Computational Continuum Mechanics
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Kinetics - 1
Lecture - 15-17
Cauchy stress tensor, Equilibrium equations, Principle of virtual work

So, now, we will discuss the second part of equilibrium equation that is the balance of angular momentum, and finally, we will see the statement of principle of virtual work ok. So, last class we were discussing the translational equilibrium. And the translational equilibrium gave us the equation of motion for dynamic bodies or equilibrium equation for static problems ok.

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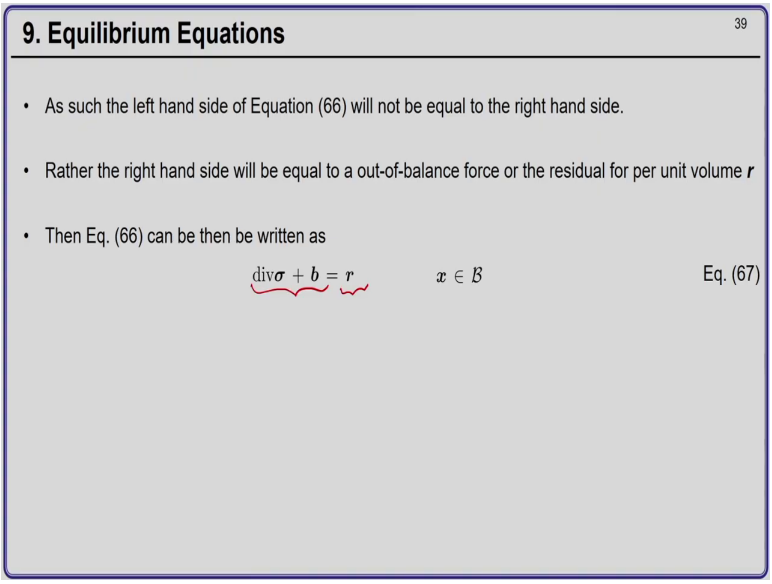

So, last class we stopped at equation 67 ok, where we said that in the context of a non-linear finite element method, the equilibrium equation will not be satisfied exactly. So, there will be an out of balance force r which is per unit volume ok. And the objective of the finite element method formulation is to reduce this out of balance force r ok.

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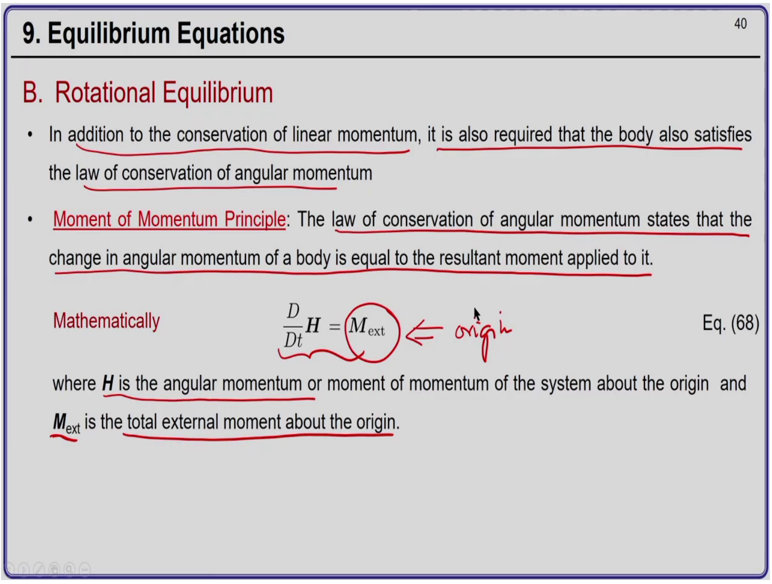

So, next we move to what is called the rotational equilibrium or the law of conservation of angular momentum ok. So, in addition to the conservation of linear momentum, it is also required that the body satisfies the law of conservation of angular momentum ok. So, first we will see what is the statement of law of conservation of angular momentum which is also called the moment of momentum principle.

So, the law of conservation of angular momentum states that the change in angular momentum of a body is equal to the resultant moment applied to it ok. Now, mathematically this can be expressed as the rate of change of angular momentum H should be equal to the net external moment acting over the body ok. Here the symbol H is a standard symbol used for angular momentum ok. And M subscript ext which is short for external; so external moment is denoted by M subscript ext and this is the total external moment and let us say it is about the origin. So, this equation we can say is about the origin ok.

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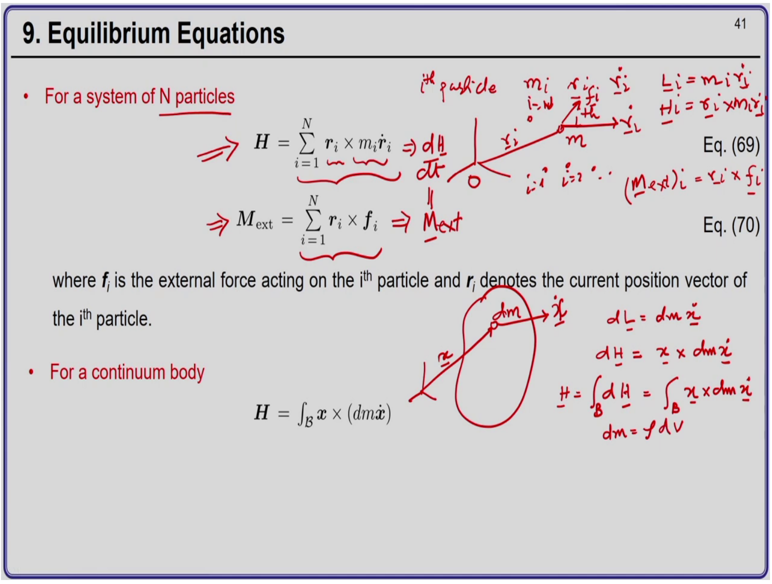

