Computational Continuum Mechanics
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Hyperelasticity - 1
Lecture – 21-22
Lagrangian and Eulerian elasticity tensors

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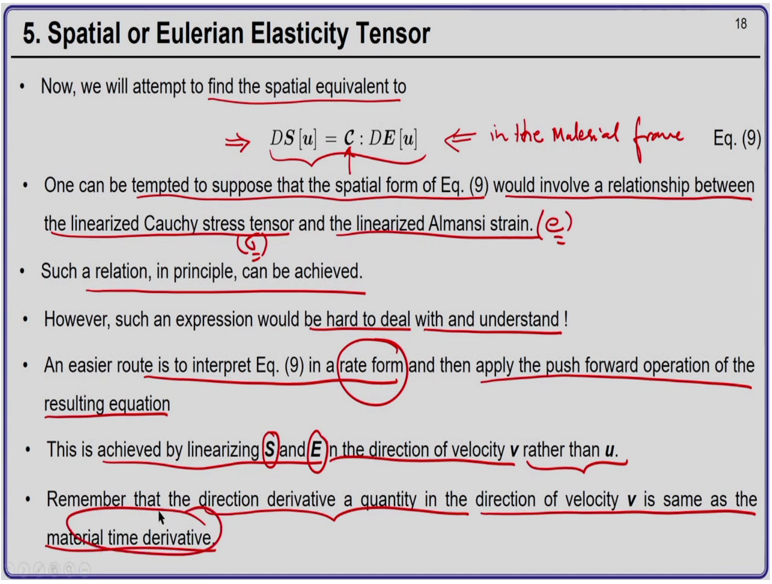

So, next, we look into the spatial or the Eulerian elasticity tensor ok. So, what we will attempt to do is, we will try to find the spatial equivalent of this equation ok, that is the directional derivative of the second Piola-Kirchhoff stress tensor in the direction of the displacement u equal to the double contraction of the Lagrangian elasticity tensor with the directional derivative of Green-Lagrange strain tensor in the direction of displacement ok.

So, this is in the material frame or the undeformed frame of reference ok, initial configuration. So, now, since the equilibrium equation is written in the current configuration or the spatial configuration, this equation has to be now transformed to the spatial configuration ok. So, you may be tempted to suppose that the spatial form of equation 9 would involve a relationship between the linearized Cauchy stress tensor and the linearized Almansi strain tensor which is e ok.

So, Cauchy stress tensor is sigma because we had second Piola-Kirchhoff and the Green-Lagrange strain tensors, two tensors which are entirely in the material configuration, it is error on not out of temptation that one can think that the spatial form of equation 9 would involve the linearized form of the Cauchy stress tensor and the Euler Almansi strain tensor which are the tensors entirely in the spatial configuration ok. But such a relation in principle can be achieved, but it would be very hard to deal with and it will be very difficult to even understand ok, how to go about.

So, what we will do is we have to take a different route and the different route is we have to interpret equation 9 in the rate form right. Now, equation 9 is not in the rate form which means you do not have velocity there you have displacement ok. If you can interpret equation 9 in the rate form, then you can carry out the push forward operation of the resulting equation ok.

Now, if you remember that we can linearize a tensor which is been linearized earlier with respect to displacement, we can also linearize that tensor with respect to the velocity ok. So, what we will try to achieve is linearize the second Piola-Kirchhoff stress tensor and the Green-Lagrange strain tensor in the direction of velocity ok; velocity v rather than the displacement.

Now, what is the advantage? The advantage is we know that the directional derivative of a tensor in the direction of velocity is same as its material time derivative ok. So, we had earlier discussed in our overview of tensors that the directional derivative of a quantity in the direction of velocity is same as the material time derivative ok.

So, remember, we needed something in the rate form ok. So, we can get the rate that is the material time derivative, when we linearize the second Piola-Kirchhoff stress tensor and the Green-Lagrange strain tensor in the direction of velocity rather than the direction of displacement ok.

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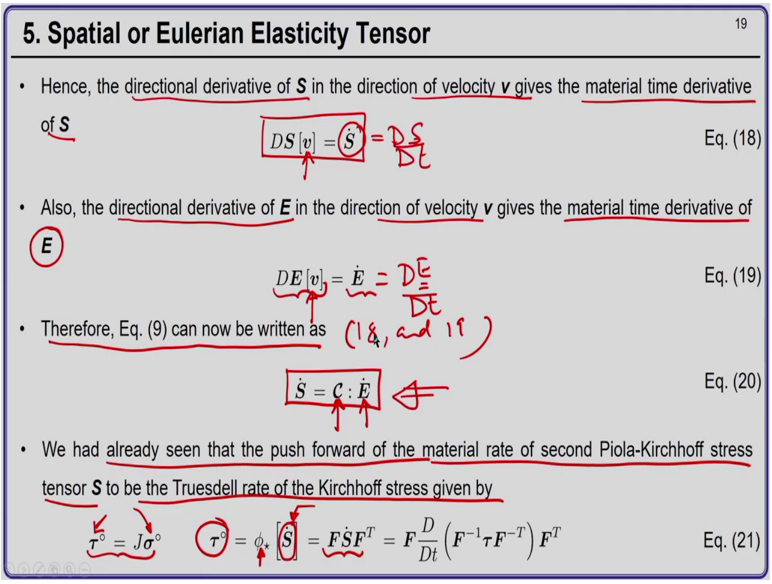

So, now let us see what happens if you take the directional derivative of the second Piola-Kirchhoff stress tensor in the direction of velocity v, it gives you the material time derivative of S that is the directional derivative of second Piola-Kirchhoff stress tensor in the direction of velocity v will give you DS by Dt which is the material time derivative of the second Piola-Kirchhoff stress tensor.

