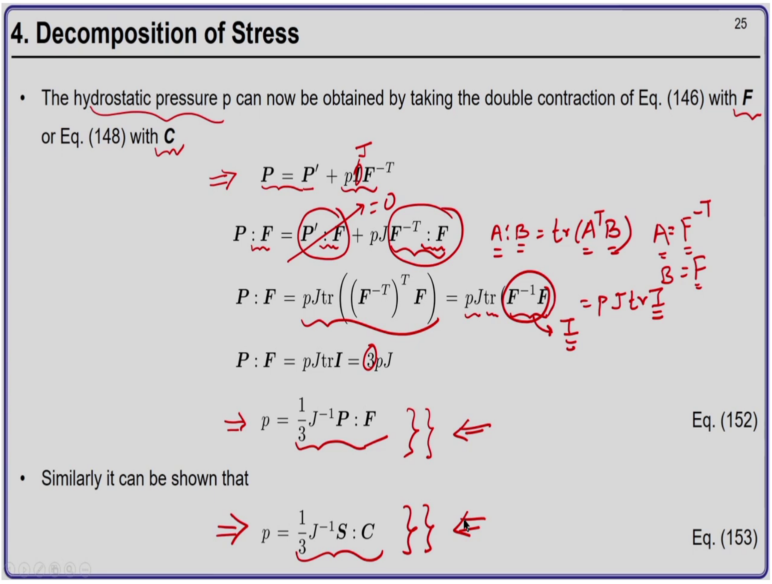
Computational Continuum Mechanics
Dr. Sachin Singh Gautam
Department of Mechanical Engineering
Indian Institute of Technology, Guwahati

Kinetics – 2
Lecture – 18-20
Work conjugacy, Different stress tensors, Stress rates

So, today we will look into finding the pressure from the first Piola-Kirchhoff or the second Piola-Kirchhoff stress tensor. Although, this is not essential for compressible hyper elasticity, however, for incompressible hyper elasticity these expressions will be helpful for the; so, for the sake of completeness we will look into how to determine pressure.

So, we have already seen the true deviatoric stress components of first Piola-Kirchhoff and second Piola-Kirchhoff stress tensor ok.

(Refer Slide Time: 01:21)



So, now to determine the hydrostatic pressure p, we have to start from taking the double contraction. Say for example, of the first Piola-Kirchhoff stress tensor with the deformation gradient tensor or the second Piola-Kirchhoff stress tensor with the right Cauchy Green tensor C ok.

So, the decomposition of the first Piola-Kirchhoff stress tensor is given by P equal to the deviatoric part P dash plus p into IF inverse transverse. So, this should be J. Now, if you take double contraction with the deformation gradient tensor on both the sides ok. So, this is double contraction you have taken from both the sides ok. So, this is what we get.

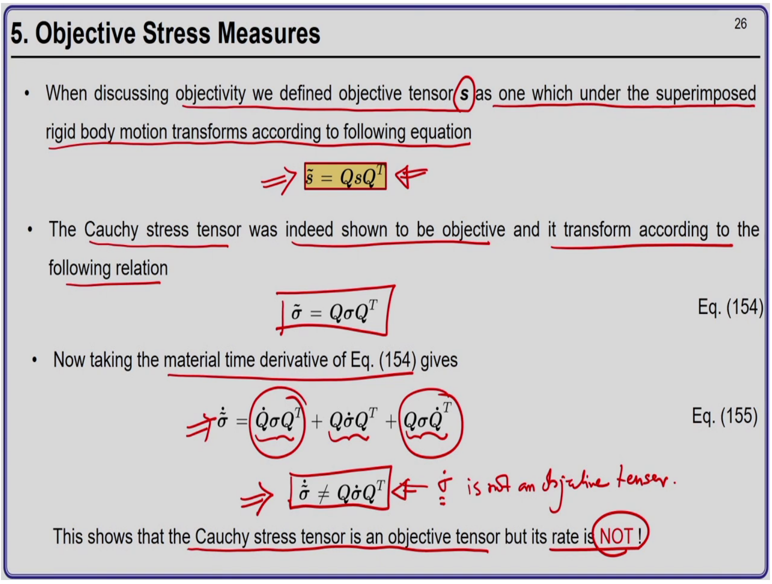
Now, we already have shown that P dash ok. So, the deviatoric part of the first Piola-Kirchhoff stress tensor when double contracted with the deformation gradient tensor is equal to 0 ok. And, then this first term on the right hand side is 0 and if I look into this term this is nothing, but A double contraction with B which is written as trace of A transpose B.

Now, so my A I can identify as F inverse transpose and B as F. So, I can write p J transpose of F inverse transpose transpose F which is nothing but equal to p J trace of F inverse F. Now, what is F inverse F? It is equal to identity and then I get p J trace of identity which is nothing but equal to 3.

So, from here I can determine the pressure as 1 by 3 J inverse the first Piola-Kirchhoff stress tensor double contracted with the deformation gradient tensor F ok. So, similarly one can take the double contraction with the right Cauchy Green tensor of the second Piola-Kirchhoff stress tensor and then you can obtain the hydrostatic pressure p as 1 by 3 J inverse S contracted with C ok. So, that is how you can determine the hydrostatic pressure ok.

So, this is as I said earlier will be helpful when you are dealing with incompressible hyper elasticity ok, but in this course because we consider only compressible hyper elasticity this will not be of much use, but for the sake of completeness we have derived these expressions ok.

(Refer Slide Time: 04:55)



So, next we moved to our important topic of objective stress measures. So, when we are discussing objectivity we had define an objective tensor s as one which under the superimposed rigid body motion transforms according to the following equation. And what is this equation? This is s tilde is equal to Q s Q transpose, where Q is the superimposed rigid body motion.

Now, also we had shown that the Cauchy stress tensor was indeed an objective tensor and then it transformed according to the relation given by equation number 154 which is sigma tilde is Q sigma Q transpose. Now, let us take the derivative of equation 154 with respect to time that is we take material time derivative, let us see what happens ok.