Now, before we move to continuum bodies, let us say first we discuss how this principle is applied to a system of particles. Let us say you have a system of N rigid body particles ok. So, the angular momentum of the system of particles will be equal to the summation over the angular momentum of all the particles ok.

So, let us say you have a ith particle ok, and let us say its mass is m i and its current position vector is r i ok. So, you have a particular coordinate system you have a mass you have a i th mass let us say this position vector is r i ok. And let us say r i dot is its current velocity ok. So, r i dot is its current velocity. So, the linear momentum will be m i r i dot ok. So, the angular momentum of this particle will be r i cross m i r i dot ok. So, the angular momentum for the i th particle will be r i cross m r i dot ok.

Now, if you have n such particles, so i equal to 1, i equal to 2, all the way up to say i equal to n, you have n such particles, then the total angular momentum for the system of n particles will be summation over the angular momentum for each particle ok. Now, let us say the particle is being acted upon by a force net external force f i ok. So, the moment of this force. So, the moment let us say the external moment on the i th particle will be r i cross f i ok. So, the total external moment will be equal to the summation of the external moment on all the particles ok.

Now, from the law of conservation of angular momentum, the rate of change of the total angular momentum will be equal to the net external moment ok. So, now, if you increase your number of particles to infinity, that means, you are going to a continuum body ok, in that case rather than taking one particle of mass m i we take ok. So, in the body we take a small chunk of mass dm, and let us say its current position vector time t is x, and then let us say its velocity is x dot ok. So, the linear momentum will be dm into x dot, so mass into velocity.

So, the external angular momentum, so the or the angular momentum will be x cross dm x dot. So, the total angular momentum will be equal to the sum of the angular momentum all such masses integrated over the volume of the body, and that will be equal to x cross dm x dot ok. Now, remember dm will be rho time dV rho times dV, where rho is the current density ok.

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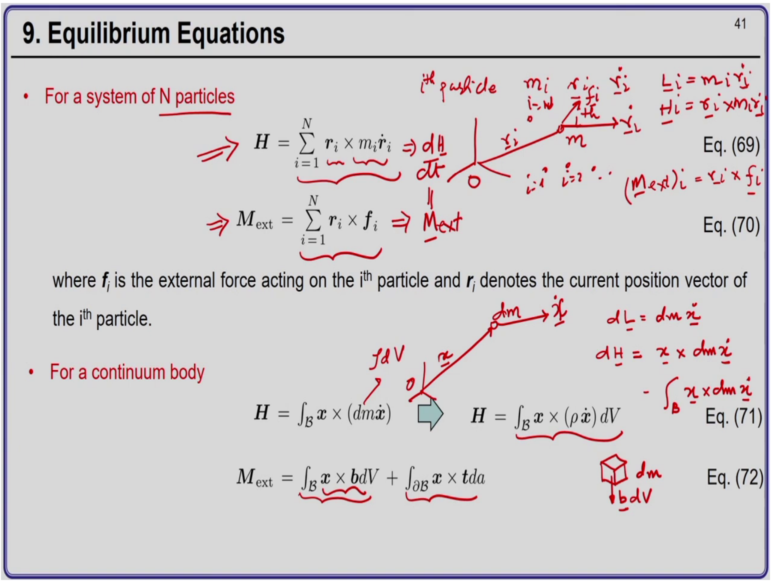

So, that is what that is what we have written here. If you replace dm by rho dV, we get x cross rho x dot dV ok. And the net external moment now becomes a little peculiar ok. So, a continuum body as we have discussed is acted upon by both the body forces and the surface tractions externally applied surface traction. So, if you have a chunk of mass dm and the body force per unit volume is say b ok, then the total body force will be rho dV, and the moment about the origin of this body force will be x cross b dV. And the total external moment because of the body force will be integration over the volume of all such body forces. Similarly, we can show that the moment about the origin of the externally applied tractions will be x cross t da that is the integral carried out over the physical surface of the body ok.