Similarly, if you take the directional derivative of the Green-Lagrange strain tensor in the direction of velocity v ok, it will give you the material time derivative of the Green-Lagrange strain tensor ok.

So, the directional derivative of E in the direction of velocity v is DE by Dt. So, that is the material time derivative. So, now, from equation 9 ok, if you remember our previous slides, equation 9 was the material time derivative of the second Piola-Kirchhoff stress tensor is the double contraction of the material elasticity tensor with the material time derivative of Green-Lagrange strain tensor ok, that was there.

So, this we already know ok. So, now, we also have seen that the push forward of the material rate of second Piola-Kirchhoff stress tensor is the Truesdell rate of the Kirchhoff stress tensor.

So, when we were discussing kinetics, we discussed that the push forward ok. So, the push forward of the material rate of the second Piola-Kirchhoff stress tensor which is shown here. So, this is the push forward operation.

It gives you the Truesdell rate of the Kirchhoff stress tensor and from this formula, we know that you can relate the Truesdell rate of the Kirchhoff stress tensor to the Cauchy rate ok. So, that is the Truesdell rate of the Cauchy stress tensor and that is how you carry out the push forward of the material rate of second Piola-Kirchhoff stress tensor ok.

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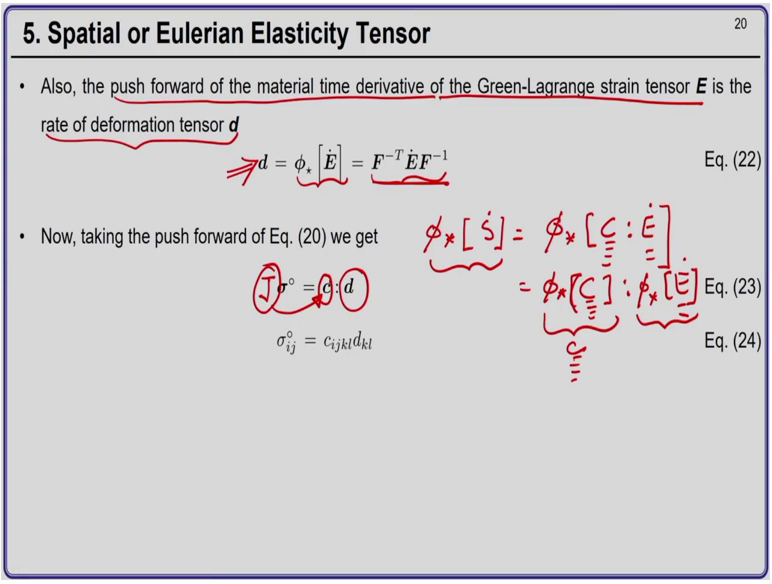

So, now, we also had seen that the push forward of the material time derivative ok. So, the material time derivative of the Green-Lagrange strain tensor is nothing but the rate of deformation tensor d ok.

If you remember from our discussion in kinematics, the rate of deformation tensor d is the push forward of the material time derivative of the Green-Lagrange strain tensor and the push forward is carried out using this particular operation ok. So, now if we take the push forward of equation 20 ok, so what is equation 20? This is the equation 20 that we obtain from equation 18 and 19.

So, if we take the push forward of equation 20, so that is if we take phi star S dot equal to phi star of C E dot ok. Then, this is phi star of say C and phi star push forward of E dot. So, phi star of S dot will be nothing but J times of sigma 0 ok; let us call this sigma 0 and the push forward of the material time derivative of the Green-Lagrange strain tensor will be d that is rate of deformation tensor and let us define the push forward of material elasticity tensor as small c ok.

So, remember I am putting 4 under bars for a fourth order tensor ok, if you have to write in direct notation. So, in initial notation ok, so what happens to this J? Why its missing? Because this has been if you we will see later, it has been absorbed in this spatial elasticity tensor c ok.

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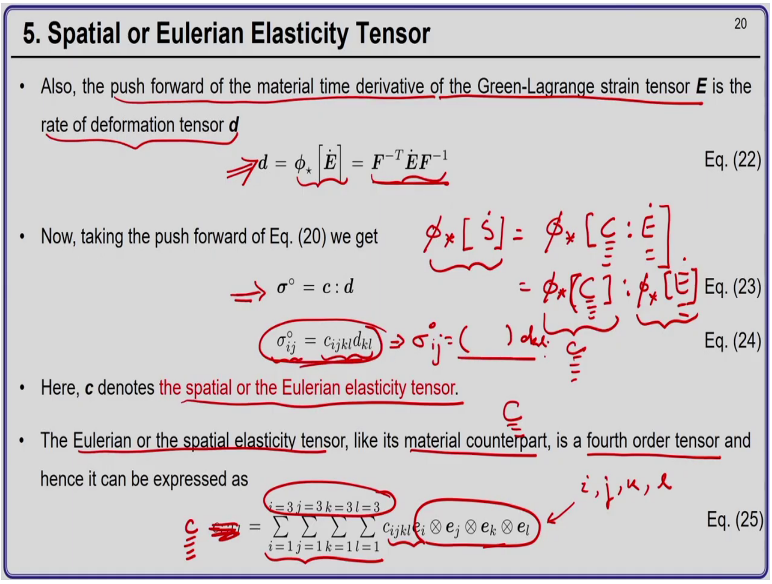