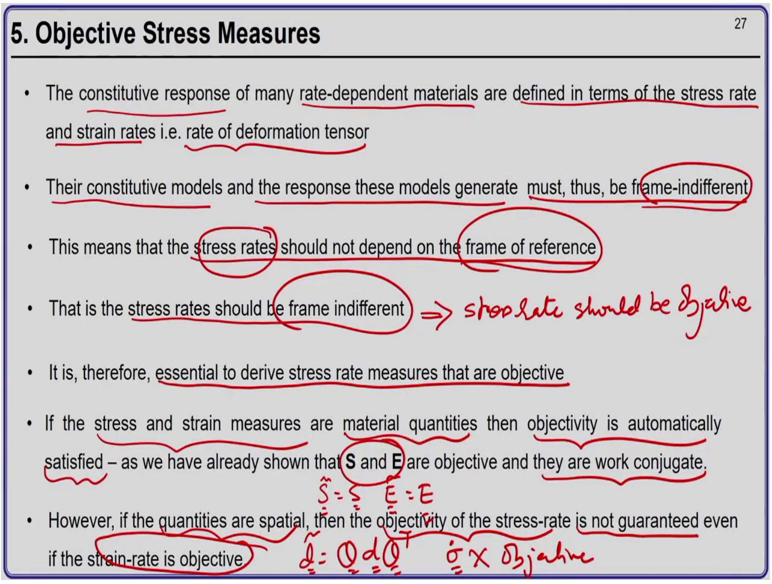
Now, if you take the material time derivative equation 154 on the right hand side we have sigma tilde dot and then we have Q dot sigma Q transpose plus Q sigma dot Q transpose plus

Q sigma Q dot transpose. We can clearly see that this expression is not equal to ok. So, the material time derivative or sigma tilde is not equal to Q sigma dot Q transpose.

So, we have two additional term; this first term and this is the third term. So, because of the presence of these two terms the material rate of the Cauchy stress tensor so, which is sigma dot that is the material rate of the Cauchy stress tensor is not an objective tensor or it is not an objective quantity if it was an indeed an objective quantity it would have transform according to relation which is given here, but did not ok.

So, Cauchy stress tensor is an objective tensor; however, it is rate is not an objective tensor. Now, this has consequence when we are dealing with rate dependent materials ok.

(Refer Slide Time: 07:30)



So, the constitutive response of many rate dependent materials are defined in terms of the stress rate and strain rates, for example, the rate of deformation tensor ok. So, if you have a rate dependent material, its response to the external loads overtime will be express in terms of stress rates and strain rate, for example, rate of deformation tensor ok.

And, we know that Cauchy stress is the true stress. So, its rate we have to use for the constitutive response, but we also have seen from the previous slide that the rate of Cauchy stress is not objective. So, there will be dependents on the material frame. So, if there is superimposed rigid body motion, then there will be some spurious increment in the stresses just because of rigid body motion which should not happen ok.

So, now, we have to deal with this situation ok. So, the constitutive models and the response these model generate must therefore, be frame indifferent ok. So, whatever response we are getting for these rate dependent materials they must be frame indifferent, they should not depend on the frame. So, what to do which means that stress rates should not depend on the frame of reference ok.

So, since the response has to be frame indifferent therefore, the stress rates that we should use should also not depend on the frame of reference or they should also be frame indifferent. That is what I have written here that the stress rates should be frame indifferent ok. So, it is essential. So, when stress rate of frame indifferent this means that that stress rates have to be objective.

So, consequence of this is our stress rates that we use in the constitutive response they should be objective and it is essential that we derive stress rate measures that are objective ok. So, our stress rate that we have seen here the rate of Cauchy stress is not objective. Therefore, we should come up with some stress rates which are objective.

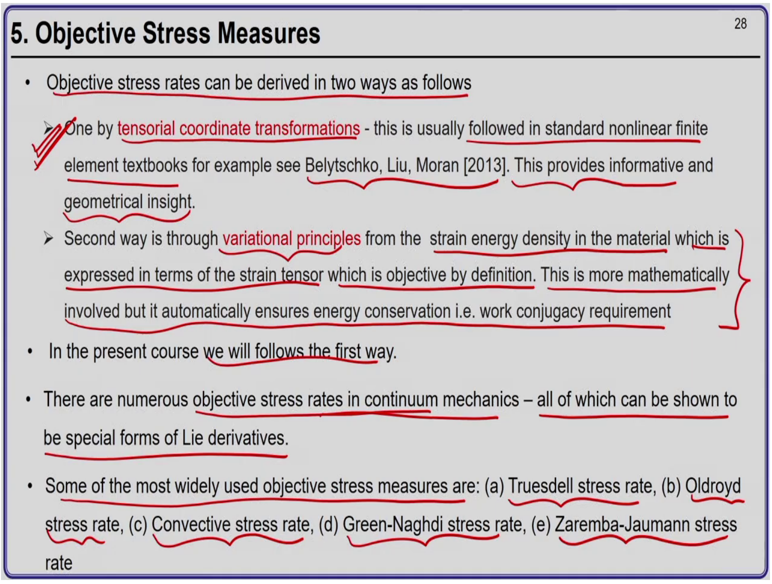
And, important point to note here is that the stress and strain measures are material quantities if your stress and strain measures are material quantities then the objectivity is automatically satisfied that we saw that the second Piola-Kirchhoff stress tensor and the green Lagrange strain tensor S and E were indeed objective ok. So, we had seen that S tilde was equal to S and E tilde came out to be same as E ok.

So, they are indifferent to the rigid body superimposed rigid body motion and then these stress and strain measures will be energy conjugate; however, if the quantities are spatial ok. Now, if the strain and stress measures that you use are spatial quantities like Cauchy stress and the rate of deformation tensor, then the objectivity of the stress rate is not guaranteed even if the strain rate is objective which means for an our case d for example, was an objective quantity ok. So, d tilde was Q d Q transpose ok.

So, the strain rate is objective, but we know that the Cauchy rate of Cauchy stress is not objective and we also know that the Cauchy stress is work conjugate with the rate of deformation tensor.

So, we now have to come up with some stress rate measure, which are objective, so that we do not have any effect or any increment in the stress when there is only superimposed rigid body motion.

(Refer Slide Time: 12:09)



So, there are two ways in which the objective stress rates can be derived. In the first way we have the tensorial coordinate transformation method and this is usually followed in standard non-linear finite element textbook for example, you can see the non-linear finite element procedures book by Belytschko, Liu and Moran and these kind of procedure they provide very informative and geometrical insight ok. You can get a lot of information and geometrical insight.

And, the second procedure is through variational principles from the strain energy density in the material which is expressed in terms of strain tensor which is objective by its definition ok. So, you can express the strain energy density say for in terms of rate of deformation tensor with indeed is objective. Therefore, if you use variational principles will get a stress measure which is objective, but this is very mathematically involved, but it automatically ensures energy conservation that is the work conjugacy requirement ok.