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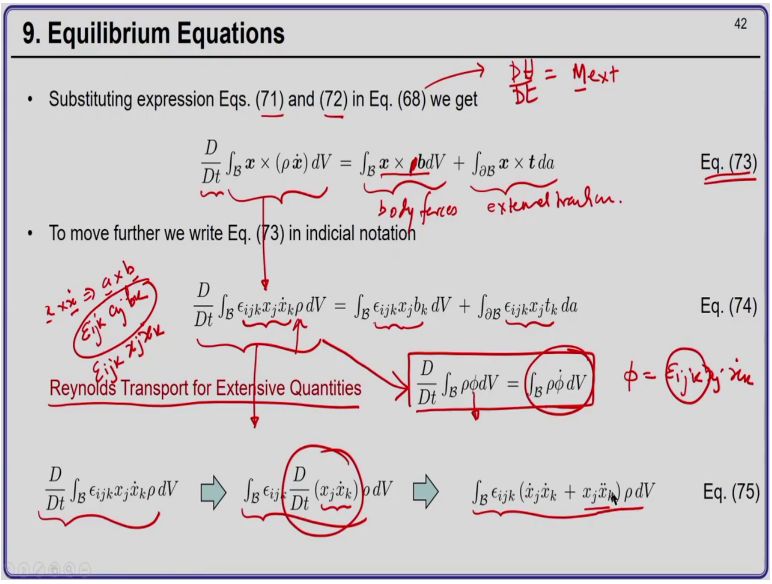

Now, if we substitute equation 71 and 72 which we had in the previous slide into the statement for conservation of angular momentum which is the material time derivative of the angular momentum that is rate of change of angular momentum should be equal to the net external moment acting over the body. So, what we get the material time derivative of the angular momentum equal to the moment of the body forces plus moment of the external tractions ok. So, this is external traction contribution ok. Now, to move further what we do is now we write equation number 73 in indicial notation ok.

So, I hope my this time you have revised or you have learnt your indicial notation and you are little familiar now. So, let us consider the first term over here ok. On the left hand side, under the integral, so we have x cross rho x dot rho being a scalar quantity can be taken outside and then you have x cross x dot ok, so which is like a cross b, where a and b are two vectors. So, in indicial rotation that will be epsilon i j k a j b k ok. And the i th component, so this is the i th component. So, our a here is x ok. So, we have epsilon i j k x j and b vector our here is x dot. So, we will have x dot k. So, that is what we have here.

Then similarly if we consider the terms on the right hand side, we can write x cross rho b as epsilon i j k, x j b k sorry. So, this rho will not be here because it was ok, and x cross t will be epsilon i j k x j t k ok. Now, we come to Reynolds transport theorem which we earlier derived for extensive-quantities. Extensive-quantities, which are dependent upon mass ok. So this over here. So, this was the Reynolds transport theorem. So, phi was a extensive quantity ok. So, the material time derivative of the integral over current volume rho phi dV is nothing, but integral over current volume rho phi dot dV ok.

So, if we compare here equation 74, our phi here is epsilon i j k x j x k dot ok. So, the left hand side expression is same as the left hand side of this equation. Therefore, this equation can be written using the right hand side of Reynolds transport theorem therefore, d by d t of epsilon i j k x j x k dot rho dV can be written as integral over current volume epsilon i j k the material time derivative of x j x k dot ok. And you can take out epsilon i j k because that is a scalar quantity. So, we just have to take the material time derivative of x j x k dot ok. So, this we can show that this is equal to epsilon i j k x j dot x k dot plus x j x k double dot rho dV ok.

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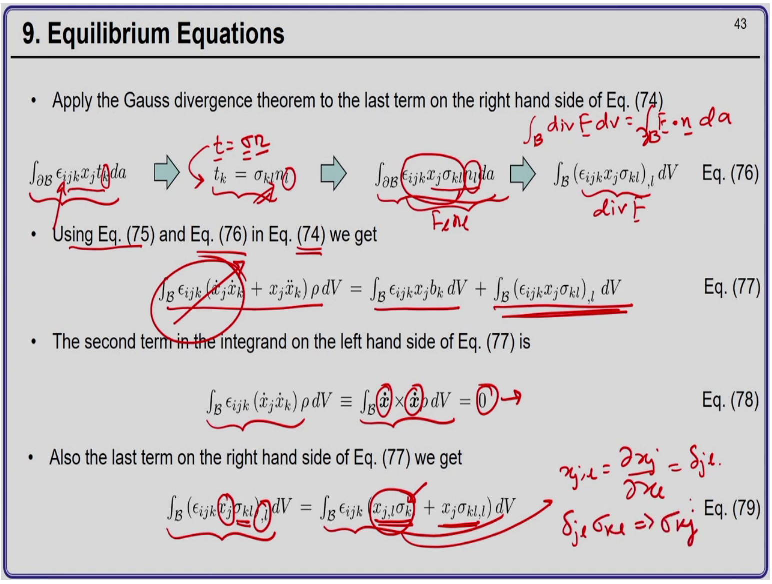

Then we apply the Gauss divergence theorem on the last term on the right hand side of equation 74. So, what is the last term on the right hand side, it is integration over the current surface epsilon i j k x j t k d a. So, this is the integration of the moment of the externally applied tractions ok. Now, I know that traction t is equal to sigma n. So, in indicial notation this is t k sigma k l n l. Why I have used k? Because I have k here ok. And because i is a free index, I cannot have i here, and also j and k are repeated twice. So, I cannot have j and k. So, I choose l ok. So, I have chosen l here. So, this when I substitute, I get integration over the surface epsilon i j k x j sigma k l n l d a ok.

And this becomes after you apply the gauss divergence theorem ok, so this and you have n l, so integration over the volume current volume divergence of f d v equal to f dot n da ok. So, our f dot n ok, so this is like f l n l ok. So, this is like divergence of F ok, this like divergence of F ok.

