Computational Continuum Mechanics

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Introduction to Tensors - 1
Lecture – 03
Tensor and Tensor Algebra – 1

Welcome to today's lecture which is going to be on Introduction to Tensors. So, following the contents of present lecture.

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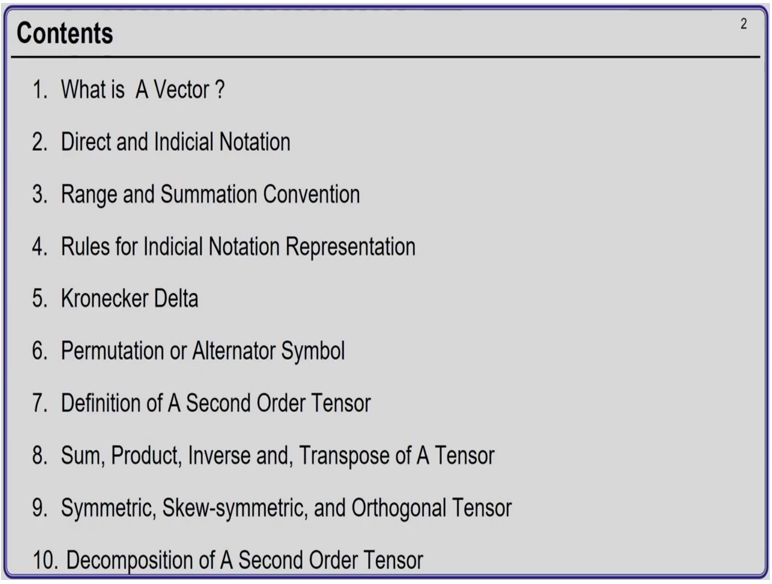

We will start by first discussing what is meant by a vector ok. Once we have defined what is a vector, we will go into discussing the different ways of denoting tensor ok, which are used to denote tensors which are direct notations and indicial notations.

This will be followed by discussion on range and summation convention which are used and then, we will discuss some rules for indicial notation representation of tensors. Followed by this, we will look into two important concepts in tensors which is kronecker delta and permutation and alternator symbol. Then, we will look into what is meant by a second order tensor ok.

And then, we will come to some of the operations that can be performed using second order tensors which is sum of two tensors, product of tensors, inverse of tensors, transpose of a second order tensor, what is meant by symmetry, skew symmetry and orthogonal tensors. Finally, we will end this lecture with decomposition of a second order tensors, how can you decompose a general second order tensor into different ways ok.

So, let us start. So, before we can discuss on what is meant by a tensor, first it is worthwhile to look into the question of what is actually meant by a vector ok.

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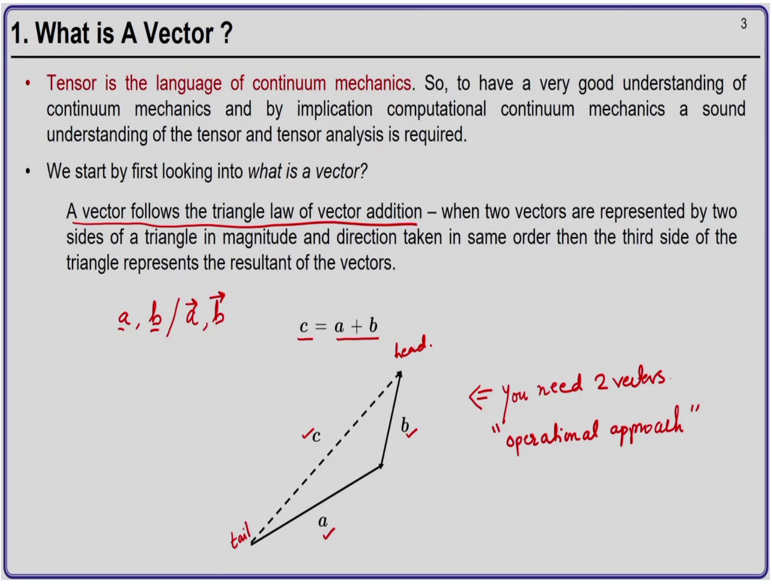

So, remember that tensors why we should focus and why we should devote on time on reading about tensors. See tensors are the language of continuum mechanics ok. Just like to speak to interact with another human being, you need to know the language with which you can speak to the other person.

Similarly, to understand so for say you want to speak to continuum mechanics, you want to speak continuum mechanics, you should learn the language in which the continuum mechanics works ok. So, this language is nothing but the language of tensors ok. So, to have a very good understanding of continuum mechanics and by its implication, computational continuum mechanics one needs to have very sound understanding of tensors and tensor analysis ok.

So, coming back to our question, what is meant by a vector? So, a vector from our school days, we know was defined as something which has magnitude and has a direction ok; but not everything which has a magnitude or a direction can be classified as a vector. Take for example, a book ok. So, a book for example, has a direction. You always read book from one side to the another side, from top to bottom; there is a procedure, you cannot read a book the other way around ok.

And also, book has a weight ok. So, you cannot assign you cannot say that book is a vector ok. So, that definition of vector that we have studied in our school is no longer fit ok. So, we need to look into another more involved definition of a vector. So, a vector follows the triangle law of vector addition ok that is what I have written here.

Vector follows the triangle law of vector addition ok. So, when two vectors are represented by two sides of a triangle in magnitude and direction taken in the same order, then the third side of a triangle represent the resultant of two vector. Say you are given two vectors a and b. Note that, I have put an under bar below the symbols a and b.

Why I have put this? Because I cannot write bold ok; when I am writing, I cannot write bold. So, there is a convention that I will put a under bar ok, the other way is you can as we used to do in school can put an arrow ok. But in the continuum mechanics courses, we do not put arrows, that is not usually followed. What is usually followed is we put a under bar. So, given two vectors a and b, the resultant of these two vectors a plus b will be another vector c. So, what will be c? This is visible in this picture ok. So, you have vector a and you have another vector b.

So, now, you put the vector a and then, you put the tail of vector b at the tip of vector a and then, the resultant vector c will be nothing but a vector which extends from the tail of vector a to the head of vector b. So, this will be your vector c ok. So, any vector which satisfy this triangle of vector addition will be called as a vector. But what is the drawback here? The drawback is you need at least two vectors to define ok, what is meant by a vector ok. So, you need two vectors ok.

So, this is what is called the and will come to it in the current lecture some point later. It is called the operational approach; operational because we are operating on two vectors; we are operating on two vectors a and b to get a third vector. So, this is called the operational approach ok. So, in operational approach, you always need more than one quantity to define another quantity. Here you have two quantities, two vectors, you add to get a third vector ok. So, we like to define vector in much more refined way ok.

