Nonlinear Vibration
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Lecture - 10
Method of Averaging

Welcome to today class of Non-Linear Vibration. So, in this module we are studying how to solve this differential equation or how to solve the non-linear differential equation used for our purpose. So, already we know different type of non-linear equations and also different methods to solve these equations.

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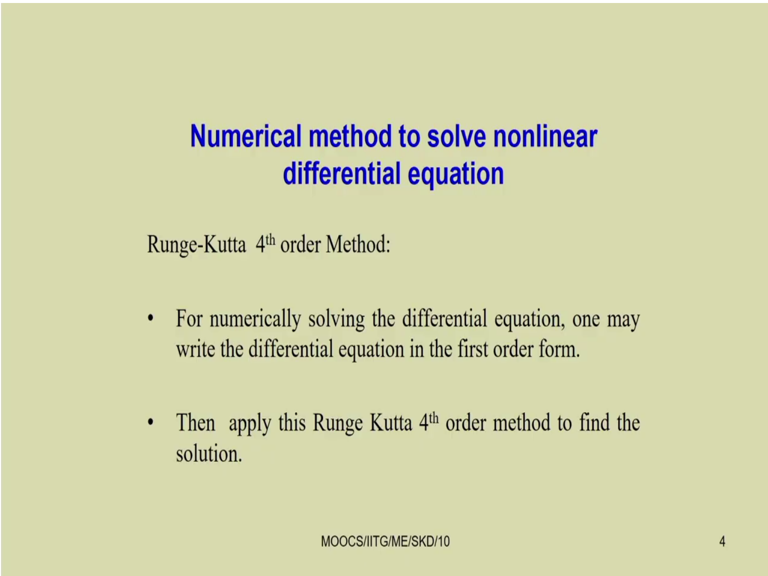

Particularly we may use this numerical method. So that is by using this Runge Kutta method, so we can solve any differential equation for example.

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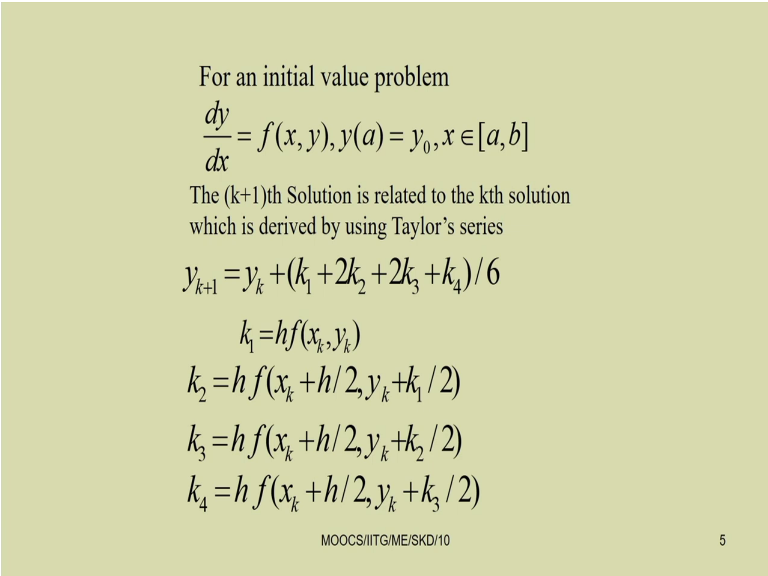

So, if you have a second order differential equation. So, we can first convert that into a set of first order equation and then by using these; 4th order Runge Kutta method. So, we can solve numerically to find the response of the system.