So, now, when we substitute equation 75 and 76 in equation number 74 what we get? This is the right hand side that we derived, there is no change in the body force term and then we have the following surface term which we have converted to the with we have converted to the volume integral using Gauss divergence theorem ok.

Now, let us look into the this term ok. If we look into this term, what we see it is epsilon i j k x j dot x k dot rho d v. Now, if I convert this to direct notation, it will be integrated over volume x dot cross x dot d V. And now because these are the same vectors, so the cross product is 0, because sin 0 is 0 ok, a cross b where a is equal to b is 0. So, this term goes to 0 ok.

So, what we get ok, and let us come to the last term over here. So, the last term now I can take the derivative with respect to x l ok. So, this comma x l actually means this is derivative with respect to x l ok. So, i get epsilon i j k ok. So, first x j coma l sigma k l x j comma l sigma k l plus x j into sigma k l comma l ok.

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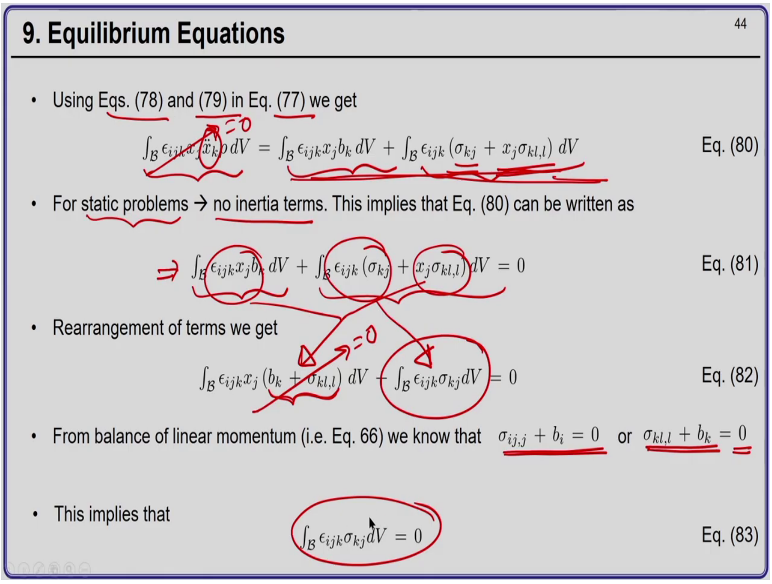

Now, you notice let me go back to you notice that this term x j comma l is if you write fully it will be delta x l. And if you recall chronicle delta then this is equal to delta j l. So, we have delta j l, and then we have sigma k l which gives me sigma k j ok. So, I can write this term as sigma k j and then I have. So, if I use equation 78, 79 in equations 77, I get the term on the left hand side equal to this term on the these two terms on the right hand side ok.

Now, let us consider a special case of static problem. Static problem means there are no inertia terms ok, you see you have acceleration here. So, in static problem the contribution of the term on the left hand side can be neglected, because it will be very small as compared to the two contributions from the right hand side. Therefore, this term goes to 0. And I known; I am only left with these two terms on the left hand side. So, that I will write again. So, this is the contributional body forces and the contribution of the external traction and that should be equal to 0 ok.

So, you see, I have this term epsilon i j k x j x sigma k l l, and I have this term. So, these two terms I can write in one bracket, I can take them together. And this term over here I can write as a separate integral ok. Now, you can realize that this particular quantity in the bracket is nothing but your statement of local statement of law of conservation of linear momentum that is your equilibrium equation ok. So, we had sigma i j comma j plus b i equal to 0 from our equation 66, therefore, or if I just change the indices sigma k l comma l plus b k equal to 0. So, this term is equal to 0. So, what I am left with now is only the second integral ok. So, this implies that integral over the current volume of epsilon i j k sigma k j integrated over the current volume is equal to 0 ok.

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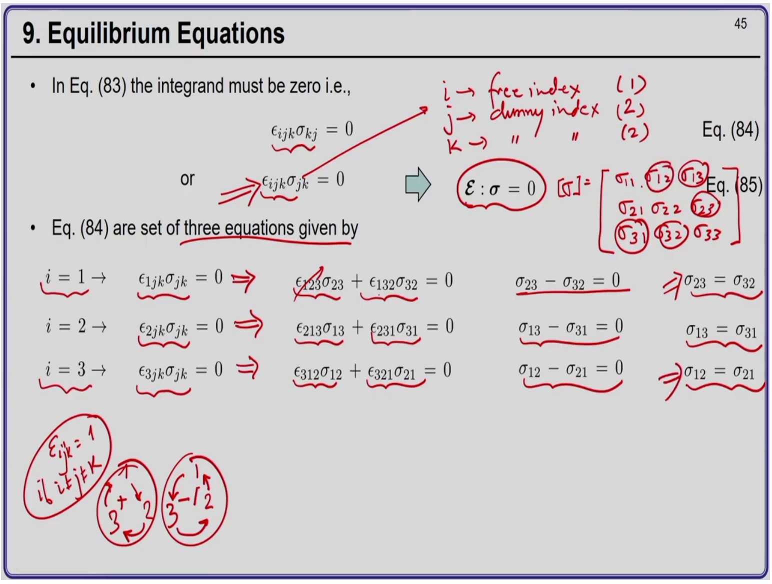

And because our volume is arbitrary; we do not know this ok, it can be anything. Therefore, for this identity to hold ok; for this condition to hold the integrand in equation 83 has to be 0, which means epsilon i j k sigma k j should be equal to 0 ok; or I can write epsilon j k sigma j k equal do 0 ok, which implies indirect notation. Epsilon does alternator symbol, the third order tensor contracted with the second order tensor sigma should be equal to 0.