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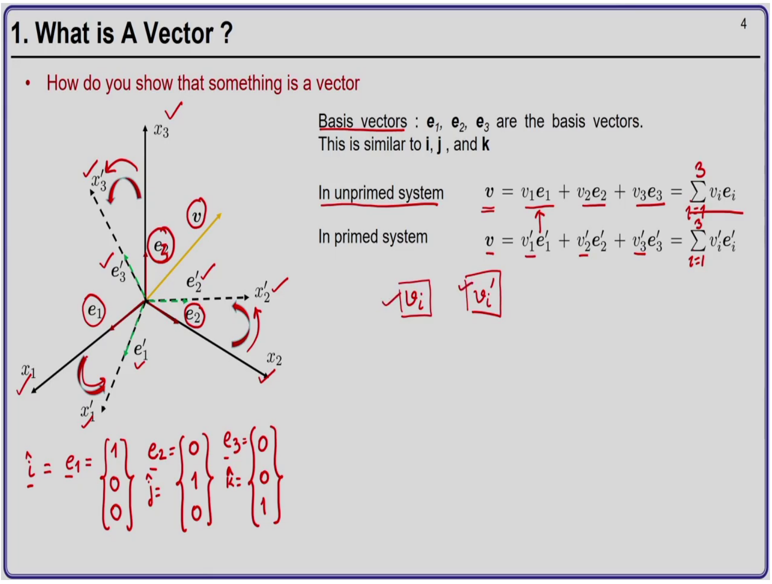

And for this, we come to how the components of a vector transform, when the coordinate system changes ok. So, in this present course, we are going to deal with only Cartesian coordinate system ok. Because of the number of lectures being limited we will not deal into curvilinear coordinate systems, will only fix our attention to Cartesian coordinate system. So, as you can see here, you have a Cartesian coordinate system shown here x 1, x 2, x 3 in bold black lines and then, the unit vectors along the directions x 1, x 2, x 3 are denoted by e 1 e 2 and e 3 ok.

This has to be e 3. So, these vectors e 1, e 2, e 3 are called the basis vectors. So, what are e 1, e 2, e 3 there? So, e 1, e 2, e 3 are like your i j k that you would have studied in your school days ok. So, e 1 is a vector because it is a vector, we put a one bar below it; e 1 is a vector which is entirely along the x 1 axis. So, what are the components of this vector? And e 1, e 2, e 3 are unit vectors.

So, the component of this vector are 1 0 0 ok. Similarly, the components of vector e 2 are 0 1 0 and the components of vector e 3 are 0 0 1 ok. So, you can see these are nothing but your unit vectors i, unit vector j and unit vector k ok. So, we use instead of using the notations i j k will use e 1, e 2, e 3 will come to it later, when we are discussing about the summation and range convention. Why this is helpful? Ok. So, once we have this. So, these e 1, e 2, e 3 called the basis vectors ok.

Now, you have a vector v ok, you see here there is a vector v which is in this represented by this arrow yellow arrow ok. Now, you can write this vector v as v 1 e 1 plus v 2 e 2 plus v 3 e 3 ok. So, the first term you have v 1 is the component of the vector v in the e 1 direction, v 2 is the component of the vector v along the e 2 direction and v 3 is the component of the vector along the e 3 direction. So, the complete vector becomes the sum of these three vectors along the x 1, x 2, x 3 direction. In short, you can write summation vi ei, where i goes from 1 to 3.

So, this is what we called as unprimed system. Now, we rotate the coordinate system this is shown here. You can see this is shown here by these red arrows. So, what you have done now? You have rotated the coordinate system. So, the new system now becomes x 1 prime, x 2 prime and x 3 prime ok. When the coordinate system is rotated the basis vectors e 1, e 2, e 3 are also rotated and they become now e 1 prime e 2 prime and e 3 prime ok. Now, you can see we have just rotated the coordinate system; we have not rotated the vector itself ok.

Now, in what is called the prime system ok, one which is obtained by the rotation of the original coordinate system. We can write the same vector in terms of its components v 1 dash, v 2 dash, v 3 dash along the x 1 dash, x 2 dash and x 3 dash directions. In short, we can write summation i equal to 1 to 3, v i dash ei dash. Now, this nothing which has happened to the vector. The vector remains there itself. What has happened is we have rotated the coordinate system.

Now, the question is can we find a relation between the components of the vector in the primes system which is v i and the unprimed system which is v i dash? So, we want to find out our relation between the components of the vector in the prime and the unprimed system.

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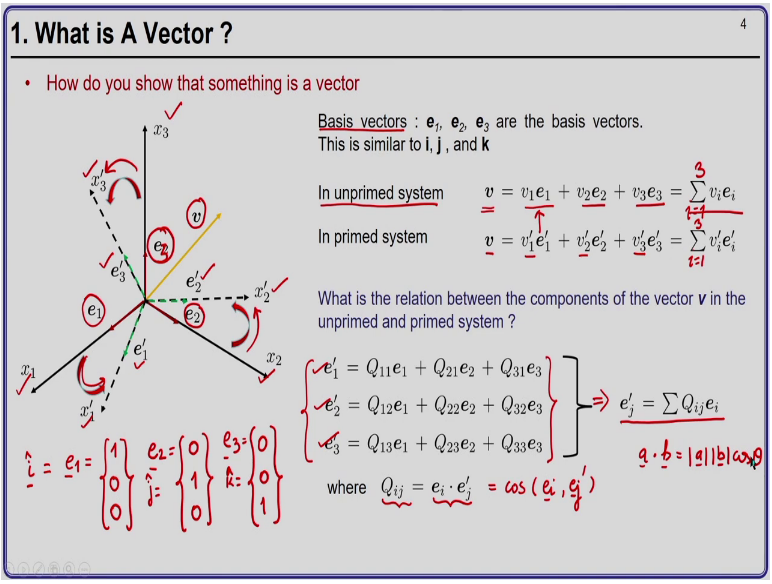

So, how can we do that? So, to do that, we first see that e 1 dash, e 2 dash and e 3 dash themselves are vectors which can be written in terms of the basis vectors e 1, e 2, e 3 and this is given by following relation ok.

You can see Q 11 e 1, Q 21 e 2, Q 31 e 3 is e 1 dash, e 2 dash is Q 12 e 1 plus Q 22 e 2 plus Q 32 e 3 like this, we can write all the three basis vectors in the prime system in terms of the basis vectors of the unprimed systems ok. So, in short, I can write e j dash as summation Q ij e i ok. I can write e j dash is Q ij e i ok, where this quantity Q ij is nothing but the dot product of the basis in the e i of the unprimed system with the basis e j dash in the prime system.

So, because both e i and e j are unit vectors, this would be nothing but the cos of the angle between e i and e j dash ok. Because you know that a dot b is nothing but magnitude of a into magnitude of b into cos of theta where theta is the angle between the vectors ok. Now, because if a and b are both unit vectors; therefore, the magnitude of both the vectors will be equal to one and then, cos theta is a dot b. So, e i dot e j dash is nothing but the cause of the angle between the basis vectors e i and e j dash ok.