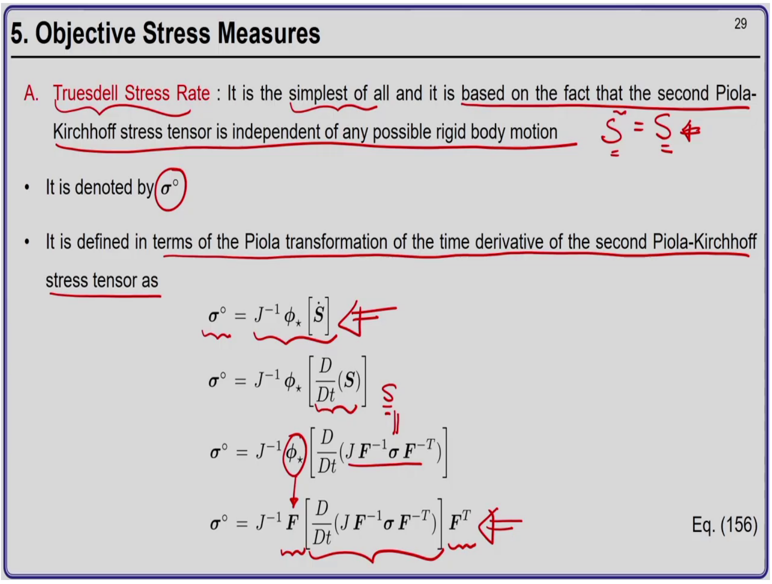
You will see that in the first approach the work conjugacy requirement is not entirely fulfilled however, but in the second this is automatically fulfilled. Now, in the present course because this is not a full course on continuum mechanics, we only look into some of the objective stress measures which are derived using the first approach which is the tensorial coordinate transformation ok. So, in the present course we follow the first procedure.

And, there are numerous objective stress rates in continuum mechanics and therefore, all of which can be shown to be special form of Lie derivatives. So, if you remember what was Lie derivative, Lie derivative was you wanted to find the rate of a certain quantity which is spatial ok. So, if you wanted to compute the rate of a certain special quantity what you did was first you pulled back that quantity to the material configuration, you took the material time derivative and then you pushed forward the this time derivative to the spatial configuration to get the rate of that particular quantity.

So, if you remember the rate of deformation tensor was nothing, but the you pulled back the Euler – Almansi strain tensor to the material configuration where it became the green Lagrange strain tensor you took the material time derivative and then you pushed it back to the spatial configuration to get the rate of deformation tensor. So, all these objective stress rates can be shown as a special form of Lie derivatives and in this course we just go very superficially we do not go much deeper because this is much more applied course ok.

So, some of the most widely used objective stress measures are the Truesdell stress rate, the Oldroyd stress rate, the convective stress rate, Green – Naghdi stress rate and Zaremba – Jaumann stress rate ok. So, some of these stress rate for example, Green – Naghdi and Zaremba – Jaumann stress rates are used in commercial finite element packages and towards the end of this lecture we will briefly touch upon the some of the recent studies which have been done on these stress measures and certain suggestions that people have been recommending for last say 5 – 6 years.

(Refer Slide Time: 16:20)



So, first we look into what is called the Truesdell stress rate. Now, the Truesdell stress rate is the most simplest of all the objective stress rates and it is based on the fact that the second Piola-Kirchhoff stress tensor is independent of any rigid body motion which means S tilde is equal to S. This we had already shown when we were discussing the effect of superimposed rigid body motion on the second Piola-Kirchhoff stress tensor and ensure that S tilde will come out to be equal to S.

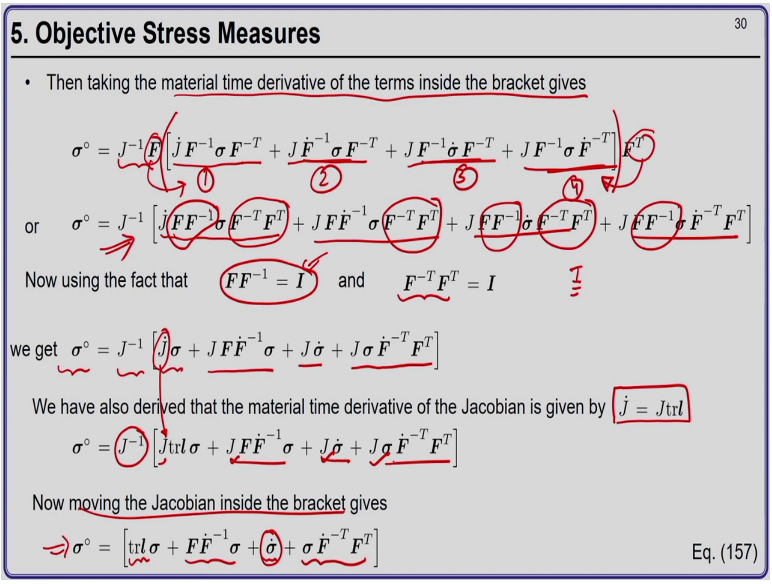
Now, this stress rate is denoted by sigma and there is a superscript with a small circle, let say sigma 0 and it is define in terms of the Piola transformation of the time derivative of second Piola-Kirchhoff stress tensor as sigma 0 is J inverse push forward of the material time derivative of the second Piola-Kirchhoff stress tensor ok.

So, the we already had studied the Piola transformation of the Green Lagrange I mean this second Piola-Kirchhoff stress tensor. So, the Truesdell stress rate is defined as the Piola transformation of the time derivative of second Piola-Kirchhoff stress tensor and this is how it is defined.

So, this means S dot is the material time derivative of the second Piola-Kirchhoff stress tensor and then we know that S is equal to. So, this is your second Piola-Kirchhoff stress tensor and this you can write in terms of t he Cauchy stress tensor which is J F inverse sigma F inverse transpose and then the push forward. So, this push forward operation will be nothing, but the term in the bracket is pre multiplied by the deformation gradient tensor and post multiplied by the transpose of the deformation gradient tensor ok.

Now, let us see what does this expression give, let us simplify this and see what expression for sigma 0 or the Truesdell stress rate we end up with.

(Refer Slide Time: 19:05)



So, now, if you take the material time derivative of the terms inside the brackets so, what we get? We get J inverse F and then J dot F inverse sigma F inverse transpose that is because the term inside the bracket had four terms 1 2 3 4 ok. So, first J dot F inverse sigma F inverse transpose, then J F dot inverse sigma F inverse transpose plus J F inverse sigma dot F inverse transpose plus J F inverse sigma F dot inverse transpose and then multiplied by F transpose ok.

Now, let us simplify it further**.** Now, I can take F inside the bracket and F transpose also inside the bracket ok. So, this I get J dot FF inverse sigma F inverse transpose transpose, then you have the similar you have the second term third term and the fourth term.

Now, I know that F F inverse is identity and if I take transpose of this I get F inverse transpose F transpose also as identity, ok. So, I can see here I have one FF inverse and I have

F inverse transpose F transpose ok. So, I have F inverse transpose F transpose I have FF inverse F inverse transpose F transpose I have FF inverse ok.