Now, let us see what is the implication of this statement; because there is only one free index here ok, so if you see this; i is a free index, j is a dummy or a repeated index, because it is occurs twice; i occurs once and k is a dummy index, because it occurs twice. So, actually equation 85 are in total three equations and they are given by when i is equal to 1, you have epsilon 1 j k sigma j k is equal to 0; when i equal to 2, epsilon 2 j k sigma j k equal to 0; and i equal to 3, epsilon 3 j k sigma j k equal to 0.

And now, from our that four rules that we discuss for indicial notation, we understand that because j and k are repeated here; the summation over j and k is implied, and also you have to remember that epsilon i j k equal to 1; if i not equal to j, not equal to k ok. And when i j k occur in this particular order, then epsilon i j k is 1; and if i j k happens to be in this particular order, then epsilon i j k is minus 1, ok. So, let us write, so what this actually means this statement over here means that for epsilon 1 j k to be non-zero, j should not be equal to k ok. So, j and k will take all values from 1 to 3 and only two set of j and k which is when j equal to 2 and k equal to 3; and when j equal to 3, k equal to 2 will be non-zero, and all other sets or combination of i of j k will lead to a 0 value of epsilon 1 j k, ok.

So, what we will have epsilon 1 2 3 sigma 2 3 plus epsilon 1 3 2 sigma 3 2 should be equal to 0. The second equation implies epsilon 2 1 3 sigma 1 3 plus epsilon 2 3 1 sigma 3 1 should be equal to 0. And epsilon 3 1 2 sigma 1 2 and plus epsilon 3 2 1 sigma 2 1 should be equal to 0. Now, epsilon 1 2 3 will be equal to plus 1 and epsilon 1 3 2, 1 3 2 will be equal to minus 1; so what this implies, sigma 2 3 minus sigma 3 2 equal to 0.

Similarly, the other two conditions implies sigma 1 3 minus sigma 3 1 equal to 0, and sigma 1 2 minus sigma 2 1 equal to 0. What this actually implies is that sigma 2 3 equal to sigma 3 2;

sigma 1 3 equal to sigma 3 1; and sigma 1 2 is equal to sigma 2 1. And now remember, sigma the Cauchy stress can be written as a 3 by 3 matrix like this ok, let me write sigma 3 2 and sigma 3 3. Now, what we have sigma 2 3 is sigma 3 2, so 2 3 is 3 2; sigma 1 3 is sigma 3 1; and sigma 1 2 is sigma 2 1 ok; what this implies finally, let me rub this.

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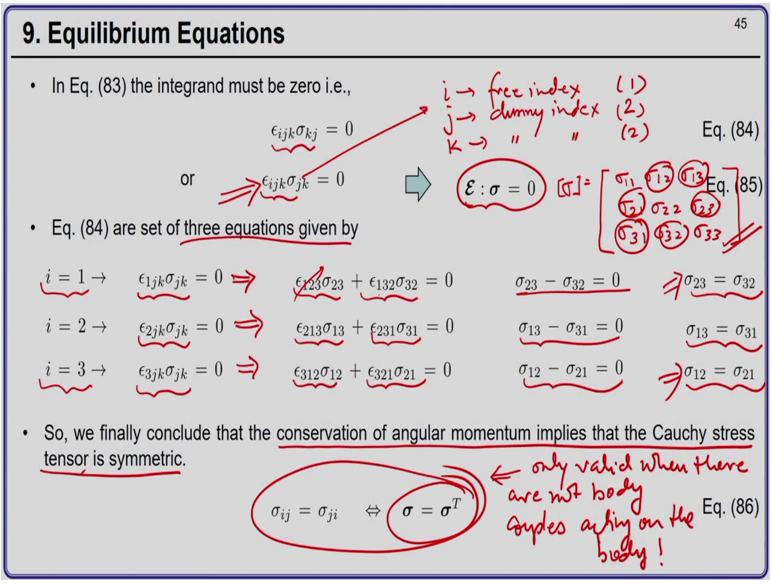

So, what this implies that the conservation of angular momentum implies that the Cauchy stress tensor is a symmetric tensor, that sigma i j is equal to sigma j i which implies sigma is equal to sigma transpose. Remember, this is only valid when you do not have ok, I will write here only valid when there are no body couples ok, acting on the body. So, if there are external couples which act on the body, then Cauchy stress will not be symmetric; it is only when, there are no body couples that the Cauchy stress tensor is a symmetric tensor ok.

