carry out material time derivative of physical quantities g which are expressed in terms of spatial coordinates a little differently ok.

And this you can carry out using equation number 82. So, I am not going into the derivation of it you can see from any textbook; how to carry out these kind of differentiation. So, what this means is you have been given a physical quantity g in terms of its spatial coordinates x and time t. This spatial coordinate x itself you would see is a function of time its a function of material coordinates and its a function of time.

So, now to take the material time derivative of this spatial quantity g this is del g by del Dg by Dt you can do by taking del g by del t. So, which means you keep the spatial coordinate fixed plus what is called a convictive term which is taking the gradient of the physical quantity times the derivative of the motion with respect to the time ok.

So, in short I can write the material derivative time derivative of a spatial quantity g is nothing, but the derivative of the spatial quantity with respect to time keeping the spatial coordinate fixed plus the gradient of g times the velocity. You can recognize that this is nothing, but velocity ok; because its here that x is fixed ok; that means, metal coordinates are fixed.

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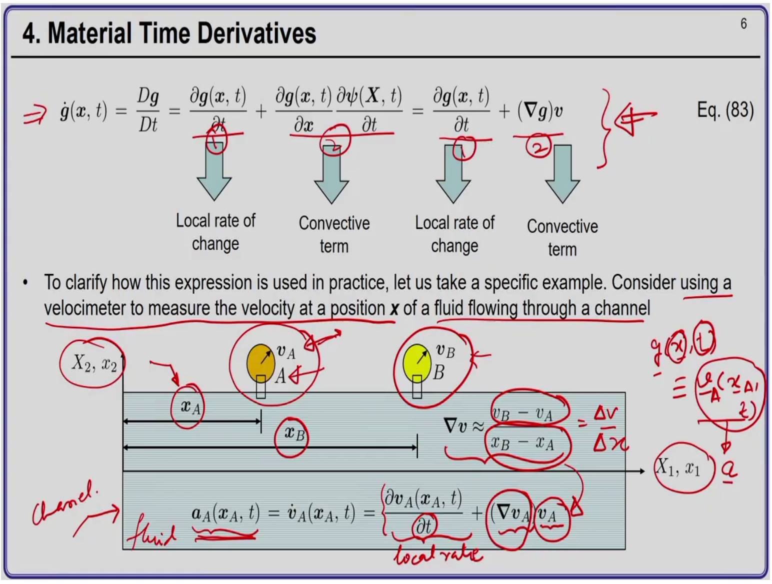

So, this equation I am reproducing here again. So, it has two terms ok. So, the material time derivative of a spatial quantity has two terms you can see the this is the first term and this is the. So, this is the first term and this is the second term ok.

So, the first term is what is called the local rate of change is the rate of change of the physical quantity currently at x ok. So, locally how its changing, now the second term is called the convective term because its like a term which is convicting with the spatial quantity.

Now, let us see to clarify this further let us see one example. So, this specific example you see here you consider a velocity meter to measure velocity at a position x of a fluid flowing through a channel. So, this blue box that you are seeing is a channel ok. And let us see this blue thing that is there is the fluid which is flowing through this channel let us fix our coordinate system ok.

Then to measure the velocity of the fluid which is flowing through this channel what you have to do? You have to put some velocity meter; velocity meters are instruments that are used to measure velocity. So, let us say you wanted to measure the fluid velocity at position A whose coordinate is given by x A. So, let us see this is your velocity meter and now somebody ask you what is the acceleration of the material particle which is currently passing through position A ok.

So, I am not saying what is the acceleration at position A; what I am saying is how do you find acceleration of a material particle which is currently passing A ok. So, its a little different I am not fixing the position x, but I am asking about the particle which is passing currently at x which is currently there at x.

So, now the velocity meter that you have ok; the velocity meter that you have here will measure the local velocity ok. This velocity meter does not measure the velocity of the particle it actually measures only the velocity at location A which means for a fixed spatial position x it is measuring the velocity ok.

So, at the time progresses different particles will come and occupy that particular position A. And when you see that velocity meter readings what you are seeing is basically velocity of different particles which are passing that position with time ok. But now we are interested in the acceleration of the particle which is currently passing that location A ok.

So, the acceleration which I have written here now you have to you only have the local information ok. So, g here x comma t for you here is your velocity x A comma t let me write v a ok. So, you only have the local measure of velocity ok.