So, therefore, if you write this expression in indicial notation sigma i j is c ijkl d kl ok. Here, this c denotes the spatial or the Eulerian elasticity tensor ok. Now, this spatial or the Eulerian elasticity tensor like its material counterpart which is this capital C is nothing but a fourth order tensor ok. So, C is a fourth order tensor, this you can show.

Hence, it can be expressed sorry this is C in direct notation can be expressed as summation over ijkl c ijkl and the basis for a fourth order tensor which is e i tensor product e j tensor product e k tensor product e l and notice, we have i, j, k and l all in lowercase, this is because the spatial elasticity tensor resides in the spatial frame and by convention, we use lowercase alphabets for spatial quantities ok. But the range of i j k l will be 1 to 3 ok.

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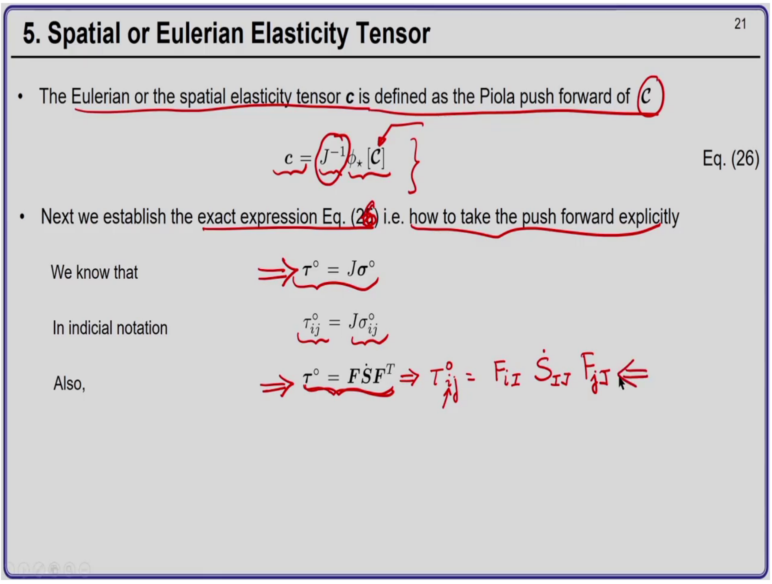

Now, this Eulerian or the spatial elasticity tensor c is defined as the Piola push forward of the material elasticity tensor C and this spatial elasticity tensor is J inverse phi star that is the push forward of the material elasticity tensor ok. So, the J that I was talking about in the previous slide is accommodated here ok.

Now, we have to next try to establish the exact expression for 25 ok, that is how to push forward sorry, this is equation 26 ok. So, right now this is only functional form which means that the push forward of the material elasticity tensor will give you the spatial elasticity tensor ok. But actually how do you carry out the push forward ok?

So, to do this let us see ok. So, let us start with the Truesdell rate of the Cauchy or the Piola-Kirchhoff stress tensor which is given by following expression ok. Then, you can write this in indicial notation tau i j is J sigma i j ok. This we already know. Again, we know that the Truesdell rate or the Kirchhoff stress tensor is nothing but FS dot F transpose. So, now, my job is now to do everything in initial rotation ok, that would be much easier because we have a fourth order tensor here.

So, the indicial notation of this will be sigma i j equal to F and because the first thing index on the left-hand side i, so I should you should have i here and then, S dot in resides entirely in the material configuration. So, I have capital I and capital J let us say. So, there should be I here and the next index here for F should be J, but because we have a transpose, this should be j and this should be J ok.

No ok; so, this transpose will not be here. So, the indicial notation for the relation between the Truesdell rate of the Kirchhoff stress tensor and the material rate of the second Piola-Kirchhoff stress tensor in indicial notation is given by following expression ok.

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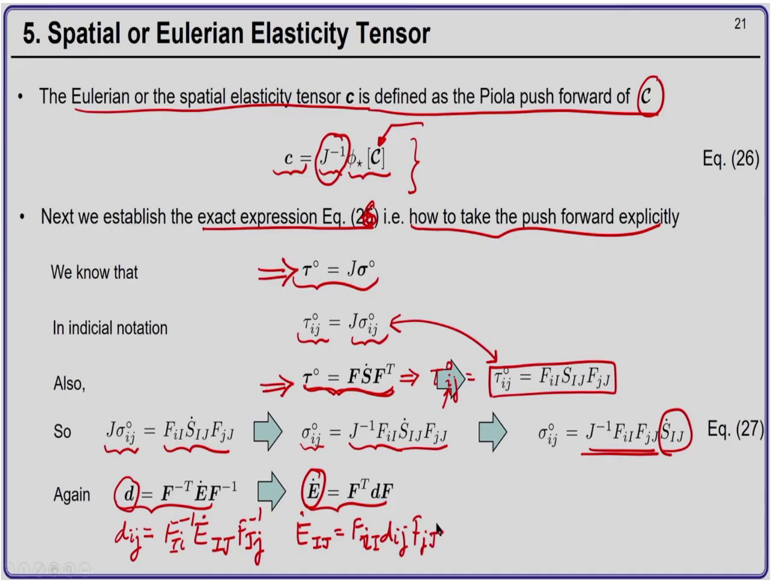