So, the implication of conservation of angular momentum is that in the absence of body couples, the Cauchy stress tensor will be a symmetric tensor. And this has huge implication, later on in constitute relation; where we can use this condition to less an our storage space in the computer. Instead of storing the 9 components of the stress tensor, I can just store the 6 components and I can save a lot of computer memory,.

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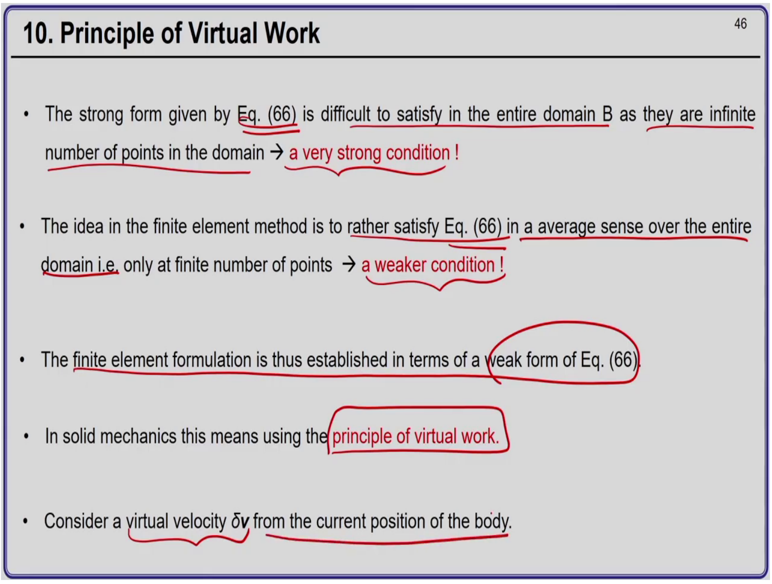

Now, coming to the last topic which is principle of virtual work ok. So, the equation 66 I said, it is also called the strong form its strong form, because it is very difficult to satisfy this condition over the entire domain of the body ok; as there are infinite number of points, where you will be required to satisfy this equation ok. So, the strong form is for all point 6 which belong to the body B ok; so if you are solving the strong form, then you should get a solution such that the equation given by equation 66 is valid for all the points.

And then this what I can call is a very strong condition ok, there is a other reason for calling it a strong form its come from the finite element, but I will not go there rather I will say, here that this is a very strong condition. And then what we do in finite element is we will rather like to satisfy equation 66 in an average sense over the entire domain ok, so we will rather like to satisfy the differential equation given by 66 only at finite number of points.

So, instead of satisfying the equations at infinite points, I will just like to satisfy those equations at finite points in an average sense; and then this is what is called a weaker condition ok. So, now going from instead of satisfying the equation at infinite point; I only have to satisfy this on an average sense over a finite number of points, and this is a much weaker condition.

And the finite element formulation is thus established in terms of the weak form of equation 66 and in solid mechanics this means using the principle of virtual work ok. So, even though our consideration in this course is static problem, but let us consider a virtual velocity del v ok. So, consider a virtual velocity del v from the current position of the body, so current position is time t ok.

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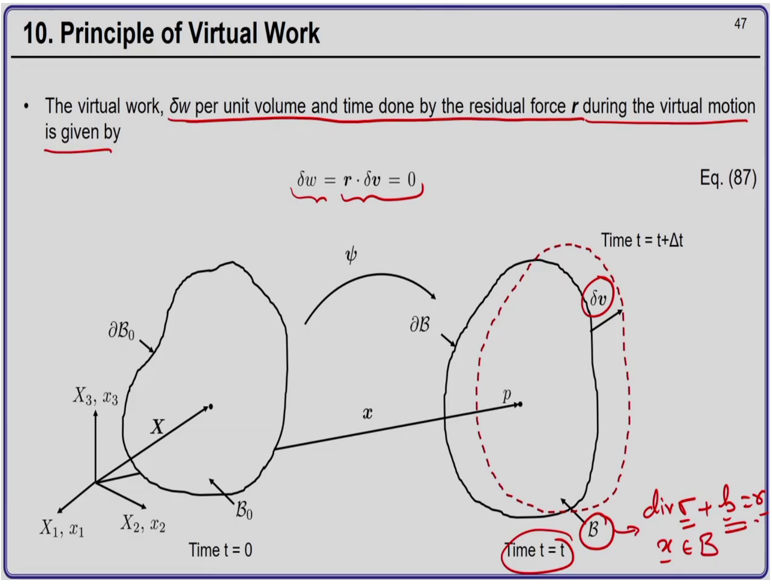

So, we have the body and the current position of the body is at time t and that is where, you want to satisfy your equilibrium equation that is where your divergence of sigma plus b equal to 0, so x belongs to B; so that is where your equilibrium equation has to be satisfied. Now, consider we give a virtual velocity del v to this configuration, it is ok. So, the virtual work d w per unit volume and time done by the residual forces during this virtual motion will be given by del w equal to r dot del v. Remember when you are using finite element, your equilibrium equation will not be satisfied and this divergence of sigma plus b will be equal to r ok. So, now the virtual work done by these residual forces will be r dot del v; so the virtual work per unit volume per unit time will be r dot del v, ok.