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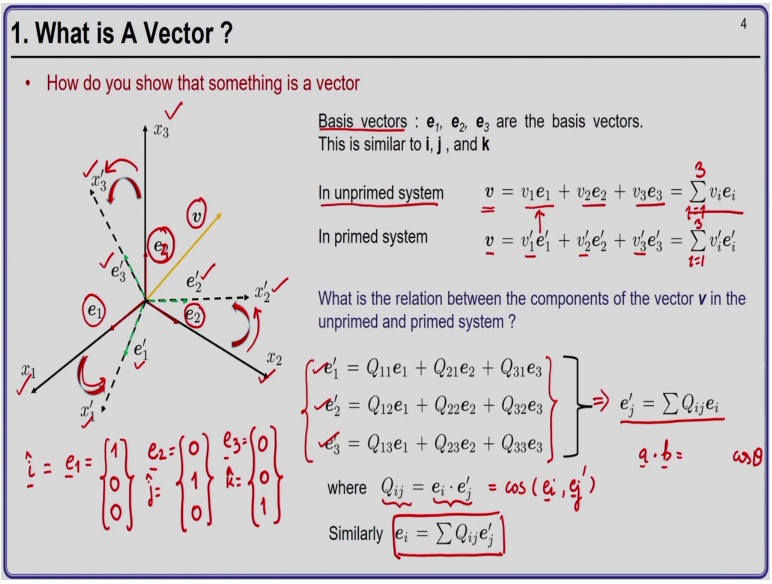

Similarly, you can turn this expression the other way around. Can you write e i in terms of e j dash? Yes, you can do that because e i themselves are vectors when you look from the prime system; you can see e i’s, where i goes from 1 to 3 as vectors. So, you can express them in terms of the basis vectors of the prime system given by following equation.

Once you have this equation, we have tools in our hand with which we can express the components of the vector v ok, we can find the relation between the components of the vector v in the unprimed and the prime system ok. To do that, we start with this definition of the vector ok.

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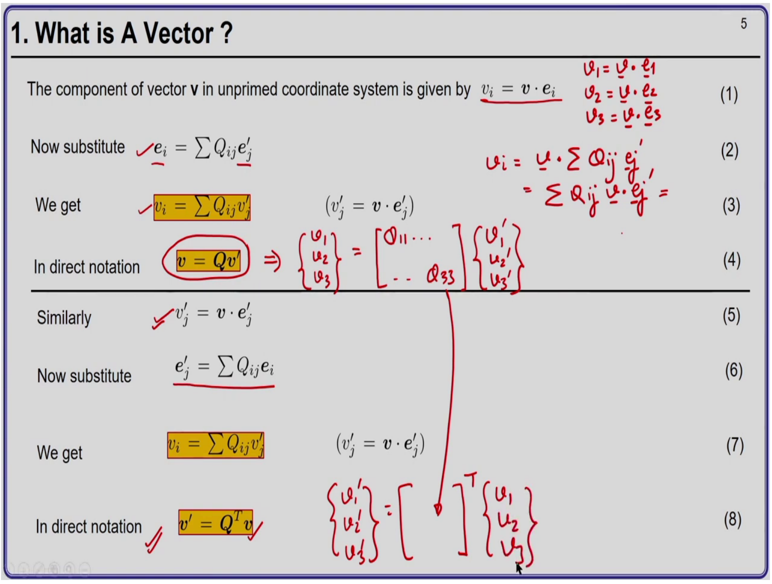

Now, you know that the component of a vector is nothing by nothing but the vector dotted with the basis vector along that direction ok, the unit vector along that particular direction. Now, so, v 1 would be nothing but the vector v dotted with e 1 right. You will get the component of the vector along the e 1 direction v 2 will be v dot e 2 and v 3 will be v dot e 3 right. So, once you have this, now you know the relation between e i’s and e j dash ok. This we saw in the previous slide. You can substitute this expression for e i, in the expression or equation 1 ok.

So, once you substitute that ok. So, vi is v dot Q ij e j dash ok. So, this I can write as sigma Q ij v dot e j dash. Now, what is v dot e j dash? It will be the component of the vector v along the j th direction in the prime system which is nothing but Q ij v j dash ok. So, this is what we get as part of our equation 3 ok. So, we can relate the component of the vector in the unprimed system which is vi to the components of the vector in the prime system which is v j dash ok. So, we can now express this relation in the matrix form ok, as vector v equal to some matrix Q times vector v dash ok. So, this is nothing but v 1, v 2, v 3 ok.

So, I am writing this equation 4 ok. We will have Q 11 like this, we will have Q 33. This will be a 3 by 3 matrix; v 1 dash, v 2 dash, v 3 dash. So, now, you have related the components of the vector in the unprimed system to the components of the vector in the prime system and Q is the matrix. You know how to compute the components of this matrix Q ok, which is nothing but the cos of the angles between the basis vectors. Now, we want to find out the relation between v dash and v ok.

So, to do that, we again can start with v j dash as v dot e j dash and now, we know e j dash is nothing but summation Q ij e i and using the procedure that we just now discussed, you can show that the relation between the components of the vector in the unprimed system and the prime system ok. So, unprimed system, the components of vectors are can be written as vector v and the prime system, we have v dash. So, v dash will be Q transpose v ok.

So, you will have v 1 dash, v 2 dash, v 3 dash equal to. So, this matrix which is here transpose v 1, v 2, v 3 ok. So, now, this is what ok, now you can check this relation with our definition of a vector which was summation a vector with satisfy the triangle law of vector addition here. So, there you needed two vectors, here we just can do we need only one vector v and using that you can derive the relation.

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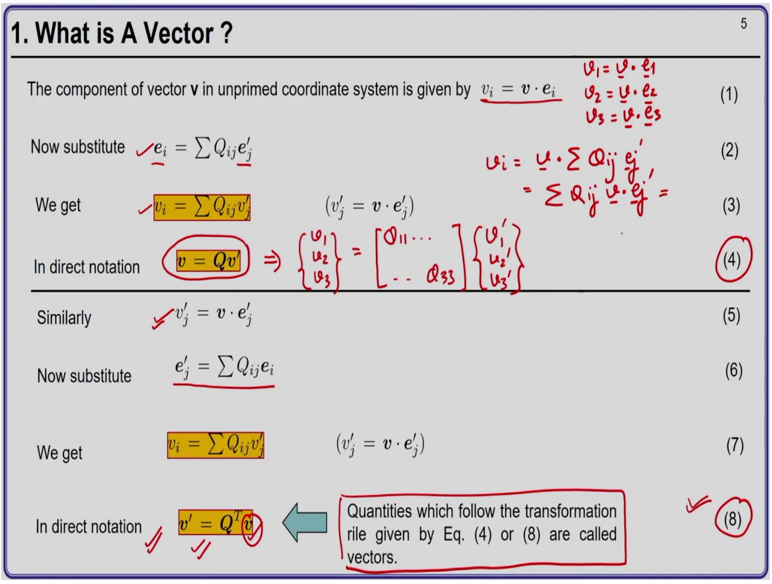

So, very important thing to note is that quantities which follow the transformation law rule. So, by given by equation 4 ok, the equation 4 and equation 8 are called vectors ok. So, any quantity if you are given and you are asked to show that whether that quantity is a vector or not. So, what you have to do? You have to show that components of that quantity transform according to the relation given by either equation 4 or equation 8.