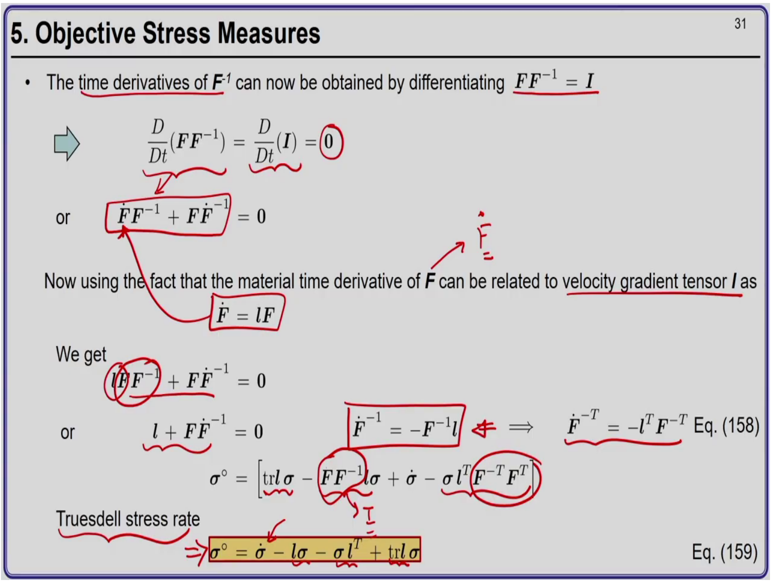
So, there are total of six terms which I can make identity ok. So, I have put identity in the circled terms and then what I get as the Truesdell rate is equal to J inverse J dot sigma plus JFF dot inverse sigma plus J sigma dot plus J sigma F dot inverse transpose F transpose ok.

Now, let us see whether we can simplify it bit further. So, we know from our discussion on the material time derivative of the Jacobian that we can show that the material time derivative of the Jacobian is nothing, but equal to J times trace of the velocity gradient tensor l. So, this is the case then I can write this J dot here as J times trace of velocity gradient time sigma and the other three terms remain same.

Now, I have J inverse outside the bracket and I have J in each of the four terms which are inside the bracket. So, I can just multiply by J inverse on all the terms which are inside the bracket ok. So, moving the Jacobian inside the bracket will give me the Truesdell rate as trace of velocity gradient time sigma plus FF dot inverse sigma plus sigma dot sigma F dot inverse transpose F transpose ok.

Now, this third term here is the rate of Cauchy stress which is not objective and till now I have still not shown that Truesdell rate is objective ok. We are first deriving the expression for Truesdell rate and then later on we will show that it is indeed an objective tensor ok.

(Refer Slide Time: 23:09)



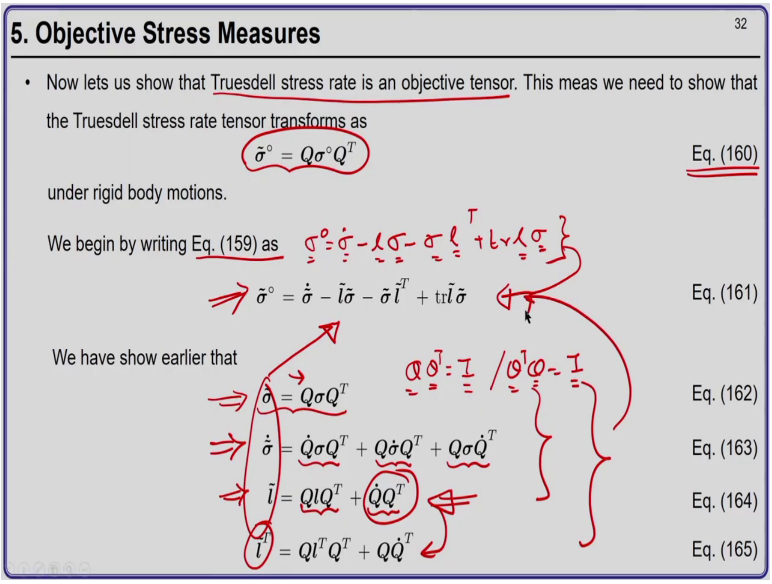
So, now what we do we can compute the material time derivative of F inverse how can you compute we know that FF inverse is identity. So, if I take the material time derivative on both the sides what I can do is the material time derivative of FF inverse is the material time derivative of identity which is nothing, but equal to 0 and then I can write. So, the left hand side is F dot F inverse plus FF dot inverse equal to 0 ok.

Now, I can use the fact that the material time derivative of F which is F dot is related to the velocity gradient tensor l as F dot equal to lF ok. So, if I substitute F dot as lF here I get lFF inverse FF dot inverse equal to 0 which means because FF inverse is identity I can get l plus FF dot inverse equal to 0 and then I can get the inverse of the material time derivative of the deformation gradient tensor as minus of F inverse times velocity gradient tensor l.

So, this now I can substitute ok. So, I can compute the transpose also and now this I can substitute in my expression on the previous slide and I can get trace of velocity gradient times the Cauchy stress tensor minus FF inverse l sigma plus sigma dot minus sigma l transpose F inverse transpose F transpose. Now, again FF inverse is identity and also F inverse transpose F transpose is nothing, but again its identity.

So, I can get the Truesdell stress rate as so, this is the Truesdell stress rate is nothing, but the rate of Cauchy stress minus l sigma minus sigma l transpose plus trace of l into sigma ok. So, this is the Truesdell stress rate and we claim right now that it is an objective measure which means sigma 0 is independent of the frame of reference or it is frame indifferent ok.

(Refer Slide Time: 26:06)



So, now let us show that Truesdell stress rate is an objective tensor ok. So, when we say have to show a certain quantity say a second order tensor is an objective tensor what I mean is under superimposed rigid body motion Q, the tensor should transform according to S tilde is Q as Q transpose ok. So, this means that the Truesdell stress rate Truesdell transform according to following relation which is given by equation 160 under rigid body motion ok

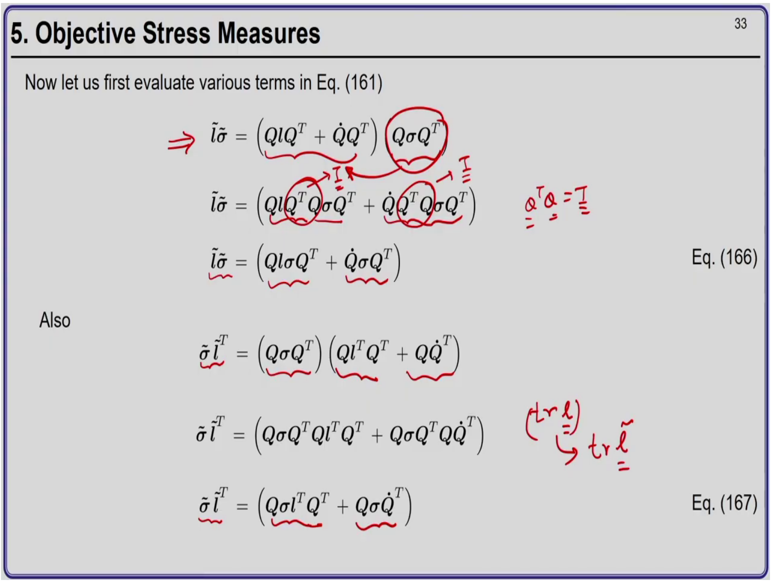
So, what we do first is we begin by writing equation 159 in the rotated configuration ok. So, we just put a tilde over our previous expression. So, our expression was sigma 0 equal to sigma dot minus l sigma minus sigma tilde l tilde transpose plus trace of l tilde sigma tilde ok. Now, I can sorry this is not tilde. So, this is the expression for the Truesdell stress rate and now under superimposed rigid body motion let equation 161 give you the Truesdell stress rate ok.