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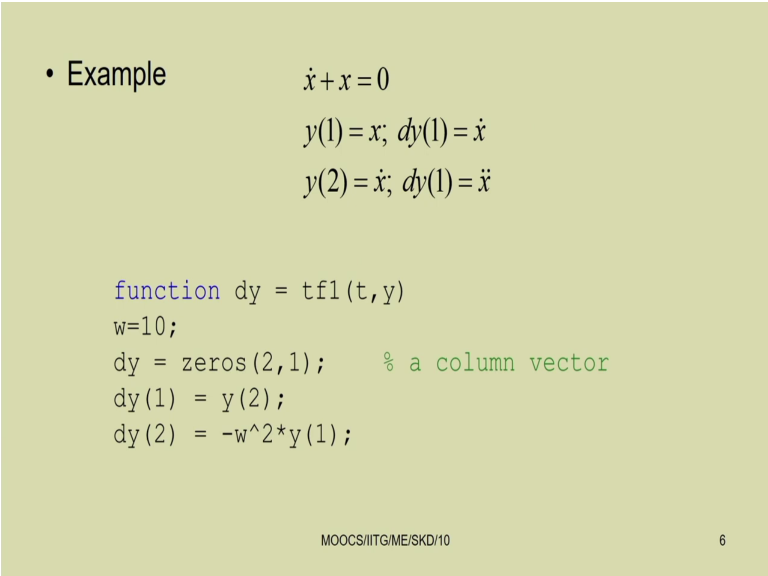

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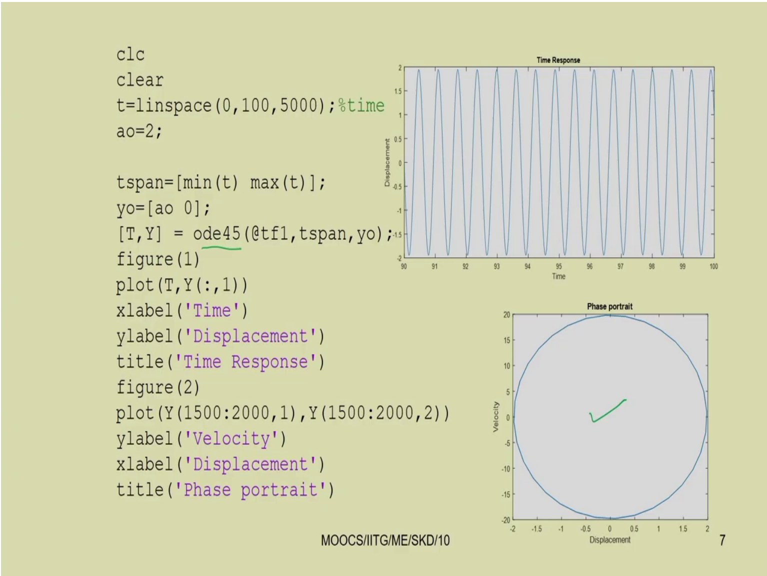

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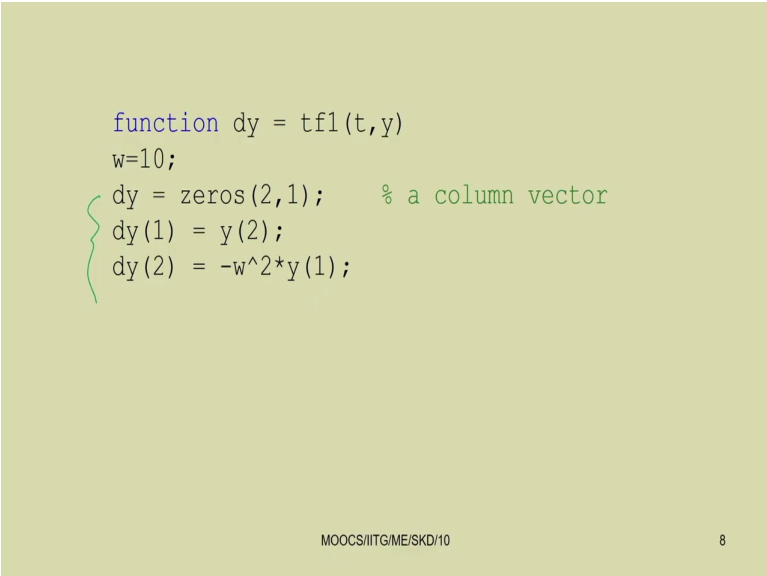

So, we have taken few examples and we have studied how we can use these Runge Kutta method for finding the response of a system. So, we I have shown also one MATLAB code; how to plot or how to solve by using this ode45. So, ode45 command in MATLAB you can use. So, you can write a function file.

So, in the function file you can write the differential equation in the form of first order differential equation. And after writing that thing then you can use this ode 45 function to find the response then you can plot the response. So, you can either plot this displacement versus time or this velocity versus displacement. So, this is known as your page portrait or state space also sometimes it is known as state space.