We frequently use equation 8 ok, because the vector v is given to us and now, we start with vector in the prime system and I will try to show that the components of the quantity transform according to equation 8. If they transform according to equation 8, then that quantity will be called a vector, if that quantity does not the components of that quantity does not transform according to equation 8, then that quantity will not be called a vector ok.

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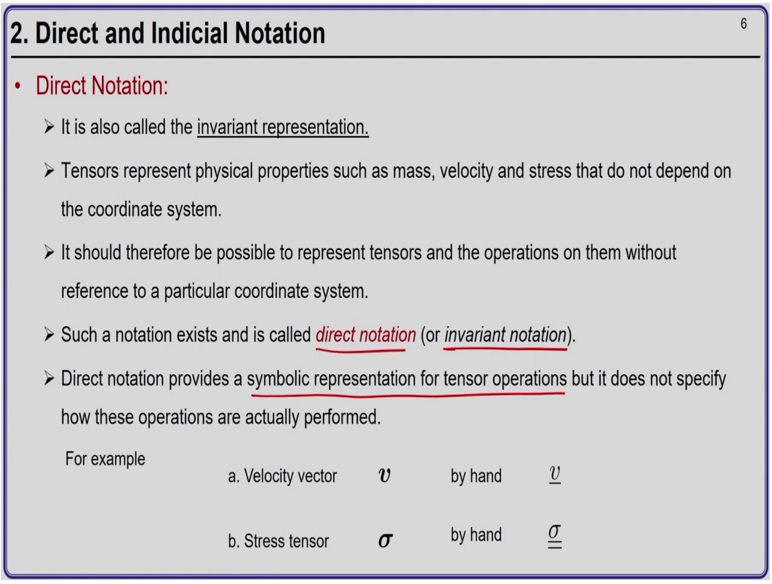

So, before we discuss more, we will try to see what is meant by direct notation and indicial notations ok. So, direct first we look into direct notation which is also called the invariant notation ok. So, it is important to point out here that tensors are physical quantities ok. They represent physical quantities like mass velocity stresses ok.

So, because they are physical quantities, they should not depend on the coordinate system ok. Because coordinate system can vary from person to person ok; one person can have one coordinate system, another person can use another coordinate system. But for example, mass; mass will not change with the coordinate system. The velocity will not change with the coordinate system ok.

So, the tensor should not change with the coordinate system in some way ok, but we do define coordinate system ok. So, there should be a way to represent the tensors and their operations without referring to any particular coordinate system ok. So, this kind of notation which is used to represent tensors and their operations without referring to the coordinate system is called what is called the direct notation or the invariant notation ok.

So, direct notation provides us symbolic representation of tensor operation ok. So, when you write the operation between tensors indirect notation, it is symbolic representation. However, it does not tell you how these operations will be actually perform ok. So, we look into some examples. For example, a velocity vector ok.

So, when we are typing on a computer is usually denoted by symbol v which is bold ok. However, when we write by hand because we cannot write a bold letter, what we do? As I told earlier, we will write symbol v and then, put a single bar we will come to it later why one bar ok.

Stress for example, when it is typed on a computer, we will put sigma which is bold. However, when we are writing the direct notation for stress will be sigma and then, there are two bars below it ok. We will come to it why we are putting one bar below velocity and two bars below the stress sigma ok.

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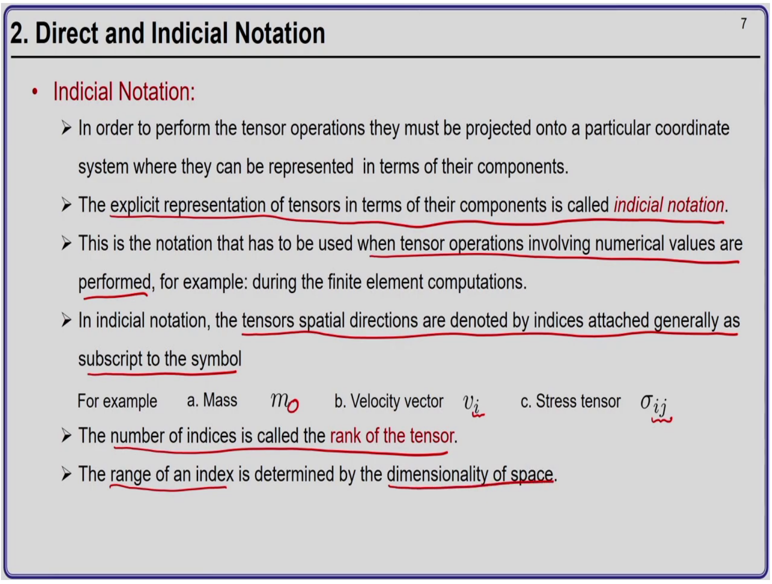

But that is how direct notation is used ok. So, there is another notation which is called the indicial notation ok. So, when you have to operate upon the tensors, when you have to do certain operations ok. So, then that time you will utilize a certain coordinate system and then, you will represent the components of tensor with respect to those coordinate system ok. So, the explicit representation ok; so, the explicit representation of tensors in terms of their components is what is called indicial notation ok.

So, these indicial notations are used when tensors operations involving numerical values are used ok. When you are doing some numerical computations, you will use indicial notations. For example: the trace of a stress tensor, when you are during finite element computation. So, when you have to do trace of a stress tensor, you have to use indicial notation ok. So, in indicial notation, the tensors, spatial directions are denoted by indices which are generally attached as subscript to the symbol ok.

For example, mass. Mass is a scalar quantity ok. So, there is no index subscript which is associated with it; velocity vector for example, there is one index subscript when we are writing velocity vector in indicial notation. So, there will be one index which is attached as a subscript. Stress tensor for example, has two indices as subscripts ok. So, the number of indices ok; so, the number of indices which are attached as subscript to the tensor when you are using indicial notation is what is called the rank of the tensor.

And what will the range ok. So, the range of this index for the velocity, we have i and tensor we have i and j two indices. So, what will the range? What values will these indices take? So, that will depend on the dimensionality of the space. For example, in 3D, I will take values from 1 to 3 ok. Similarly, in 3D, the stress tensors both i and j will take values 1 to 3. So, stress tensor in 3D will have total 9 components; sigma 11, sigma 22, sigma 33 all the way up to sigma 32 like this ok, there will be total 9 components.