Now, we had earlier shown that the Cauchy stress under rigid body motion transforms according to relation Q sigma Q transpose and remember Q is orthogonal which means Q transpose equal to identity or Q transpose Q is equal to identity which means and also we have shown that the material time derivative of the Cauchy stress tensor is Q dot sigma Q transpose plus Q sigma dot Q transpose plus Q sigma Q dot transpose ok.

And, if you remember from our discussion on objectivity that the velocity gradient tensor in the after applying the superimposed rigid body motion was given by Q l Q transpose plus Q dot Q transpose. Now, we had shown that l the velocity gradient tensor l is not an objective quantity because of the presence of this term; however, be just recapitulate this equation ok.

Now, if you see all these equations 162, 163, 164 we can just substitute the left hand side in terms of the right hand side in equation 161 and that is what we are going to do ok. So, l tilde transpose just take the transpose of this and now, we have to just substitute all these four here ok.

(Refer Slide Time: 29:39)

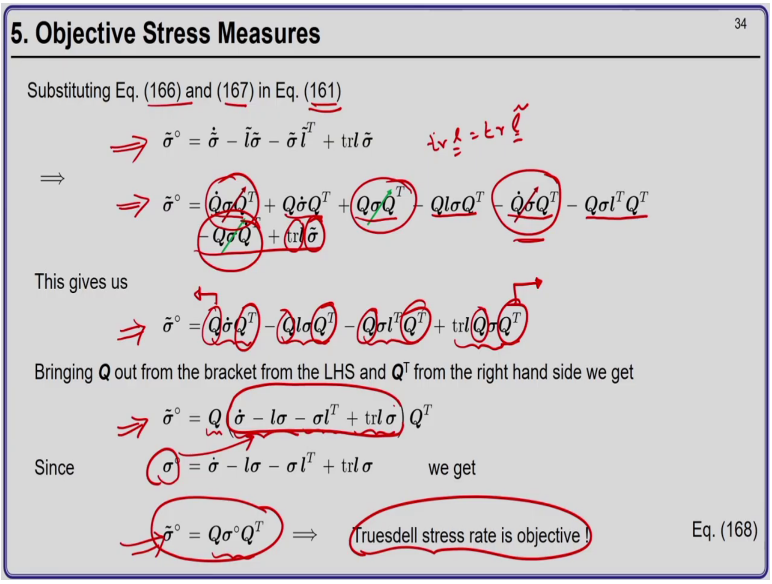


So, not to cause the clutter of writing a very long expression let us first calculate l tilde sigma tilde ok. So, l tilde is QlQ transpose plus Q dot Q transpose into sigma tilde is Q sigma Q transpose. Now, I can open up the bracket I can take Q sigma Q transpose inside the bracket and multiply by both the terms and I get QlQ transpose Q sigma Q transpose plus Q dot Q transpose Q sigma Q transpose ok.

And, I know that Q transpose Q is identity because Q is an objective tensor therefore, this quantity over here is equal to identity and then I get l tilde sigma tilde as Q l Q transpose plus Q dot sigma Q transpose ok.

Similarly, I can compute sigma tilde l tilde transpose which is Q sigma Q transpose plus Q l transpose Q transpose plus Q Q dot transpose ok. If you simplify what we get sigma tilde l tilde transpose as Q sigma l transpose Q transpose plus Q sigma Q dot transpose ok. Once we have this and also just to mention that trace of l it is a scalar quantity. Therefore, trace of l will be same as trace of l tilde because trace of l itself is a scalar quantity and scalar quantities have no effect of rigid body motion. Therefore, superimposed rigid body motion so, trace of l will be same as trace of sigma tilde ok.

(Refer Slide Time: 31:40)



So, if you substitute equation 166 and 167 in equation 161 what we get? So, this is your equation 161. So, this implies that the Truesdell stress rate after rigid body superimposed rigid body motion will be Q dot sigma Q transpose plus Q sigma dot Q transpose plus Q sigma Q dot transpose minus Ql sigma Q transpose minus Q dot sigma Q transpose minus Q sigma l transpose Q transpose minus Q sigma Q dot transpose plus trace of l sigma tilde, where I have used the fact that trace of l same as trace of l tilde because it is a scalar quantity ok.

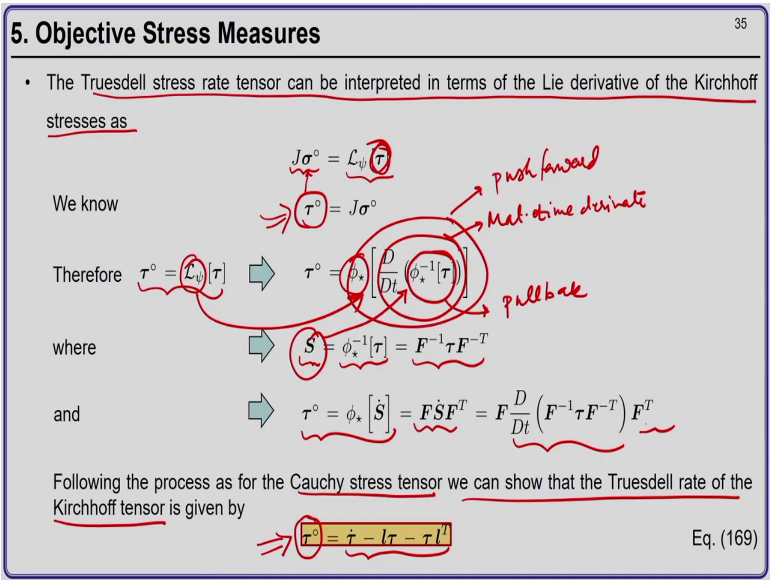
And, now we notice that this term cancels out with this term ok. So, this is the positive, this is the negative term. So, these two terms cancel out and the third term and this term over here they also cancel out ok. So, these four term cancel out and what I am left with is the Truesdell stress rate after superimposed rigid body motion is given by Q sigma dot Q transpose minus Ql sigma Q transpose minus Q sigma l transpose Q transpose plus trace of l and sigma tilde is Q sigma Q transpose.