Now, from this you want to find out the acceleration of the particle at A. Now, this local information is velocity and then to find the acceleration of the particle that is passing at A; you have to take the local rate of change of velocity ok, plus the gradient of the velocity at A times the velocity at A.

So, the local rate of change of the velocity can be obtained by this velocity meter its always possible in real life application you can say after say you want to find out the rate of change of velocity. So, say you fix 1 second as your delta t. So, you see how much the velocity at A has changed that will give you approximation of the rate of change ok.

So, you see the velocity meter you note the reading at t 1 and then after delta t you re measure the velocity and then you get the velocity change rate of change of velocity. Now, the problem occurs and you also at time t we will know the velocity at A that is the direct reading of the velocity meter. The problem is how to find out the gradient of the velocity.

Now, to find out the gradient of the velocity ok; you can fix another velocity meter at position B a little further into the stream let its position be x B ok. And then you can approximate the velocity gradient at a as the difference in the velocity measurement of the two velocity meters divided by the difference of the positions ok. So, this is nothing, but delta v by delta x ok.

Now the closer you put this velocity meter to position A the more accurate you can get the measure of the gradient of the velocity. Now, if you have these quantities you can compute the acceleration of the particle that is passing current position A ok. So, what equation 83 shows and this example shows that it is perfectly possible to compute the material time derivative using only the current information; information at the current location which is in terms of the spatial position x and time t.

So, this expression helps you to find out the material time derivative of a spatial quantity using only the spatial information ok. So, you can get the material information using the spatial information ok.

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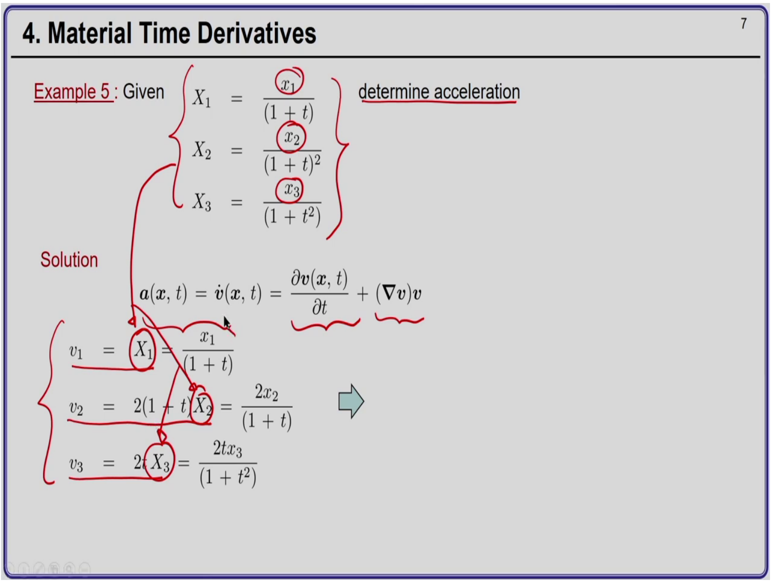

Now, let us see one example; we take example number 4 and we invert the motion ok. So, we had that motion spatial position in terms of material position and time. Now, what we do? We write the material position in terms of the spatial and the spatial position and the time and now we wish to determine the acceleration.

Now, previously in example 4 acceleration was directly obtained because the material coordinate x was fixed. Now, we have been given the information in terms of spatial position and we want to find out the acceleration. So, we can write acceleration as the local rate of change of velocity plus the gradient of velocity times the velocity itself.

Now, the velocity we have found out in an example number 4 as v 1 is equal to X 1, v 2 equal to 2 1 plus t X 2 and v 3 is 2 t X 3. Now, we can substitute for X 1, X 2 and X 3 from our motion here we can substitute from our motion and then we can get the velocities ok. So, once we do this we can get our velocities in terms of the spatial coordinates ok. So, that is the velocity that we have in terms of spatial coordinates.