So, if you are writing if you are for example, you have a you have 3 equations or 4 equation for first order equation. So, any two state vector you can take and you can plot. So, that way any state vector you can take and plot the time response also. So, you can plot the time response you can plot the page portrait or state space and you can study the system. So, this way you can use the numerical method to solve the differential equation.

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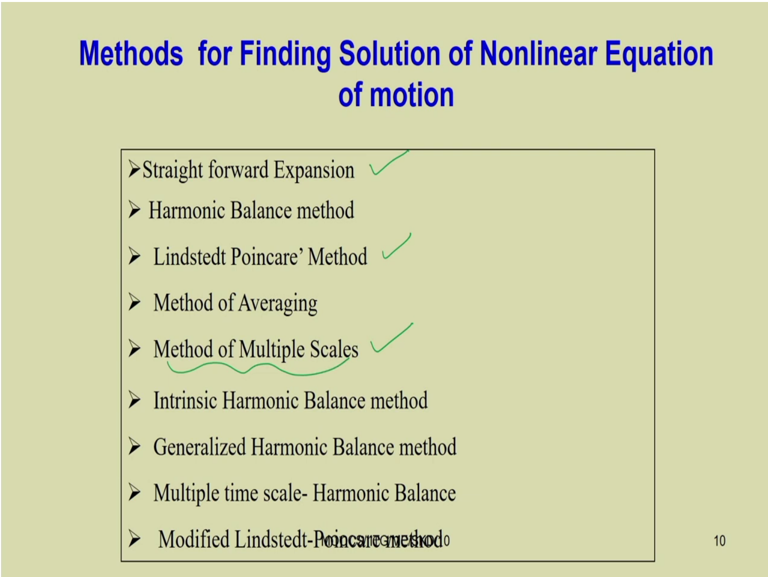

And there are several methods also I have told you there are several methods for finding the solution of this non-linear equation of motion in a approximate way. So, out of that thing we have already studied these straightforward expansion, so you can use a function. So, or you can expand the displacement term or the state vector and by substituting that displace for that expanded form you can study the response of the system.

So, we have found that; so in this method that even it is the presence of the secular term and one has to eliminate those secular term by using some other method like these Lindstedt

Poincare method and method of multiple scales also. So, also we have seen the Harmonic Balance Methods; so where you can take several harmonics the solution of the state variable.

And then by substituting that harmonics and collecting the coefficient of the sin and cosine part; so we will get a set of algebraic equation. So, by solving those set of algebraic equation, so you can find the response of the system. So, today class particularly we will be interested to study this method of multiple scale.

So, already we have studied this harmonic straight forward expansion, harmonic balance method and Lindstedt Poincare method which briefly also we will review.

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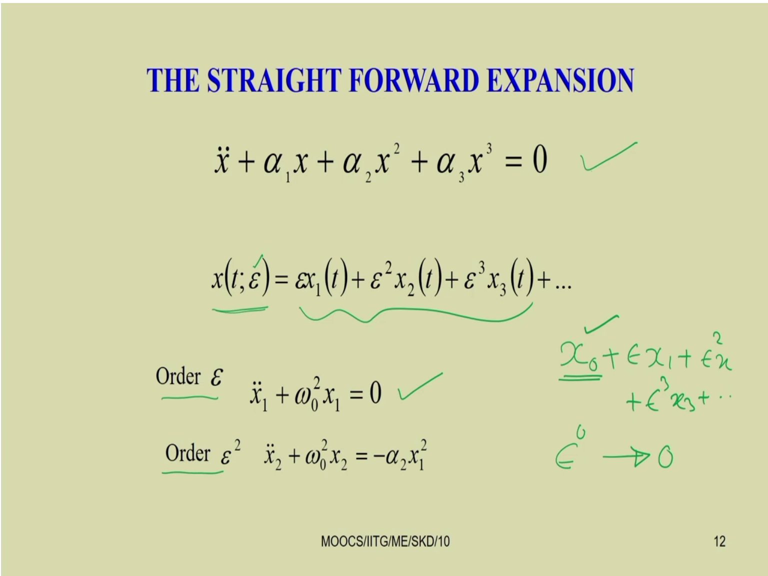

So, in straightforward expansion for example, you have taken this equation. So, here we have expanded this state variable x t. So, you just see it is written x t semicolon epsilon. So, the semicolon is written to separate a variable with a parameter. So, epsilon is the parameter and that is why we are putting the semicolon otherwise if it is a variable then we can put a comma there.