So, let me just rub this and this is what you can see here ok. So, that is what you can see here. Now, if I match these two expressions; so, I can write J sigma i j is F iI S dot IJ F jJj which means that sigma i j is J inverse F iI S dot F jJ ok. Now, I can just rearrange and take S dot IJ on the right hand side ok, then the next thing is this is the push forward of the material rate of the Green-Lagrange strain tensor to give the rate of deformation tensor.

So, the indicial notation of this would be d i j because d is a spatial tensor. So, lowercase, F inverse and because i is the first index let me put i here ok; but because this is inverse I put F iI E dot IJ F inverse J j ok. So, and I can write the material time derivative of the Green-Lagrange strain tensor given by this expression. So, the indicial notation of this will be E dot IJ equal to F and this would be lowercase iI d i j F j J ok.

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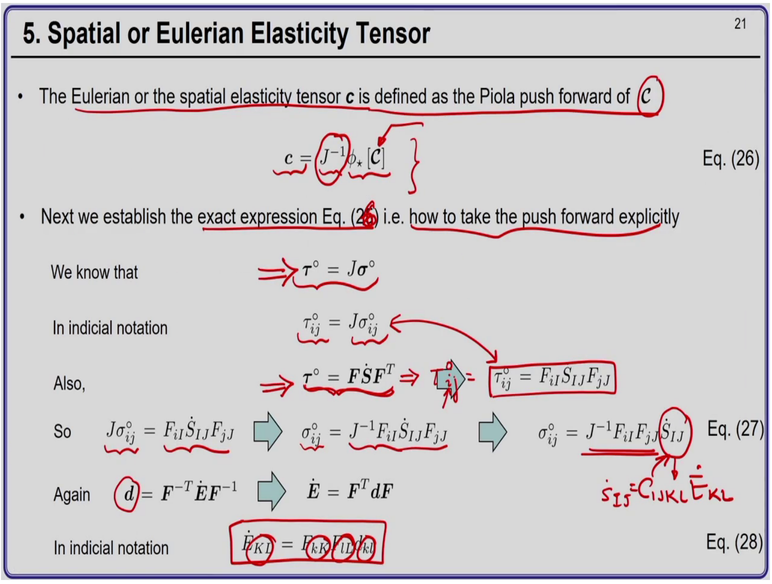

So, that is what we have in indicial notation over here ok. And now because I already have an IJ here, so what I do? I replace IJ with another index KL. So, according to our rules of indicial notation, it is perfectly possible ok. So, I have KL and small k l ok. Now, what I do? I know this relation between the material time derivative of the second Piola-Kirchhoff stress tensor and the material time derivative of the Green-Lagrange strain tensor ok.

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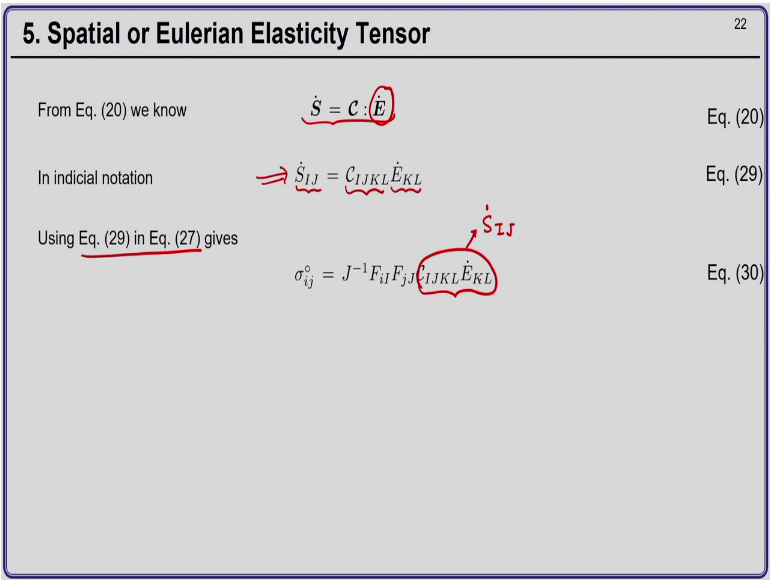

So, what I now do is I write this in indicial notation which is S dot IJ is C IJKL E dot KL and now, in my previous slide, I have got the expression for E dot KL ok. So, what I do now is if you see here, I had equation 27, where I had S dot IJ and now, S dot IJ is C IJKL E dot KL ok. So, this I can substitute here ok, so that is what we are doing ok. So, we have C IJKL E dot k lKL which is nothing but S dot IJ. So, using equation 29 in equation number 27, I get this particular expression ok.