Now, if I look closely on the left hand side I always have Q ok, the left most term on each sub term is Q and the rightmost term in each sub term is Q transpose ok. So, I can take Q outside the bracket from the from all the terms on the left hand side and this Q transpose I can take out from the right hand side and what I get? I get the Truesdell stress rate after superimposed rigid body motion as Q times sigma dot minus l sigma minus sigma l transpose plus trace of l sigma into Q transpose.

And, I can immediately identify that this term inside the bracket is nothing, but the Truesdell stress rate before the superimposed rigid body motion were applied. Therefore, if I substitute it here I get the Truesdell stress rate after superimpose rigid body motion is equal to Q times the Truesdell stress rate into Q transpose ok. And, we see that this relation is relation which has to be satisfy by any second order tensor which has to be an objective tensor. Therefore, we can say now that the Truesdell stress rate is an objective tensor ok.

So, we have derived one of the objective stress measure which is the Truesdell stress rate you are derived expression and also now we have shown that indeed the Truesdell stress rate it is an objective quantity ok. So, there will be no effect of superimpose rigid body motion ok.

(Refer Slide Time: 35:30)



So, now we will look into some more details about the Truesdell stress rate tensor. So, the Truesdell stress rate tensor can be interpreted in terms of the lie derivative of the Kirchhoff stress ok. So, Kirchhoff stress is tau ok.

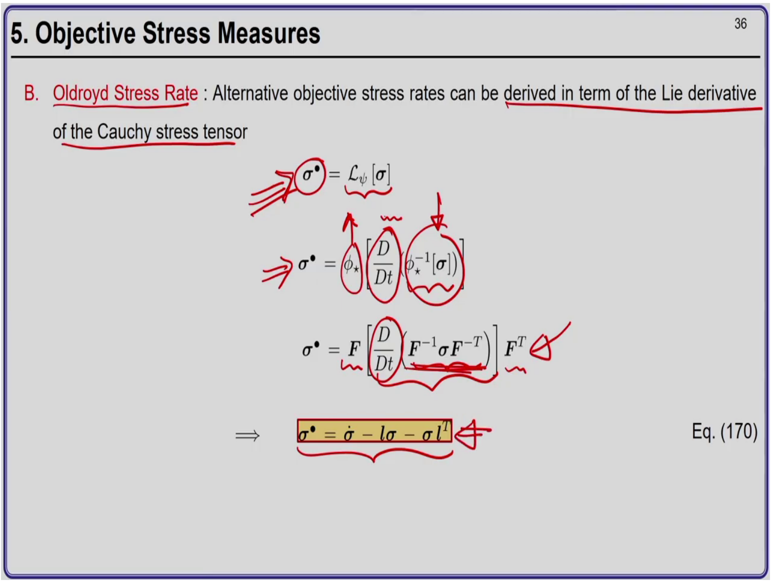
Now, the J times Truesdell stress rate is the lie derivative over the current deformation of the Kirchhoff stress which means here I can write J sigma 0 as tau 0. Therefore, tau 0 is the lie derivative of the Kirchhoff stress tau where this lie derivative explicitly is carried out like this tau 0 is push forward of the material time derivative of the pull back of the Kirchhoff stress tensor ok.

So, this is the pullback, this is the material time derivative material time derivative and this is the push forward ok. So, the Truesdell stress rate can be treated as a lie derivative of the Kirchhoff stresses. So, here the second Piola-Kirchhoff stress tensor we know is nothing, but the pullback of the Kirchhoff stress tensor which is explicitly written as F inverse tau F inverse transpose ok.

Now, if I can put it here I can put this here and then tau 0 is the push forward of the material time derivative of the second Piola-Kirchhoff stress tensor explicitly F S dot F transpose ok. So and S dot is nothing, but D by Dt of F inverse tau F inverse transpose and then F transpose.

So, if you follow a similar process that we did for the Cauchy stress tensor, we can also show that the Truesdell rate of the Kirchhoff stress tensor ok. So, earlier we derived the Truesdell rate of the Cauchy stress tensor sigma 0, now if you follow a similar procedure we can show that the Truesdell rate of the Kirchhoff stress tensor tau 0 is given by the material time derivative of the Kirchhoff stress tensor minus l tau minus tau l transpose ok. So, this you can show ok.

(Refer Slide Time: 38:47)



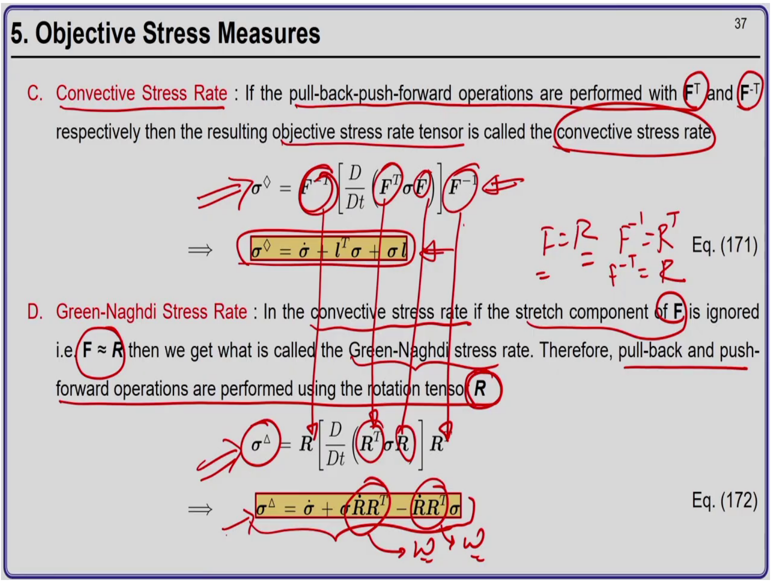
So, now let us move on to some other stress measure because Truesdell stress measure is not the only objective stress measure, there are some other types and the first one is the Oldroyd stress rate. This is another stress measure and if this can be derived ok. So, this is denoted by this symbol and this can be derived in terms of the Lie derivative of the Cauchy stress tensor.

So, this is the Oldroyd stress rate is nothing, but the lie derivative of the Cauchy stress tensor ok. So, what it means is you pullback the Cauchy stress tensor in the material configuration, take the material time derivative. So, this is the pullback to the material configuration then you take the material time directive and then you finally, push it forward. So, this is the push forward.

So, first you do the pull back, you carry out this material time directive and then you push forward to the spatial configuration to get the Oldroyd stress rate ok. So, the pullback is F inverse sigma F inverse transpose and then push forward, then you have the material time directive, the push forward is F times the material time derivative of the pullback of sigma times F transpose.