So, x ts semicolon epsilon so that thing we have expanded in this form epsilon x 1 t, epsilon square x 2 t, epsilon qx 3 t and substituting it in the original equation and then by separating the terms with different order of epsilon. For example, order epsilon to the power 1, epsilon to the power 2, epsilon to the power 3.

Sometimes you may write this equation also by putting a constant x 0 plus epsilon x 1 plus epsilon square x 2 plus epsilon q x 3; so this way also you can write. So, here you can have a constant term, so then in that case you have to separate the term with epsilon to the power 0 that is the constant term and equate it to 0. So, this term also we have to equate to 0.

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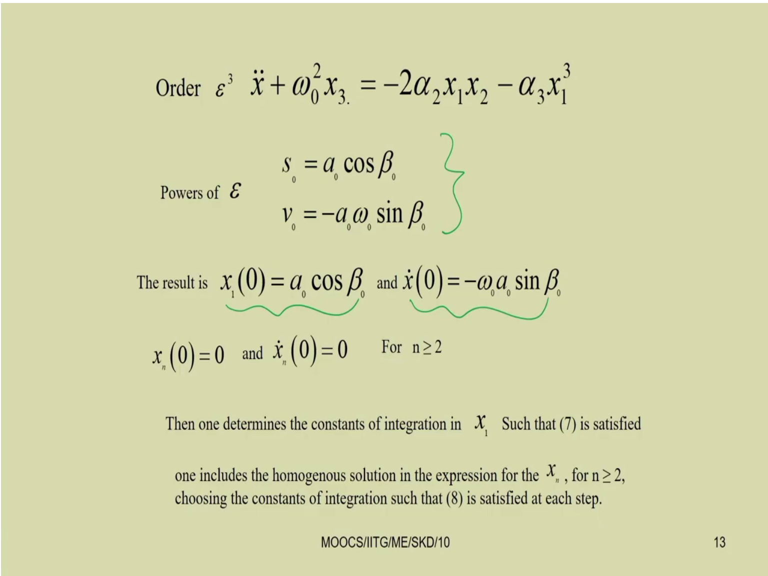

Similarly, order of epsilon; order of epsilon square and order of epsilon cube so you can separate. And then so from each term so you can get for example, order of epsilon so you know the solution of this thing. So, in this case simply this x 1 double dot will be equal to minus omega 0 square x 1. So, the so this x 1 double dot is similar to your acceleration and x 1 is displacement the motion must be harmonic.

So, as acceleration is proportional to this displacement and takes place in a direction opposite to that of displacement. So, the motion is harmonic. So, the solutions you can write in a harmonic form I using either the sin or cosine term or you can use these exponential function to express the solution of x 1. Then after getting that solution for example, you can write this way x 0 or this is your x 1 you can write.

So, now you can put this x 1 0 equal to this thing and x 1 dot equal to these and substituting them in the other order then you can find the solution.

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The general solution of (3) can be written in the form $x_1 = a\cos(\omega_0 t + \beta)$

$$\ddot{x}_2 + \omega_0^2 x_2 = -\alpha_2 a^2 \cos^2(\omega_0 t + \beta) = -\frac{1}{2}\alpha_2 a^2 \left[1 + \cos(2\omega_0 t + 2\beta)\right]$$

$$x_2 = \frac{\alpha_2 a_0^2}{6\omega_0^2}\left[\cos(2\omega_0 t + 2\beta_0) - 3\right] + a_2 \cos(\omega_0 t + \beta_2)$$

$$x_2 = \frac{\alpha_2 a^2}{6\omega_0^2}\left[\cos(2\omega_0 t + 2\beta) - 3\right]$$

$$x = \varepsilon a_0 \cos(\omega_0 t + \beta_0) + \varepsilon^2 \left\{\frac{a_0^2 \alpha_2}{6\omega_0^2}\left[\cos(2\omega_0 t + 2\beta_0) - 3\right] + a_2 \cos(\omega_0 t + \beta_2) - \right\} + o(\varepsilon^3)$$

$$x = \varepsilon a \cos(\omega_0 t + \beta) + \frac{\varepsilon^2 a^2 \alpha^2}{6\omega_0^2}\left[\cos(2\omega_0 t + 2\beta) - 3\right] + o(\varepsilon^3)$$