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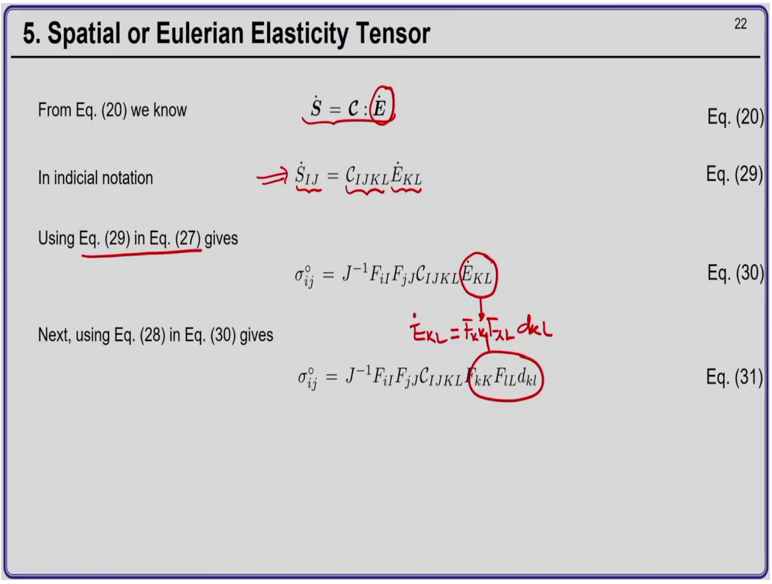

Now, I have E dot KL and from my previous slide, if you look closely, I have E dot KL is given by F kK F lL d k l ok. So, this I can substitute here F kK ok. So, F kK F lL d k l ok, that is your E dot KL ok.

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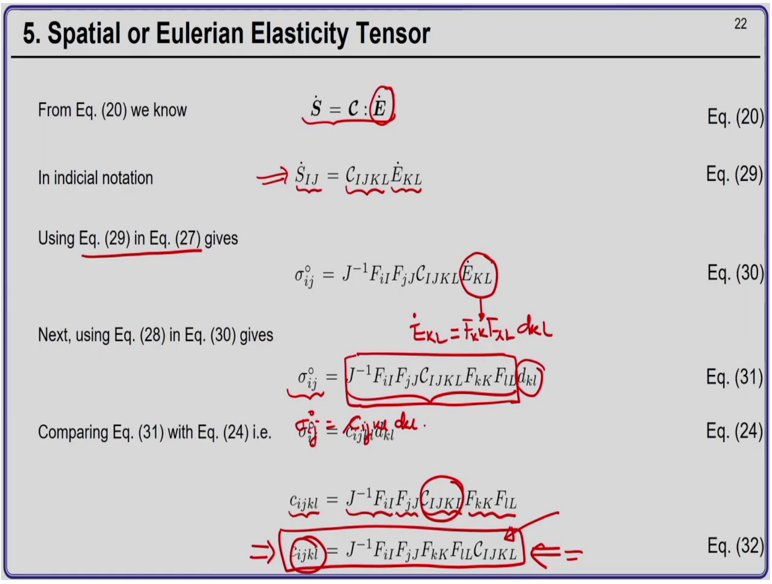

So, now, what I get? I get an expression for relation between sigma i j and d k l and then, I have this particular term ok. So, this is say coefficient of d k l ok, if you see this relation over here ok, let me go back; if you see this relation over here, so we had this relation and also now, we have got sigma 0 i j equal to one big expression d k l.

So, if I compare both the sides ok, if I compare both the sides ok. So, if I compare c ijkl is small c ijkl d kl, I can immediately identify this term in the bracket to be equal to c ijkl ok. So, c ijkl is j inverse F iI F jJ C IJKL F kK F lL ok.

So, if I take c ijkl totally on the other side ok, on the right side, so I get a very seemingly long expression for the competent ijkl'th component of the spatial or the Eulerian elasticity tensor as this J inverse F iI F jJ F kK F lL C IJKL. So, now if you know the expression for C IJKL ok, so the tensor expression for C IJKL, you would use equation number 32 to get the spatial components of the Eulerian elasticity tensor ok.

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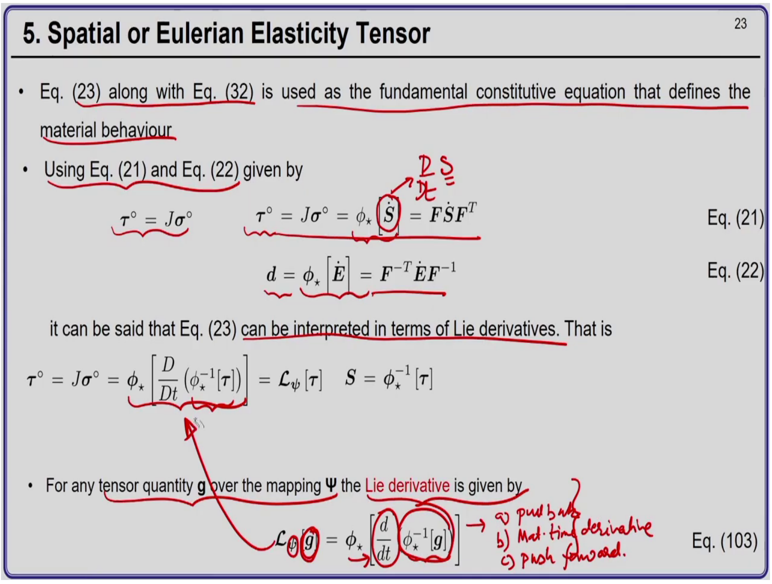

So, equation number 23 along with equation 32 is used as the fundamental constitutive equation that defines the material behavior ok. Now, if you use equation number 21 and equation 22 ok, we can see something ok, we can relate whatever we got in the previous slide to the lie derivatives ok.