And, if you carry out so, in Truesdell rate we have already shown how to compute these kind of expressions. So, all you need to do now is take the material time derivative inside. So, you will have three terms and then you do a similar procedure that we did for the Truesdell stress rate and then finally, you will arrive at the following expression for the Oldroyd stress rate which is given by equation 170 ok.

(Refer Slide Time: 40:51)



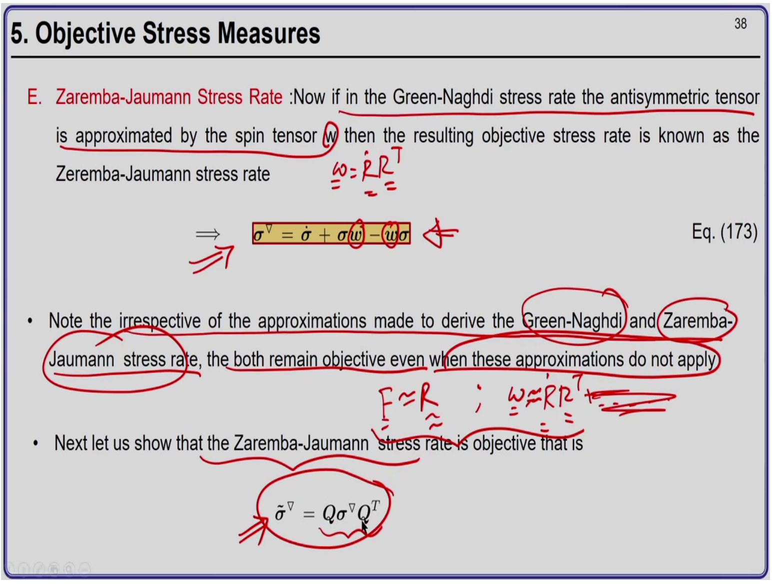
So, the next rate is the convective stress rate ok. So, if you carry out the pull back and the push forward operations with F transpose F inverse transpose. In the previous in the Oldroyd instead of the usual we do it with F transpose F inverse transpose, then we get an objective stress rate tensor which is called the convective stress rate and this is the expression. And, if you carry out this operation so, if you just do these operations what you will get is the following expression which is given in equation 171 and this is called the convective stress rate ok.

The next one is a Green – Naghdi stress rate it is used widely in many of the commercial packages and now, in this convective stress rate if this rate the stretch component of the deformation gradient tensor F is ignored which means that if you take the deformation gradient tensor is nearly equal to the rotation tensor R orthogonal rotation tensor R, then we get what is called the Green – Naghdi stress rate ok.

So, in Green – Naghdi stress rate the pullback and the push forward operations are performed using the rotation tensor R ok. So, here F is taken as R therefore, F inverse is R inverse which is nothing, but R transpose and F inverse transpose will be nothing, but R ok. So, F inverse transpose get replace by R F inverse gets replaced by R transpose and F transpose is R transpose and F is R. So, that is what we have.

So, in the convective stress rate if you have I mean instead of going with the pull back and push forward by F transpose F inverse transpose, we take the pull back and the push forward using the rotation tensor R then what we get is the Green – Naghdi stress rate which is shown in equation 172. And, if you do the simplification, this is left as a trivial exercise I mean you just have to take the carry out the directives and put it there, you get the Green – Naghdi stress rate as sigma dot plus sigma R dot R transpose minus R dot R transpose sigma ok. This is called the Green – Naghdi stress rate.

(Refer Slide Time: 43:35)



And, the another and the last one that we consider in this course is the Zaremba – Jaumann stress rate this again is used in many commercial packages. So, here what you do if in the Green – Naghdi stress rate the antisymmetric tensor is approximated by the spin tensor w ok.

So, we had shown that omega is R dot R transpose ok. If omega was R dot R transpose you can see R dot R transpose you have an under special condition we had derived that under rigid body motion omega is same as R dot R transpose this we had derived when we were discussing why omega is called the spin tensor ok.

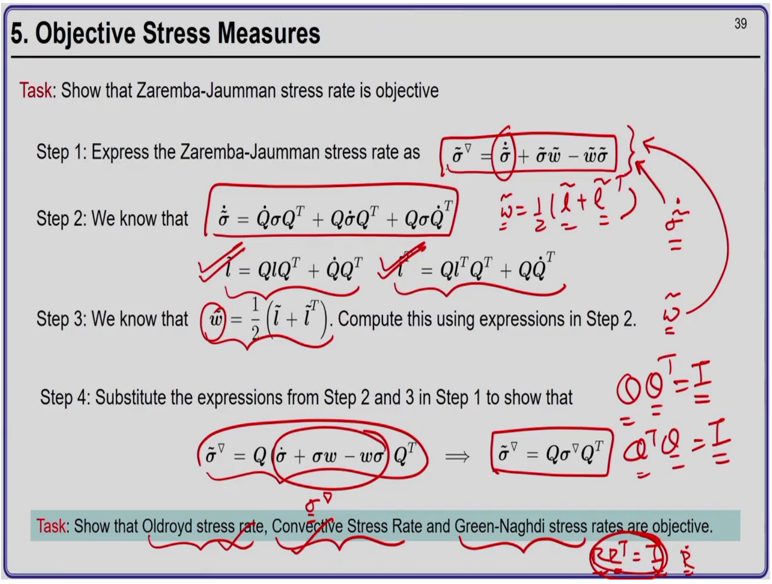
If you recall that then omega is omega replaces R dot R transpose and then the rate that we get is called the Zaremba – Jaumann stress rate given by equation 173 ok. So, these are the five different stress rates that are commonly used.

Now, the important point to note again is that irrespective of the approximations made to derive the Green – Naghdi or the Zaremba – Jaumann stress rate that is equation 172 and 173 both will remain objective even when these approximations do not apply. In Green – Naghdi we had F was nearly equal to R and in the second one we approximated omega as R dot R transpose ok.

If you remember we derived omega equal to R dot R transpose only when we considered that there is rigid body motion. If there was no rigid body motion there were some extra terms after R dot R transpose. So, here what we have taken as omega is R dot R transpose. So, both these approximations even if they do not apply even then the Green – Naghdi stress rate and the Zaremba – Jaumann stress rate will remain objective and this you can show.

So, now let us show that the Zaremba – Jaumann stress rate is objective and now, I will not detail out the expressions ok, but I will give you the steps and you can carry out these operations and get the derived result. So, what you want to show is the Zaremba – Jaumann stress rate after the superimposed rigid body motion is given by Q times the Jaumann stress rate times Q transpose.

(Refer Slide Time: 46:33)



So, how do we show the first step is you express the Zaremba – Jaumann stress rate by following expression ok? So, what you do? You just take the expression for Zaremba – Jaumann and just put a tilde over each term.

Once you have this the first term sigma tilde dot we already have derived this is the material rate of the Cauchy stress tensor. This we already have derived and to get omega tilde omega tilde is 1 by 2 l tilde plus l tilde transpose ok.