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$$a = A + \varepsilon A_2 + ..., \qquad \beta = B_0 + \varepsilon B_1 + ...$$

Then

$$\varepsilon a \cos(\omega_0 t + \beta) = (\varepsilon A_1 + \varepsilon^2 A_2 + ...)\left[\cos(\omega_0 t + \beta_0)\cos(\varepsilon B_1 + ...) - \sin(\omega_0 t + B_0)\sin(\varepsilon B_1 + ...)\right]$$

$$= \varepsilon A_1 \cos(\omega_0 t + B_0) + \varepsilon^2\left[A_2 \cos(\omega_0 t + B_0) - A_1 B_1 \sin(\omega_0 t + \beta_0)\right] + o(\varepsilon^3)$$

$$= \varepsilon A_1 \cos(\omega_0 t + \beta_0) + \varepsilon^2 \left(A_2^2 + A_1^2 B_1^2\right)^{1/2} \cos(\omega_0 t + \theta_2) + O(\varepsilon^3)$$

Where $\theta_2 = B_0 + \tan^{-1}\left(\frac{A_1 B_1}{A_2}\right)$ We can choose $A_1 = a_0, B_0 = \beta_0$

A_2 And B_1 Such that $\left(A_2^2 + A_1^2 B_1^2\right)^{1/2} = a_2$ and

$$\beta_0 + \tan^{-1}\left(\frac{A_1 B_1}{A_2}\right) = \beta_2$$

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So, you can see that sometimes the straightforward expansion contain the term which will be unbounded.

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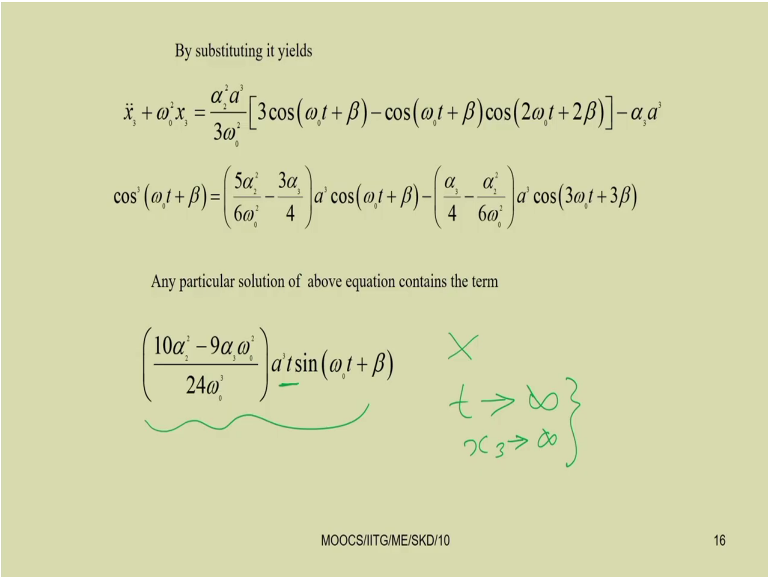

So, you can see this term particularly these term you just see the solution of this term as this contained this t term. So, as t tends to infinity so the x 3 tends to infinity, but in actual case the response is bounded. So, the direct straightforward expansion is not giving the correct result; so it has to be modified.

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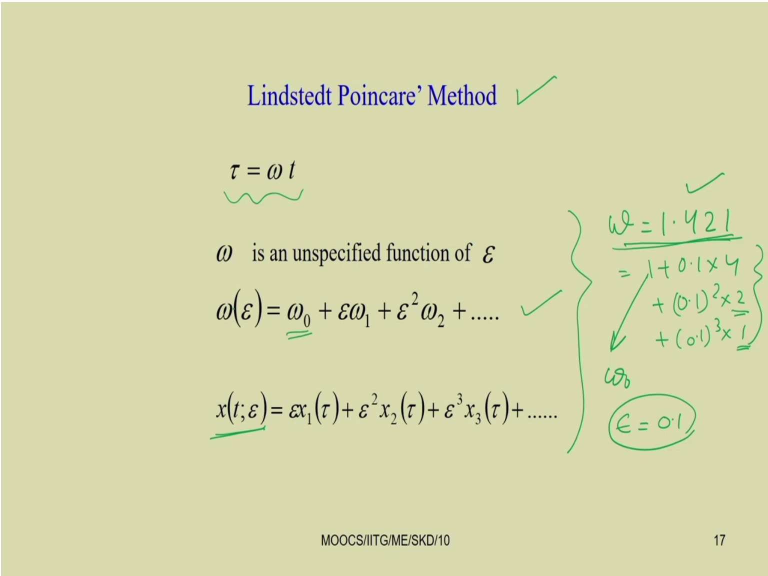