I hope you remember the concept of lie derivatives ok. So, the Truesdell rate of the Kirchhoff stress is J sigma 0 and now, because the Truesdell rate is equal to the push forward of the material rate of the second Piola-Kirchhoff stress tensor ok. So, you have this. Also, d that is the rate of deformation is the push forward of the material time derivative of the Green-Lagrange strain tensor given by this following expression.

Therefore, equation number 23 which is the expression for the component of the spatial elasticity tensor in terms of the deformation gradient tensors and the component of the material elasticity tensor can now be interpreted in terms of the lie derivatives. And what is lie derivative? Just to recapitulate, what we had discussed in kinematics, for any tensor quantity g over the mapping psi, the lie derivative of the tensor quantity g ok.

So, g is a spatial tensor quantity. So, the lie derivative of this spatial quantity g over the mapping psi will be nothing but the push forward of the material time derivative of the pullback of the spatial quantity ok.

So, what it means? It is a three step process, the first the; so, the lie derivative is a three step process; the first is the pullback, then you take the material time derivative and then, you have the push forward ok. So, using this and you knowing that S dot can be written as D by Dt of S and S is nothing but the pullback of the Kirchhoff stress tensor ok. So, if you look closely this expression over here, it matches well with our definition of lie derivatives.

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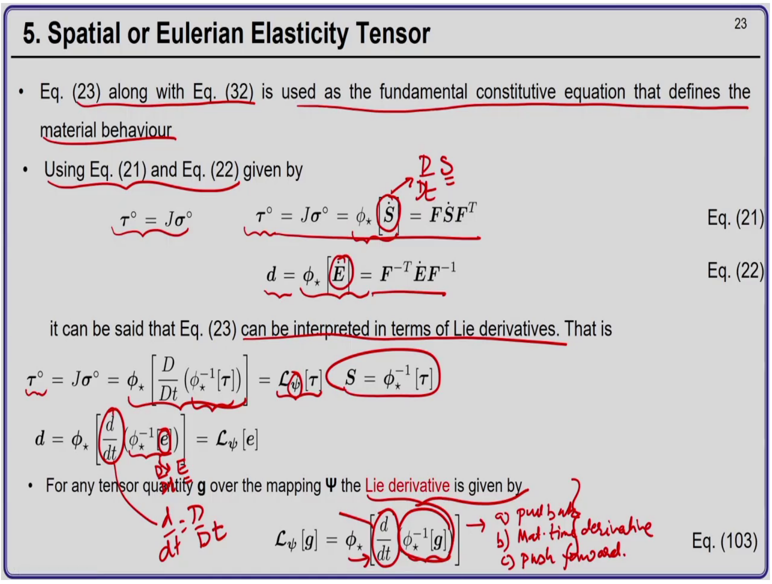

So, the Truesdell rate of the Kirchhoff stress tensor is nothing but the lie derivative of the Kirchhoff stress over the mapping psi ok, where this expression has been used. Similarly, we know that the pullback of the Euler Almansi strain tensor is nothing but the; so, this is a pullback this is nothing but the Green-Lagrange strain tensor.

So, E dot here is nothing but D by Dt of E. So, this d by dt is nothing but the material time derivative ok, although is written as small d by dt is it same, the meaning is same here as capital D by Dt ok.

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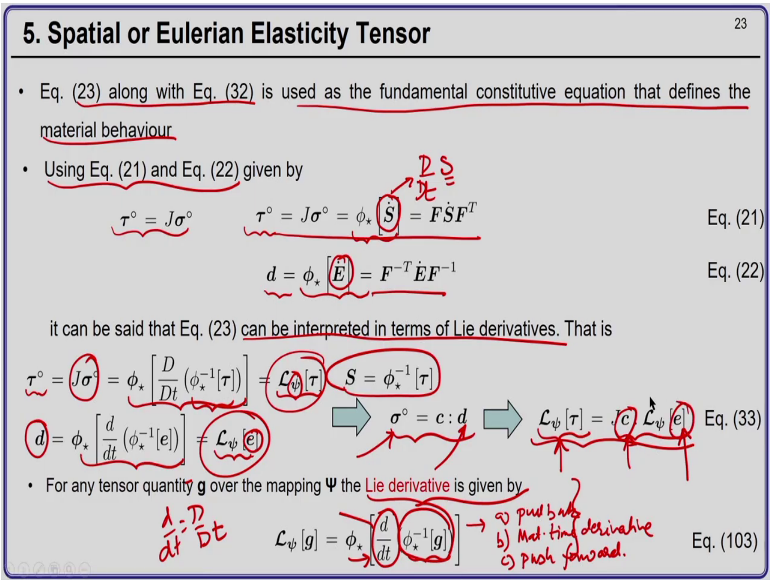

So, if we look again closely this expression, you will realize that this is nothing but the lie derivative of the Euler Almansi strain tensor over the mapping psi and this gives you the rate of deformation tensor. So, using these two ok, using this and this in our expression over here ok.