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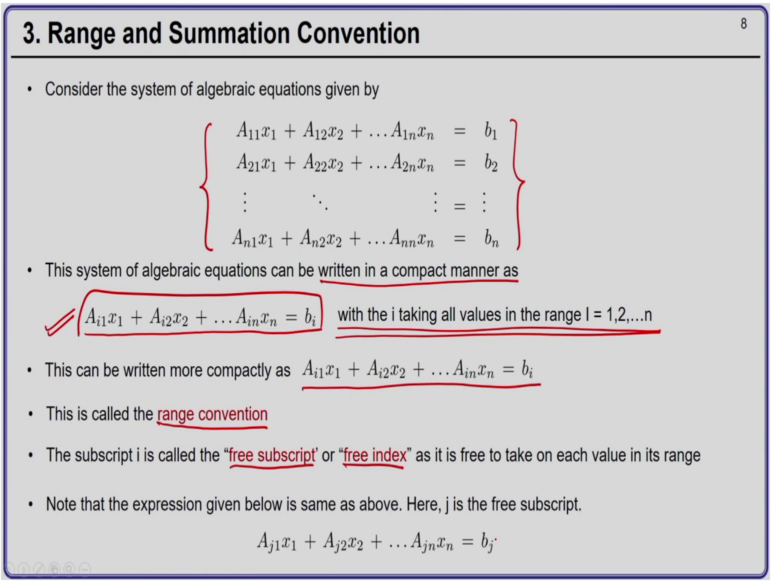

Now, we come to what is called the range and summation convention ok. Now, to study about the range and summation convention; it is important that instead of going to theory, we will start with an example and this will make you understand better how to write expressions using indicial notations ok. Because when you are doing computations and we are simplifying the expressions, you will be mostly writing in indicial notation. Once you have done the indicial notation, you will transform the indicial notation back to the direct notation ok.

So, suppose you are given a system of algebraic equation, n equations are given to you which are shown here ok. Now, if you have to write these equations many a time ok, it will be very cumbersome and very tedious to write all of these equations every time that you have to do certain computation.

So, would not it be better, if we can somehow condense all these equations into one small expression ok. So, this can be done using what is called the range and summation convention. So, the first step that you can do is we can write these equations in a compact manner given by A i1 x 1 plus A i2 x 2 all the way up to A in xn equal to bi with the i's taking all the values in the range 1 to n ok.

Now, n can be anything ok; n can be 10, n can be 3, n can be 1000 whatever. So, I will take all values. Now, you can see instead of these n equations, we just now have one equation and this is given here ok. Now, what we can do? We can drop this line ok, which we have to explicitly mentioned and we can just write the expression here, with the understanding that i will take its value in the range 1 to n. We need not specifically or explicitly tell that I will need to write that i will take values 1 to n ok; just writing this expression suffices. So, this is called range convention.

Because i takes value in its range 1 to 3; the range that is why name range convention ok. So, this index i is what is called the free subscript or the free index, add this because its free to take values in its range 1 to n ok. So, somebody can ask, why i, can I use some other index? Very well, you can use another index for example. If I use j, I can write A j 1 x 1 plus A j 2 x 2 all the way up to bj. The meaning of the expression remain same ok.

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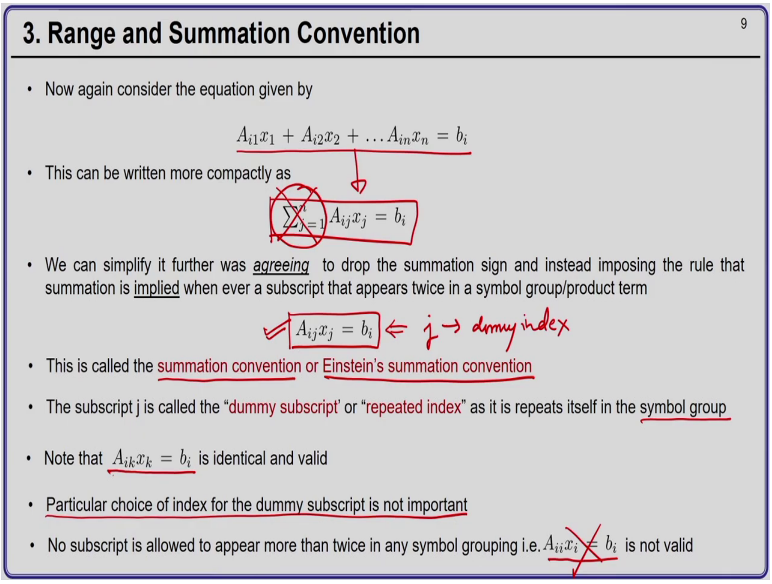

Now, I can further compress this expression, I mean this expression over here is still long, I mean it will be tedious to write. So, what we can do? I can use the summation sign and I can write summation j equal to 1 to n Aij xj equal to bi ok. Using this, I can write the top expression in a compact manner; but still there is a summation symbol. So, I can just compactly raise this expression by dropping the summation symbol ok.

So, this summation symbol over here can be dropped and I can simply write Aij xj equal to bi with the understanding that whenever there is an index which is repeated, then there is a summation which is implied ok. So, j for example, is repeated here. So, I can understand there is a summation which is implied over j ok. So, this convention is called the summation convention or Einstein summation convention ok. So, j here in this expression over here, j is called the dummy index ok; j is the dummy index or the repeated index ok. And Aij xj is called; so, each product term in an expression is called a symbol group ok. So, there are two symbol groups here Aij x j and bi ok.

So, each product term in an expression, it will be a symbol group. So, anybody can ask why j, can I use some other symbol? Yes, you can use instead of j, you can use k. So, you can write instead of Aij xj, you can write Aik xk equal to bi and this itself is were identical to the previous expression and its valid expression ok.

So, a particular choice of index for the dummy subscript is not very important ok. Now, you also should know note that no subscript is allowed to appear more than twice in any symbol grouping ok. So, you cannot write Aii xi equal to bi, this is not a valid expression ok; you should note that you cannot write an index more than twice ok.

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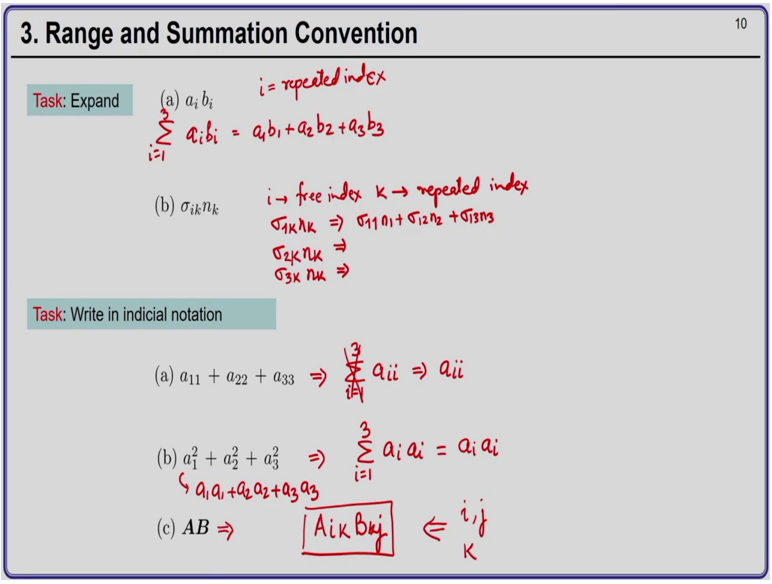

So, this brings us to some example. The first one is can you expand the first one is ai bi and I want to expand this expression ok. So, i for example, here is a repeated index and there are no free index here ok. So, i is a repeated index.