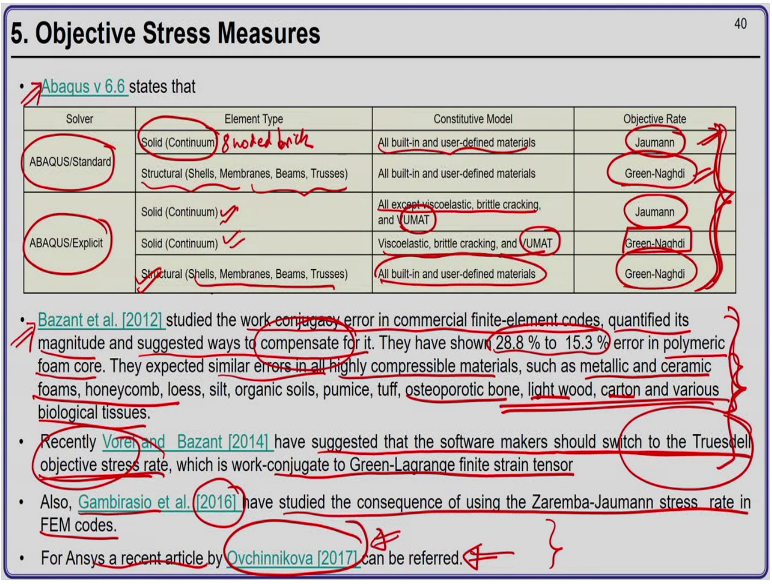
So, l tilde is Q l transpose Q dot plus Q dot Q transpose l tilde transpose is Q l transpose Q transpose plus QQ dot transpose ok. Therefore, step 3 I can compute omega tilde which is the symmetric part of the velocity gradient tensor and then I can compute omega tilde using these two expression for the velocity gradient tensor after superimposed rigid body motion.

Once I have calculated this I can substitute omega tilde from step 3 and sigma tilde dot from step 2 in this expression over here ok. I can substitute it here and then finally, I will come to this expression and then, this term inside I can identify it as the Jaumann stress rate ok. So, I will be able to show that the Zaremba – Jaumann stress rate is indeed an objective tensor ok.

So, this I leave it to you as an exercise ok. You can follow the four steps it is very easy to show there will be long expression, but you can and all everywhere you just use this condition that Q Q transpose equal to identity or Q transpose Q is equal to identity ok.

So, using similar procedures you can show that the Oldroyd stress rate, the convective stress rate, and the Green – Naghdi stress rates are all objective quantities and this is left for you as an exercise ok. And, it is very easy to show for all these quantities except for Green – Naghdi where you have to use the fact that RR transposes identity. So, to get R dot you have to take the material time derivative of RR transposes identity and then from there you derive R dot. That is the only hint. The other two are very simple to do, the third one this particular held that I am giving you can use ok.

(Refer Slide Time: 49:50)



So, finally, we come to the last part of this module on kinetics which is the some of the observation or some of the work that I think you guys should know.

So, there are in large some 10 years a lot of papers have come lot of literature as come we discusses what are the implications of various objective stress measures in commercial finite element codes. Say for example, when I looked into the Abaqus 6.6 manual which is available online. There what is say is that if you are using ABAQUS Standard which means the implicit Abaqus ok.

And, if you choose a solid element which is continuum like you use 8 noded brick element which is a solid element and if you choose a constitutive model which is all inbuilt models which are there in Abaqus or any user defined material model, then the objective stress rate they will use you see the Zaremba – Jaumann or simply the Jaumann stress rate ok. If you use any structural elements like shells, membranes, beams and trusses, the Green – Naghdi stress rate is used ok.

On the other hand, if you are using ABAQUS Explicit where no Newton Raphson method is used for the solution for example, in high velocity impact factor kind of problems. Then for solid elements for all material models except viscoelastic, brittle cracking and in VUMAT except for all these three cases they use Jaumann ok.

For solid continuum for viscoelastic, brittle cracking, and in the VUMAT subroutine the Green – Naghdi stress rate is used ok. And, for structural elements like shells, membranes, beams and trusses for all inbuilt and user defined materials the Green – Naghdi stress rate is used ok.

So, you can see Abaqus as per version 6.6 there are current versions also so, but what I am stating is in the previous version the use either Jaumann or the Green – Naghdi stress rate. So, you can check in the current manual of Abaqus what kind of stress measure are being used.

Now, as I said there I mean some studies on how these stress measures cause problem in certain kind of simulation. For example, Bazant and coauthors they studied the work conjugacy error in commercial finite element codes ok, they quantified it is magnitude that is the error of work conjugacy and they suggested ways to compensate for it and they through examples have shown that there can be an error of 15 to 28 percent in the polymeric foam core, if you use the some of the inbuilt stress measures with the finite element codes commercial finite element packages have ok.

So, they said they expected similar errors in case of highly compressible material such as metallic and ceramic foams, honeycomb like bones, wood, carton or different kind of biological tissues. So, what the implication of this study is, if you are using a commercial package with their inbuilt objective stress measure then if you have these kind of materials which they have stated, then you can expect that your results will not be accurate ok. Even though these stress measures are objective there will be certain errors and these errors if you want to go into more detail you can read the paper by Bazant ok.

And, recently they have also suggested that the software makers should switch to the Truesdell stress rate. So, instead of using the Jaumann, Green – Naghdi stress rates the software makers should switch to the Truesdell objective stress rate which is work conjugate with the Green – Lagrange finite strain tensor. So, that is what they suggest and through many examples they have shown that what kind of errors you can get there are more papers by Bazant and coworkers and you can obviously, look into them.

So, Gambirasio in 2016, he studied the consequence of using the Jaumann stress rate in finite element codes and he showed that there are many spurious fluctuations in various stress rates when you are using the Zaremba – Jaumann stress measure ok. So, again you can go through this paper and have a understanding ok. In 2017, Ovchinnikova presented a paper where they have studied a number of objective stress measures which are available in Ansys.

And, they show that there are certain stress measures which are present in Ansys, but a certain different the results that are obtained can only be obtained by a different stress measure which is not actually mention in their manuals. So, such a kind of reference is made in this particular paper of Ovchinnikova in 2017. So, this paper also I suggest you can read and there are many papers by Bazant and coworkers which is with detail out the effect of different stress measures ok.

So, as I said here as per their suggestion the or their observation the Truesdell stress rate is the most accurate one and the software maker should use it and also in some of their papers, they have shown if you are using a commercial package where the stress measures like Jaumann and Green – Naghdi are used what you can do to compensate ok. So, they compensate for the errors that might occur because of these stress measures ok.

So, with this we come to the end of the theory part of kinetics ok. So, next we go to some solved examples and try to wrap it up this module.