So, J sigma 0 is a lie derivative of Kirchhoff stress and d is the lie narrative of Euler Almansi strain tensor and if you use these two here, you will realize that the lie derivative of Kirchhoff stress over mapping psi is J sigma J c double contracted with the lie derivative of Euler Almansi strain tensor over the mapping psi ok.

So, the c establishes a relation between the lie derivatives of the Kirchhoff stress tensor and the Euler Almansi strain tensor ok. So, initially, we said if somebody was tempted to think, there will be a relation between the Kirchhoff or the Cauchy stress tensor and the material time derivative of the Cauchy stress tensor and the material time derivative of the Euler Almansi strain tensor will give you the spatial elasticity tensor, it is not like that.

It is actually the relation between the lie derivatives of the Kirchhoff stress and the Euler Almansi strain tensor respectively ok.

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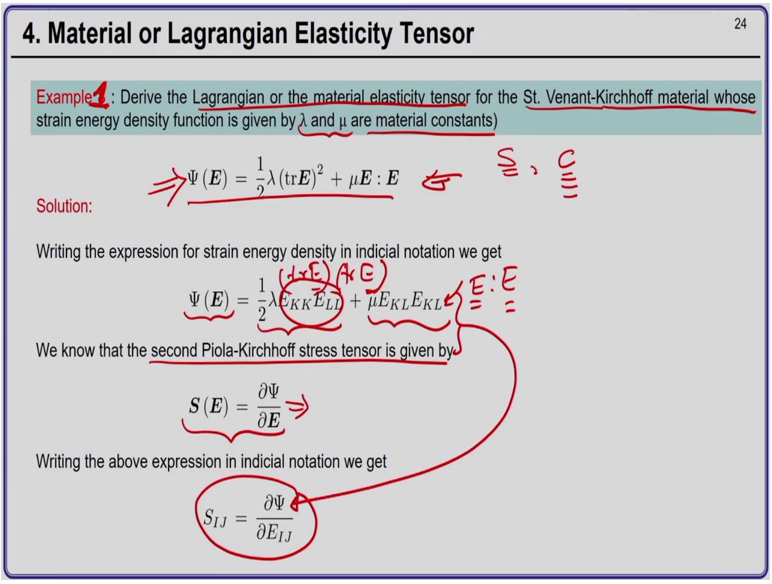

So, let us derive the Lagrangian and the material elasticity tensor for us very simple material ok, very simple class of material which are hyperelastic and the model is given by the St. Venant-Kirchhoff material model and for these material, the strain energy density function is given by this expression, where lambda and mu are the material constants ok. From lambda mu are here the material constants. So, your job is now to get the second Piola-Kirchhoff stress tensor expression and also get the expression for the material elasticity tensor ok.

Now, how do we proceed? The first thing is we write this expression in the indicial notation and remember, Green-Lagrange strain tensor is in the material frame. So, all the indices will be in uppercase. So, the strain energy density potential is written as 1 by 2 lambda E KK E LL because stress of E is E KK. So, stress of E square will be E KK into E LL plus mu times E KL E KL. So, this is nothing but E double contracted with E and this is nothing but trace of E into trace of E ok.

Now, we know that the second Piola-Kirchhoff stress tensor is given by following expression ok. If this is given by following expression, then what we need to do is we need to first write this in indicial notation ok. So, the indicial notation is S IJ del psi by del E IJ and now, what we will do? We will put expression here into this particular relation ok.

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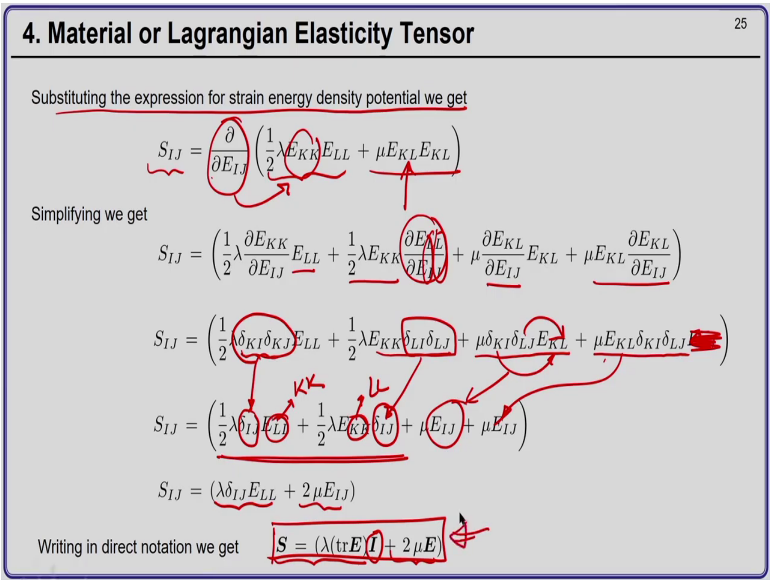

So, S IJ. So, we substitute the expression for strain energy potential in the expression for the second Piola-Kirchhoff stress tensor and then, S IJ will be del by del E IJ of 1 by 2 lambda E KK E LL plus mu times E KL E KL. And now, you can move this derivative inside and carry out the derivative.