Then we have modified that thing by using this Lindstedt Poincare method. So, in this Lindstedt Poincare method now you are using a non dimensional time; non dimensional time tau; tau equal to omega t and here so you can expand this omega in this form. Previously you have expanded only x now you are expanding this frequency also.

Frequency we are writing equal to omega 0 plus epsilon omega 1 plus epsilon square omega 2 physically to interpret these things. So, for example, let you assume that your omega equal to 1.421. So, in that case this omega you can write equal to this 1 plus 0.1 into 4 plus 0.1 square into 2 plus 0.1 into 3 into 1; so this way you can write you can expand.

For example if your actual omega equal to 1.421 so in that case; so this 1 represents the omega 0; so this is omega 0. So, then this 4 is omega 1, then this 2 is omega 2 and 1 will be omega 3 and here epsilon will be nothing but this 0.1. So, this way you can expand you can write these omega equal to omega 0 plus epsilon omega 1 plus epsilon square omega 2 plus epsilon cube omega 3 that way you can go on expanding.

So, when you are not expanding or you are writing only omega 0. So, there is a chance that we may miss some of the higher order terms. For example, these 2 and 1 we may miss or only we can write d equal to 1.4; so this way by expanding this writing this omega equal to omega 0 plus epsilon omega 1 plus epsilon square omega 2 and x t epsilon equal to epsilon x 1 plus epsilon square x 2 plus epsilon qx 3.

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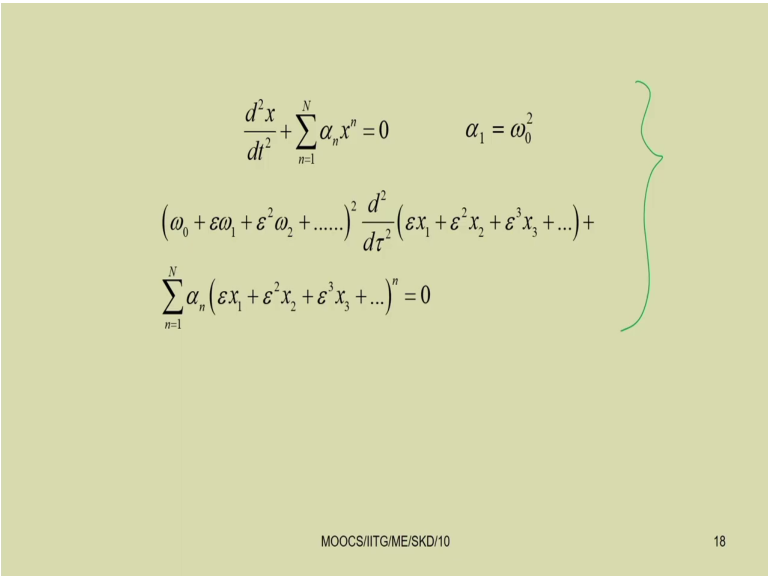

Then taking any differential equation given differential equation, so we can substitute there.