So, by a summation convention, it is implied that there is a summation which should be carried out over i and i should take all values from 1 to n. If something is not mentioned, then we usually take the rain as 3 ok. So, i will go from 1 to 3. So, I can write this as this actually means i equal to 1 to 3 ai bi is which is nothing but a 1 b 1 plus a 2 b 2 plus a 3 b 3 ok.

So, there is one single expression which is a 1 b 1 plus a 2 b 2 plus a 3 b 3. So, the next example is sigma ik nk ok. So, we note here that i, here is a free index and k, here is a repeated index. So, i will take values in its range 1 to 3 and k will take its values in its range 1 to 3, as I mentioned. If nothing is mentioned, we will take that the range goes from 1 to 3 ok.

So, since i is a free index and I will i takes value 1 to 3. So, this is basically three equations sigma ik nk, sigma not ik sigma 1k nk, sigma sigma 2k nk and sigma 3k nk. Now, each of these expression, you can further use the summation convention to write sigma 11 n 1 plus sigma 12 n 2 plus sigma 13 n 3 ok. So, the rest two, I leave it to you. So, you will can write the other two expression. So, there are total three expression, each expression has three symbol group or the product terms ok.

Now, we go the other way around given an expression a 11 plus a 22 plus a 33 ok. I can write summation i equal to aii and using summation convention, I can write I can drop the summation symbol and I can write simply aii ok. So, that is my expression in the indicial notation. Now, the second expression is a 1 square plus a 2 square plus a 3 square.

So, this I can write summation i equal to 1 to 3. Now, one thing which is never done when writing in indicial notation is we never write the power, all these term have a power 1 ok. So, here we have been given power which is 2. So, basically this expression is nothing but a 1 into a 1 plus a 2 into a 2 plus a 3 into a 3 ok. So, I can write this as ai into ai and this is nothing but ai ai ok. So, as I mentioned; so as I mentioned, we do not write power ok, all terms have power 1. So, we do not write ai square, we have to write ai into ai ok.

Now, the last expression is you have say a matrix A and another matrix B ok. These are 3 by 3 matrix. So, product of two will be a 3 by 3 matrix. Now, suppose you have to write in the indicial notation ok. So, this is written as A ik B kj. So, now, for the last one, I leave it to you; you write explicitly all the expression, all the terms for product of matrices A and B and finally, reduce.

So, if you have A as 3 by 3 and B as 3 by 3; you will have three equations with each equation having three terms ok. You will have three expressions and you should be able to compress using the range in summation convention, the three expressions into what I have written here A ik B kj ok. So, this expression has two free indices which are i and j ok.

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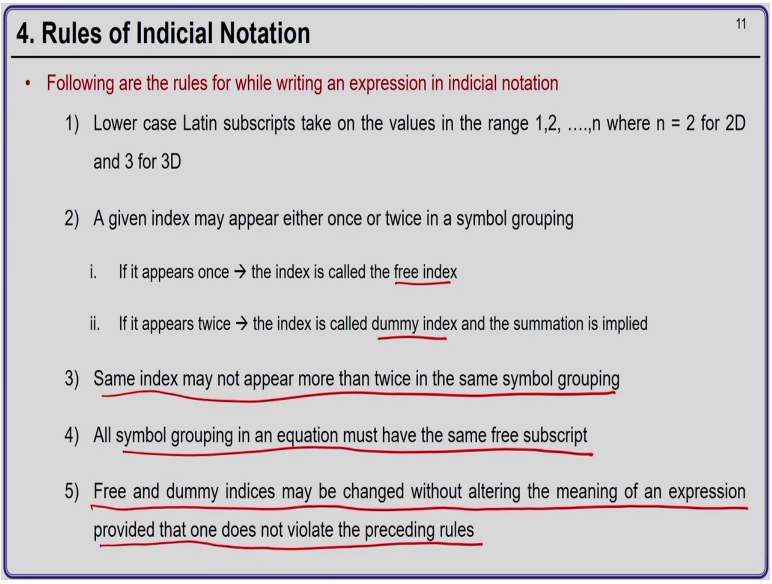

And there is on one repeated index which is k. So, with this, we move to our rules for indicial notation. So, when you are using indicial notation there are certain rules that you have to follow and these rules are stated as five points here in this slide. The first point is the lower case Latin subscripts take the values in the range 1 to n; where, n will be equal to 2 for 2D problems and 3 for 3D ok.

A given index may appear either once or twice in a symbol grouping or a product term ok, say every expression will have product terms which are also called symbol grouping. So, a given index can appear once or it can appear twice, not more than that. If it appears once, the index is called a free index.

And if it appears twice, it is called a dummy index or the repeated index and then, summation will be implied the same index may not appear more than twice in the same symbol grouping. So, in one symbol grouping a particular index cannot occur more than twice ok. So, either it can be a free index which is it can occur once or twice which is then, it means that that index is a repeated index or a dummy index. So, all symbol grouping in an equation must have the same free subscript ok.

So, it may happen that your expression is long and you have multiple product terms ok, but in each product term the free index should be same ok. In the first product term, if the free indexes i, in the second product term also it will be i, the third product term also it will be i like that and you can have as many free index as you want.

But it can occur only once within each product term and for all the product terms, it can the free index should be same ok. And the free and the dummy indices maybe change without altering the meaning of an expression provided that one does not violate the preceding four rules ok. You can interchange the free and dummy indices you can make i equal to j, j equal to k, whatever j equal to i; but you should keep in mind that the first four rules should not be violated ok.