So, what you get, if you move it inside? 1 by 2 lambda del E KK by del E IJ into E LL plus 1 by 2 lambda E KK del E LL E IJ plus mu times del E KL E IJ E KL plus mu times E KL del E KL del E IJ ok. And now, we know that E del E KK by del E IJ will be delta KI delta KJ E LL and this will be E KK delta LI ok. So, delta LI delta LJ, then mu delta KI delta LJ E KL plus mu E KL delta KI delta LJ E KL ok; sorry.

So, once you have this, you can use the substitution property of the Kronecker delta ok. So, if you see look closely here K is common. So, if you use the substitution property, it will be equal to delta IJ ok. Here also, if you look closely here L is common. So, what you will get? Delta IJ and here K will be replaced by I and L will be replaced by J.

So, what you get is, E IJ and similarly, in the last expression you get E IJ ok. So, these are; so, we can add these two expression L and K are repeated index. So, it does not matter; you can replace this L with KK and what you will get or you can replace this KK with LL and what do you get? Lambda delta IJ E LL plus 2 mu E IJ ok.

So, if you write this indirect notation, you will get S is lambda trace of E into second order identity tensor I plus 2 mu times the Green-Lagrange strain tensor E ok. So, that is the relation for the second Piola-Kirchhoff stress tensor. But you have been asked to get the Lagrangian elasticity tensor for which again you have to take the derivative of this expression with respect to E ok.

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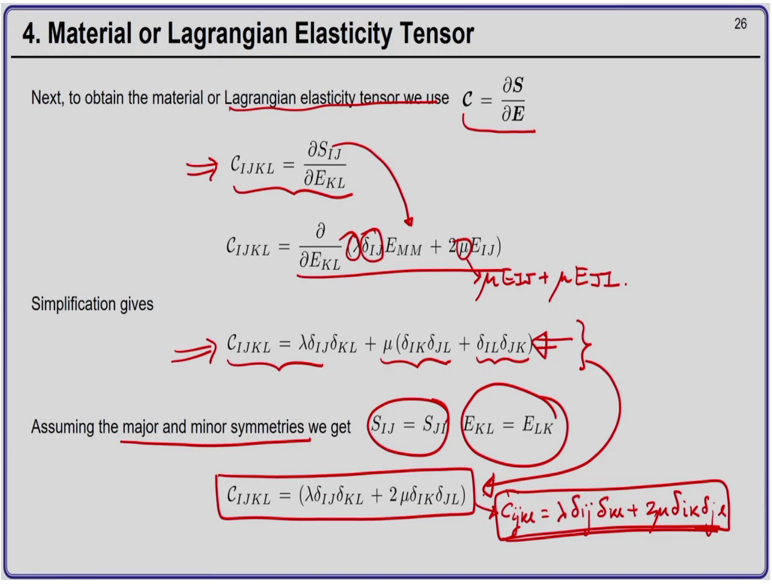

So, the Lagrangian elasticity tensor will be del S by del E. So, if we write indicial notation, we will have C IJKL del S IJ by del E KL ok. Now, I can substitute the expression for second Piola-Kirchhoff stress tensor from the previous slide and then, I can open up the bracket. So, now, you know how to take out the take the derivative? Delta is constant ok, lambda is constant, mu is constant ok; if you do this and realizing that because E IJ is symmetric, this is like mu E IJ plus mu E JI ok.

So, we will on the simplification, we will get C IJKL is lambda delta IJ delta KL plus mu delta IK delta JL plus delta IL delta JK. So, this expression here is without the consideration of the major and minor symmetry.

If you consider both the major and minor symmetries that is symmetry of the second Piola-Kirchhoff stress tensor that is S IJ is JI and the Green-Lagrange strain tensor that is E

KL is E LK, then this following expression will boil down to C IJKL equal to lambda delta IJ delta KL plus 2 mu delta IK delta JL ok.

So, this relation for small deformation problem, where the material configuration and the spatial configuration do not differ by much. In that case, we can write this as c ijkl is lambda delta ij delta kl plus 2 mu delta ik delta j ok. So, you will realize this, if you are done a course on advanced solid mechanics, you will realize that this is nothing but the Fourth order elasticity tensor that we studied ok. This is only when the material configuration and the spatial configuration do not differ by much ok.

So, with this we end the first part of hyper elasticity. So, next we are going to start the derivation of the expression for the second Piola-Kirchhoff stress tensor and the material and spatial elasticity tensor for a specific class of hyper elastic material which is Neo Hookean material model ok. So, that is the most simplest form of model that can be used and we will derive the expression for that particular model and the later on, the finite element formulation also will be based on this particular material model only.

There are many more hyper elastic material model; but the duration of the course does not allow us to go much deeper. But whatever we will discuss in the next two lectures will help you to get the expressions for the second Piola-Kirchhoff and the material and the spatial elasticity tensors for other kind of hyper elastic material models ok. So, we are going to discuss the derivation for the Neo Hookean material model.

Thank you.