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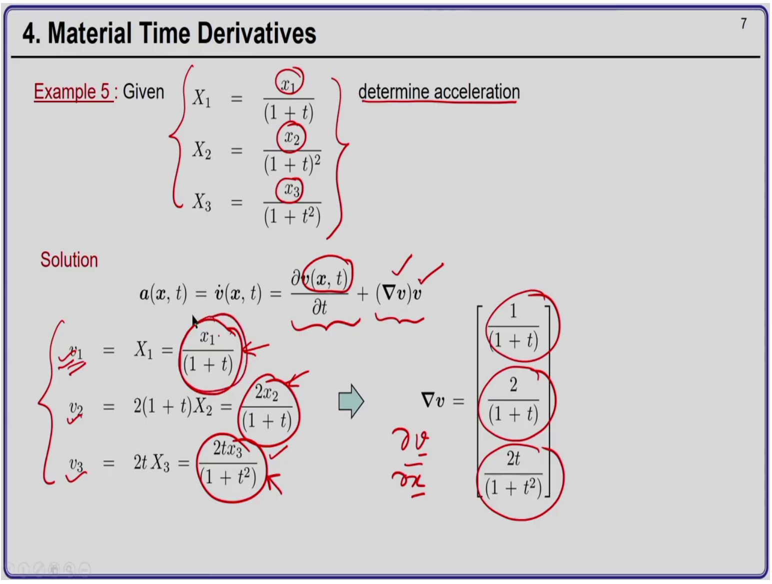

And now; so you see here we have to have the velocity in terms of spatial coordinates and these are the velocity in terms of spatial coordinate. Now, the gradient of velocity which means del v by del x you can compute ok. So, del v by del x you can compute and because v 1 only depends on x 1 v 2 only depends on x 2 v 3 only depends on x 3 we get del v 1 by del x 1 as 1 upon 1 plus t del v 2 by del x 2 will be 2 by 1 plus t and del v 3 by del x 3 will be 2 t by 1 plus t square.

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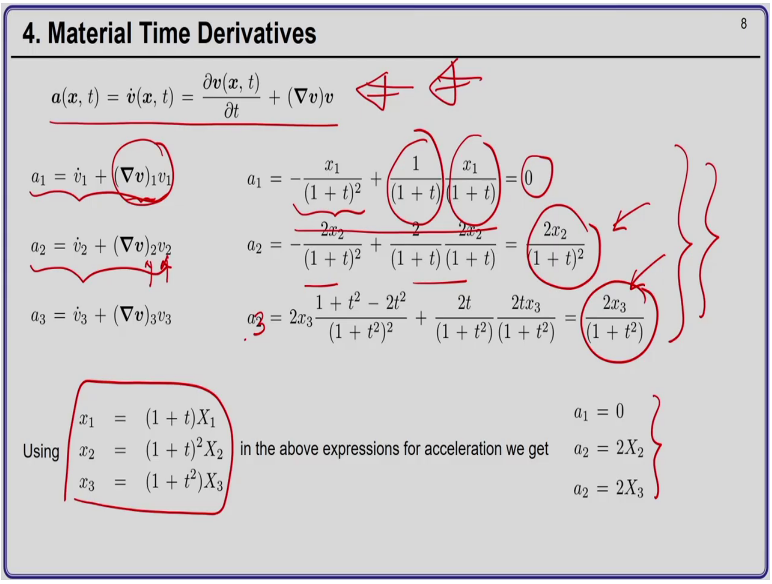

So, now we have the velocity, we have the gradient of the velocity. And now we can compute the first component of the acceleration as v 1 dot ok. So, this vector equation we can write in component form. So, the first equation will be a 1 equal to v 1 dot plus the first component of the gradient times v 1 ok. This actually will be a del v 1; del v 1 v 1 plus del v 2 v 2 plus del v 3 v 3, but because the other two terms of 0 we only have del v 1 into v 1 ok.

You can take the derivative of this quantity with respect to time. So, it will be minus x 1 upon 1 plus t the whole square and the first component del v 1 the first component of the gradient is 1 upon 1 plus t. So, that is 1 upon 1 plus t and this is the first component of the velocity and when you simplify this you will get 0. So, you remember our acceleration when we computed in example 4 was also the first component was equal to 0.

Similarly, I leave it to you to compute the other components of the acceleration ok. So, A 2 will be v 2 dot del v second component into v 2 and everything is known you substitute and you will get 2 x 2 by 1 plus t the whole square. Similarly, the third component you will get sorry this is a 3 the third component will be 2 x 3 upon 1 plus t square.