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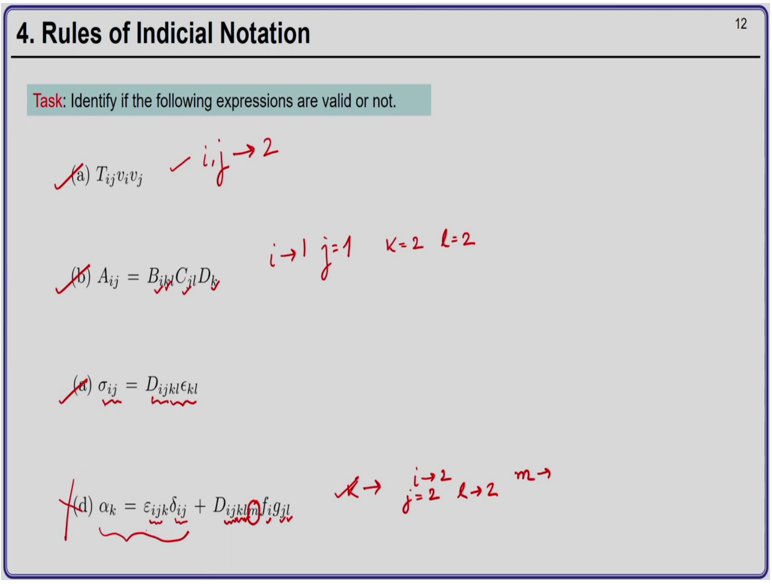

So, with this, we move to few examples and the four examples and we wish to identify whether these expressions are valid or not. The first is T ij vi vj, whether this expression is valid or not? So, we can see that both i and j. So, there is one symbol grouping here only one product term T ij vi vj and both i and j are repeated twice.

There are two i's and two j's and there are no free indices ok. So, all the four rules that we mentioned in the previous slide are valid. Therefore, this is a valid expression. Now, we come to the next expression which is Aij equal to Bi kl Cjl Dk. Here, we have two product terms or two symbol grouping; first is Aij and the another is what is on the right hand side.

So, we count the number of free indices ok. So, i occurs; so in the first symbol group, i occurs once and j occur one ok. So, we should check in the second symbol group whether we have i and j and they should occur only once. So, i occurs in the subscript of B once, fine enough j occurs in c once ok. So, that is fine.

Now, we have k which occurs in the second symbol grouping twice and we have l in the second symbol grouping which occurs twice ok. So, we have free indices i and j occurring in all the symbol groupings and the repeated index k and l in the second symbol group repeated twice. So, all the four rules are followed. So, this expression also is a valid expression.

The next expression is sigma ij equal to D ij kl epsilon kl. This is similar to the second example, here also i and j are the free indices. You can see here and kl n are the dummy indices. Therefore, this also is a valid expression. So, the last expression now is alpha k equal to epsilon ijk delta ij plus D ijklm f i g jl ok. Now, this expression has three symbol groupings or three product terms; one on the left hand side and two on the right hand side ok.

So, if you look into the first term alpha k. So, k occurs here once. So, k has to be a free index ok, obviously because it occurs once in the. So, we should check in the other two symbol grouping whether k occurs once. So, in the second expression, we can see k occurs here and here also k occurs here only once ok. So, that is valid k is a free index. Now, we check for whether we have the repeated index occurring properly. So, i and j in the second symbol grouping occur twice ok. So, that is fine. They are repeated index.

In the second symbol grouping on the right hand side i j k l ok, we have i j k l, also we have i, we have j and we have l ok. So, you can see here ok. So, k l occur and now, you have m ok. So, in the second expression also you have i which is occurring twice, j which is occurring twice, you have l which is occurring twice, remember k was a free index.

Now, the only index which is left is m which is occurring only once. If index occurs only once, it has to be a free index, so it should occur in the other symbol group; but if you check on the other two expressions, there is no m involved ok. So, because m is a free index for the third expression; but it does not occur in the other symbol grouping.

Therefore, this is not a valid expression. Because m being a free index in the second expression is not present in all other terms ok. So, if you look into these example, the first three are your; the first three are your valid expression and the last one is your invalid expression ok.

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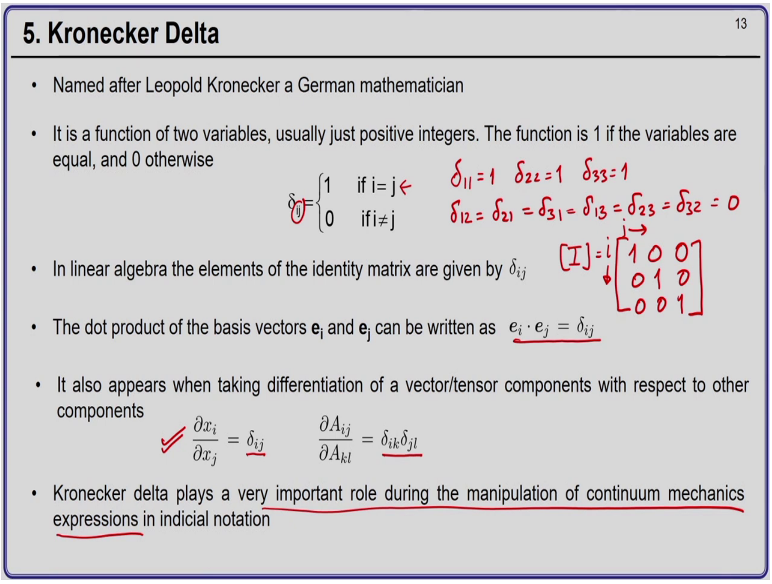

The next concept that we look is called the Kronecker delta. It is named after the German mathematician Leopold Kronecker and it is a function of two variables; usually just the positive integers and the value of this function is 1, if the indices ok, if the values, if the variables are equal and 0 otherwise ok. A Kronecker delta is usually denoted by the symbol delta followed by 2 indices i and j ok.

So, if i is equal to j, then the value of delta ij will be equal to 1 and if i is not equal to j, then delta ij will be equal to 0 and i and j will take values from 1 to 3. Therefore, delta 11 will be equal to 1, delta 22 will be equal to 1 and delta 33 will be equal to 1, delta 12 equal to delta 21 equal to delta 31 equal to delta 13 equal to delta 23 equal to delta 32 will be equal to 0 ok.

So, you can see if the indices are repeated if i is equal to j, then we have the value of Kronecker delta which is 1, otherwise it is equal to 0 ok. So, in linear algebra, the elements of identity matrix can be written in this form delta ij ok. So, i an identity matrix of order 3 will be 0 1; 1 0 0, 0 1 0, 0 0 1. So, you can see when row is equal to column, when i is equal to. So, if you consider i's as your rows and j as your column. So, when i is equal to j, you have the value as 1 ok. Sorry. The last one should be 1; 0 0 1.

So, for 11 22 and 33, you have value of identity matrix which is equal to 1, otherwise it is 0 ok. So, the indicial notation for a identity matrix is nothing but delta ij ok. Also, the dot product of any two basis vectors ei and ej; ei ej can be anything e 1, e 2, e 3 ok. So, ei dot ej will be nothing but delta ij in Cartesian coordinate system. So, if you take i equal to j equal to say 1. So, e 1 dot e 1 will be nothing but 1 ok. But e 1 dot e 2, i and j; i is 1, j is 2.