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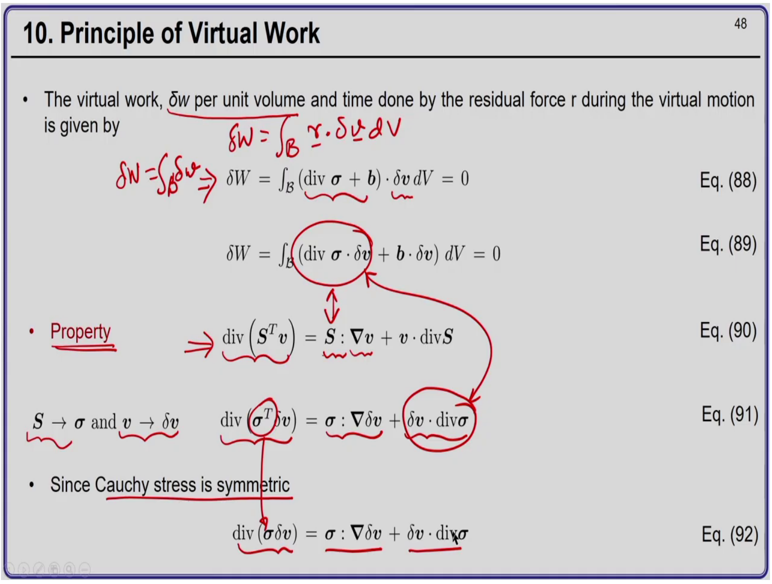

Now, let us say that total virtual work per unit volume per unit time will be nothing but del capital W will be integral over volume del w, and this is given by del W equal to integrated over integration of volume del v dv, ok. Now, r is nothing but divergence of sigma plus b and then you have dot dV virtual velocity integrated over volume dV equal to 0. So, the virtual total virtual work done by the residual forces across the applied virtual velocity del v will be equal to 0.

Now, I can take the dot product with virtual velocity is inside, and I can write divergence of sigma dot del v plus b dot del v equal to 0, ok. Now, I use this property and I leave that as a task for you to show this property ok. This property states that the divergence of a second order tensor, transpose of its second order tensor with a vector v is nothing but the double contraction of the second order tensor with the gradient of the vector plus the vector dotted with the divergence of the second order tensor S.

Now, if I compare this term ok, I can see that S is equal to sigma and v is dot del v if I compare these two expressions. So, divergence of sigma transpose del v will be sigma contracted with gradient of virtual velocities plus the virtual velocity dotted with divergence of sigma ok. So, I can get this relation ok, so this relation these relations are same; therefore and also because Cauchy stresses symmetric this becomes sigma. So, I can write divergence of sigma del v equal to sigma contracted with gradient over virtual velocities plus del v dot divergence of sigma, ok.

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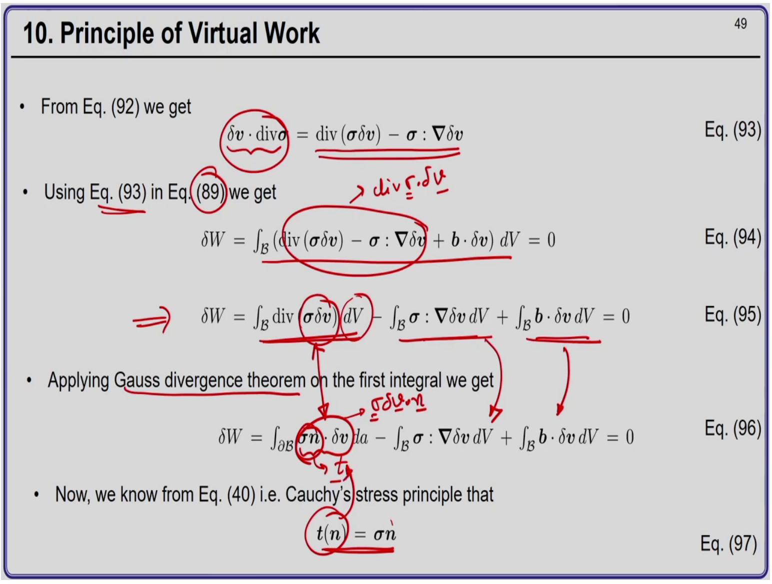

Now, I can get del v dot divergence of sigma as this quantity I can take everything on the right hand side, and now I can substitute this ok; I can use this expression in 89, and I will get this following expression ok. So, this is what was originally my divergence of sigma dot del v,. Now, I what I can do? I can split all the terms ok, so I can take the integral of each individual term and now let focus only on the first term. Now, this is again where I can use gauss divergence theorem ok, so this is integral over the volume divergence of a vector field, integrated over the volume, so divergence of f dV will be f dot n d S ok; so I will have sigma n dot del v.

And now, this is I leave it to you as an exercise to show that you have, so what you will actually get is sigma del v dot n and this is same as sigma n del v ok, this is as a task for you, I will leave. And then you have this second term and the third term, remain as it is ok. And now from the Cauchy stress principle, I know that sigma n is nothing but the externally applied traction ok, see t is equal to sigma n therefore, I can substitute t here in equation 96.