Now, what you can do check whether this solution is correct; what you can do? These spatial position x 2 x 3 you can write in terms of material positions capital X 1, capital X 2, capital X 3 in terms of this motion. So, this motion is given to you. So, if you substitute this here in this expression you will get a 1 as 0 a 2 as 2 X 2 and a 2 as 2 X 3 which is exactly what we obtained in example number 4 ok. So, you see this expression for the material time derivative and one which was originally there are one and the same.

In the first case what we discuss? The spatial quantity or the physical quantity was given in terms of material coordinates and time t and in the second case the physical quantity was given in terms of spatial coordinates and time t. And then to find out the material time derivative you have to see whether your physical quantity is given in terms of material coordinates or spatial coordinates in.

If it is given in terms of material coordinate you just simply take derivative with respect to time. If the physical quantity is given in terms of spatial coordinates, then you have to use this particular expression ok. So, this particular expression you have to apply and you have to get the material time derivative ok.

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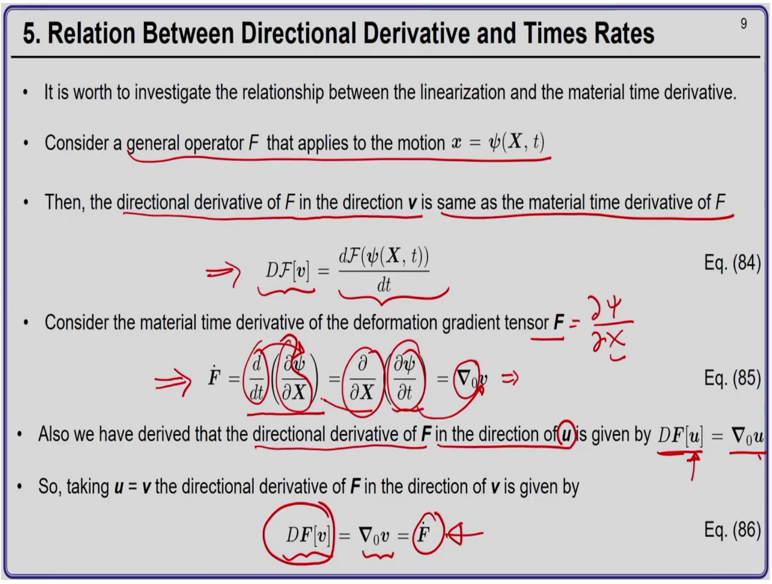

Now, there is a definite relation which exists between the directional derivative and the material time derivative ok. So, consider you have a general operator F that applies to the motion given by x equal to psi of X comma t.

So, you can show and this proof I am not doing in this particular course because its not part of this applied course. You can show that the directional derivative of the general operator F in the direction v is same as the material time derivative of the general operator F. So, which means that the directional derivative of F in the direction v is same as the; material time derivative of F.

So, we will take show this using one example which we take as the material time derivative of the deformation gradient tensor F ok. So, the material time derivative of F is F dot which is nothing, but d by dt of del psi by del x F is nothing, but del psi by del X ok.

Now, I can move del psi by del x outside the bracket and I can move d by dt inside. So, I can rewrite this expression as del by del X into del psi by del t and now psi is given in terms of material coordinates. So, I can directly take its time derivative and this is nothing, but your velocity v and del by del X is nothing, but del 0 ok. So, the deformation gradient tensor F which is given by F dot is nothing but del 0 v or the gradient of velocity with respect to material coordinates.

Now, let us see the other way around. We have derived in our previous lectures the directional derivative of the deformation gradient tensor in the direction of displacement u and it was given by directional derivative of F in the direction u is nothing, but the gradient of u with respect to the material coordinate del 0 u.

Now, I can take the directional derivative of the deformation gradient tensor instead of taking in the direction of u I can also take in the direction of velocity; because velocity is also a vector u was also vector. So, instead of u now if I take in terms of velocity what do I get? I can simply replace u by v and then the directional derivative of F in the direction of velocity will be nothing but gradient of velocity with respect to material coordinates and this from equation 85 you can see is nothing but F dot ok.

So, what we have shown here is that the directional derivative of an operator F here the deformation gradient tensor taken in the direction of velocity is nothing but the material time derivative of that operator here deformation gradient tensor.

So, next we move to our next topic.