So, e 1 dot e 2 will be equal to 0 because cos of the angle between e 1 and e 2 is 90 degree. Therefore, cos 90 is 0; so, e 1 dot e 2 is 0. So, you can write all this in a compact notation as ei dot ej equal to delta ij that is one another place, where Kronecker delta shows up. The other places where it appears is when you are taking differentiation of a vector or a tensor with respect to its own component.

For example, if you write del xi by del x j, it can be written as delta ij ok. If you have a second order tensor or rank two tensor which has two indices. So, del A ij del A kl; A ij correspond to ij'th component of the tensor A. So, del A ij by del A kl will be nothing but delta ik delta jl. So, you remember that Kronecker delta plays a very important role during the manipulation of continuum mechanics expression in indicial notation.

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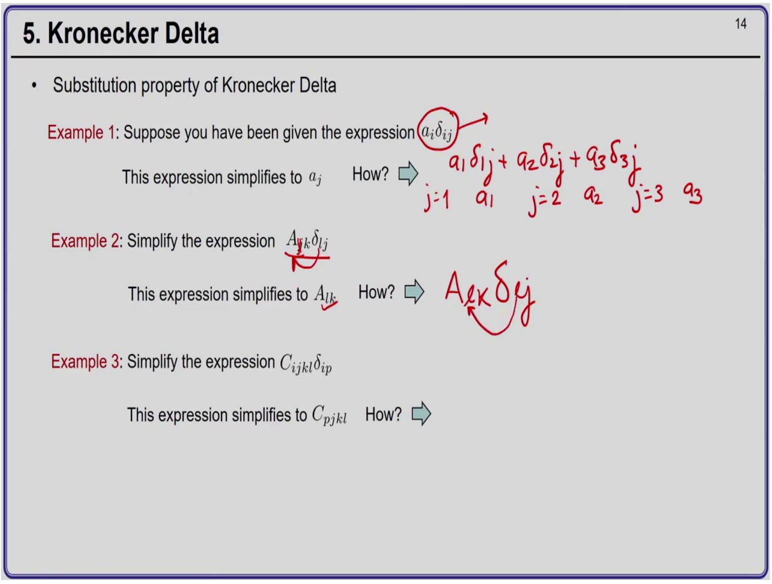

It will occur frequently and you have to use its property which is the substitution ok. So, Kronecker delta is used for substitution ok. So, now, we look into three examples ok. Suppose, you have been given the expression A ij delta ij ok. So, this expression simplifies to a j how? A ij delta ij, so this has i as a repeated index. So, there is a summation which is implied and i will take value 1 to 3. So, we will write delta a 1 delta 1 j plus a 2 delta 2 j plus a thr3ee delta 3 j ok.

Now, when j has value of 1, when j has a value of 1, then delta 11 will be equal to 1, but delta 21 delta 31 will be equal to 0. So, when j equal to 1, we are only left with a 1. When j is equal to 2, we are left with only a 2. When j is equal to 3, we are left with only a 3 ok. So, you can see in this particular expression Kronecker delta has replaced the indices of a by j that is called the substitution property. The property of j has been substituted ok. I mean the i has been substituted with j. Now, if you have another expression a j k delta l j ok.

So, this again you can see the common index between the Kronecker delta and a is j. So, the final expression will be this j, from a will be taken out and l will go and take its place. So, the final expression will be A lk ok. So, you have a jk delta lj ok. So, what will happen? This j will be taken out. And then, this l will go and fill its substitute for that j over here and then, Kronecker delta also goes away. Similarly, the third expression C ijk; C ij kl delta ip ok.

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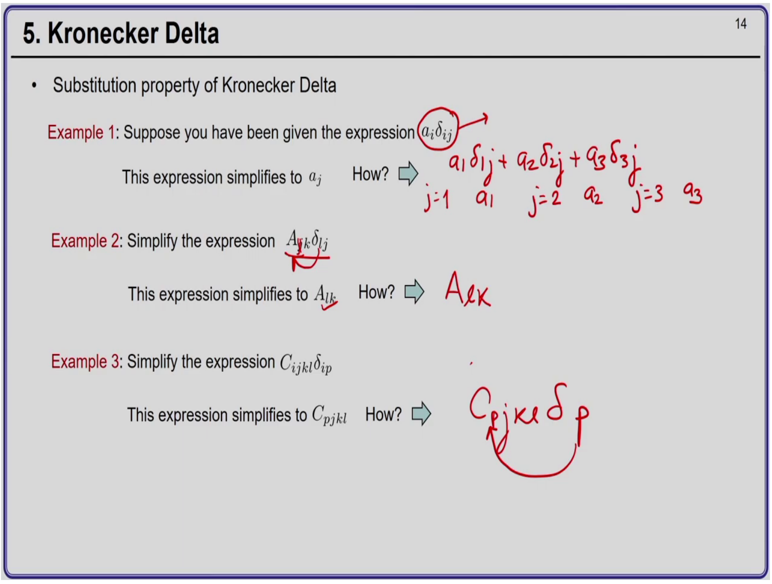

Here, you have i which is repeated ok. We have C ij kl delta ip. So, what will happen? i will drop off and p will go and take the place of i in C. So, you will have p here and then, delta drops off ok.

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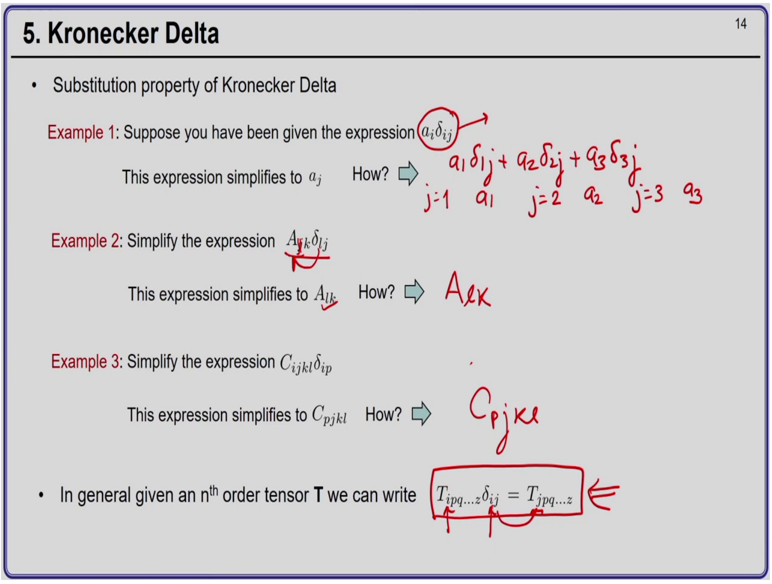

So, in general, these substitution property for Kronecker delta in general can be written in this particular is given in this particular expression ok, where i is replaced by j. You can see i has been replaced by j ok, where T is any nth order tensor; it can be vector, it can be second order tensor, fourth order tensor whatever. The repeat the index which is common between the Kronecker delta and the term which is coefficient of the Kronecker delta gets replaced by the other index of the Kronecker delta which is j here ok.