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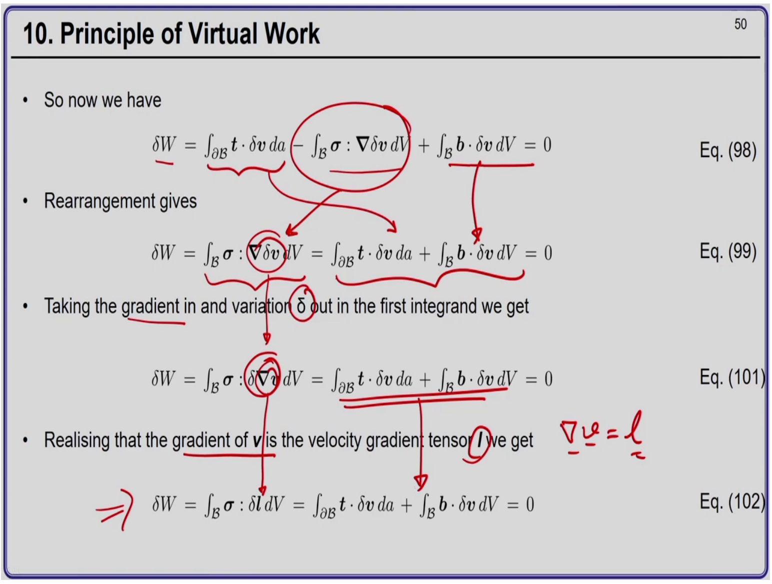

And then what I get is del W is integral over the surface of the virtual work of the externally applied traction minus these two term ok. Now, I focus on I can rearrange, I can bring this term on one side; and I can bring the other two terms on the other side, and the virtual work will be equal to this term or equal to the virtual work of the external body forces and the external tractions, ok.

Now, this I can interchange; the gradient and the variation symbols can be interchange. So, gradient of del v I can write as del of gradient of v, and now I know that gradient of v is nothing but the velocity gradient ok. So, del v is nothing but l ok, so I can substitute l here and this is what I get. So, these two term remain same, they do not change.

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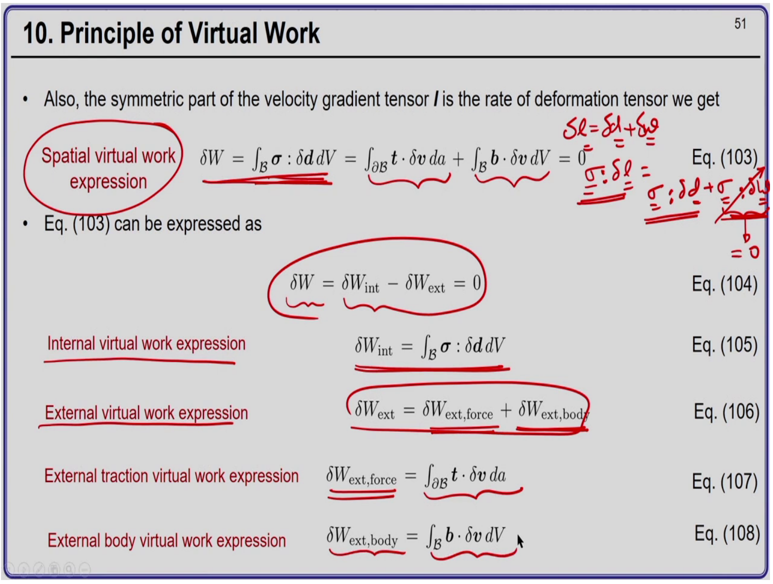

Now, the next thing I notice the velocity gradient can be decomposed as a sum of a symmetric part that is the rate of deformation tensor d and a anti symmetric part that is the spin tensor w, ok. So, l will be equal to d plus w; so del l will be equal to del d plus del w ok. And now, sigma contracted with del l will be sigma contracted with del d plus sigma contracted with del w. Now, you can show and you also would know that the double contraction of a symmetric tensor which is sigma here, and a anti-symmetric tensor which is del w here will be equal to 0, so this term will drop away.

Therefore, sigma contract with del l will be same as sigma contracted with del d and that is what we have it here. And this is called the spatial form of the virtual work expression, which says that the virtual work of the internal stresses should be equal to the virtual work of the externally applied tractions plus the virtual work of the body forces acting over the complete volume of the body ok. So, this equation number 103, I can write del W as del in minus del W external ok; where the internal virtual work expression is given by integration of the double contraction of the Cauchy stress tensor with the virtual rate of deformation tensor.

And the external virtual work is split into two parts; one because of the external tractions and the other because of the body forces, where the virtual work of the external forces will be this and the external body virtual work will be equal to this, equation 107 and 108 ok, so with this we end our this module on kinetics ok. So, we have covered the Cauchy stress principle; we have covered what is meant by Cauchy stress tensor; we have shown how objectivity plays an important role and I have also shown that Cauchy stress tensor is an objective tensor.

And finally, we have derived the equilibrium equation by satisfying the law of conservation of linear momentum, and using the law of conservation of angular momentum we have shown that Cauchy stress is a symmetric tensor ok. And finally, using the principle of virtual work we have derived the necessary equation which will be used later during the finite element formulation, ok. So, those the equation that we derived in the previous slide will be used for the derivation of the finite element matrices, ok. So, with this I end today’s lecture, ok.

Thank you.