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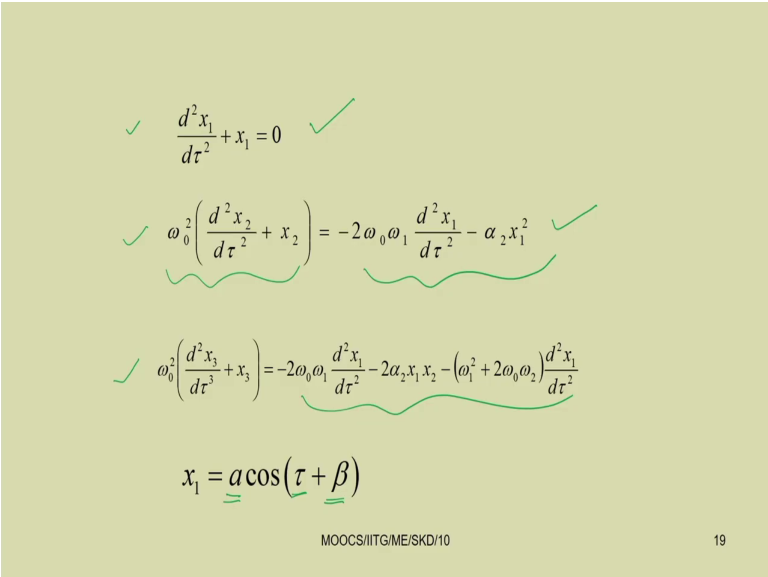

And then following the similar method as that of the straightforward expansion; so we can write this d square x 1 by d tau square plus x 1 equal to 0, so that is order of epsilon. So, order of epsilon; so this is order of epsilon this is order of epsilon square and this is order of epsilon cube.

So, now the solution of order of epsilon that is d square x 1 by d tau square plus x 1 equal to 0 is nothing but x 1. So, this is nothing but these x 1 equal to a cos tau plus beta here the coefficient is 1 as the coefficient is 1, so here the coefficient of tau is also 1. So, the response becomes x 1 equal to a cos tau plus beta. So, this is second order differential equation; so the solution must contain two constants.

So, a and b are constant; so which can be obtained from the initial condition. So, after getting this x 1 or knowing the expression for x 1 we can substitute that thing in the second equation that is order of epsilon square. You just see we have to write it in such a way that in the left side we have; so in the left side we can put all the terms with x 2 and right side all the known terms because x 1 is already known, so right side we are putting all the x 1 term.

Similarly, now by substituting this x 1; so we can find the particular integral of this thing so particular integral to represent this x 2. Now, substituting the expression for x 1 and x 2 in this third equation, so we can also find the expression for x 3 knowing this expression for x 1 x 2 and x 3 then we can find the solution.

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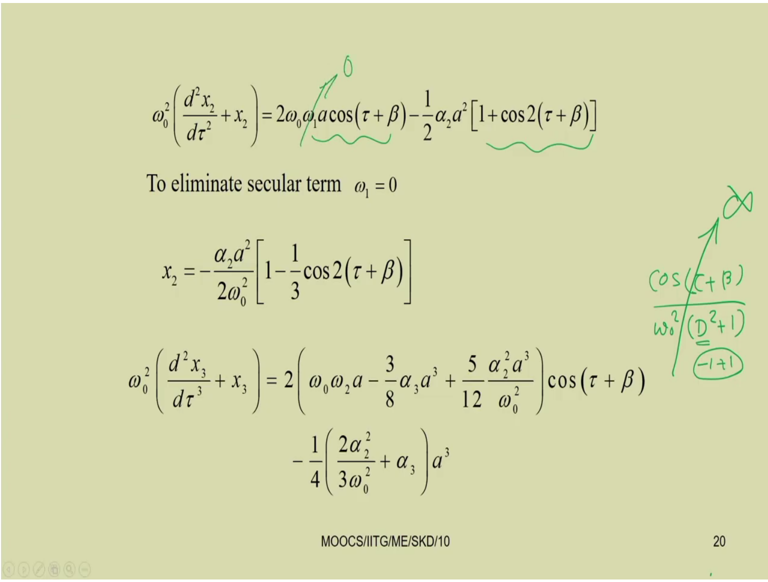

So, here you just see; so when we have substituted the first term. So, we have a term here at omega 0 square d square x 2 i d tau square plus x 2 equal to 2 omega 0 omega 1 a cos tau plus beta and this term. So, you just note that the particular integral of this part that is equal to as it contains a term cos tau plus beta. So, the particular integral, so which will be divided by omega 0 square into D square plus 1.

So, in place of D square we have to substitute minus. So, this is minus omega square here omega equal to i; so this is cos tau plus beta. So, for these things so we have to substitute these D square equal to minus 1. So, minus 1 plus 1 so this becomes 0 so this is 0. So, the denominator we have 0; so these term tends to infinite.

So, due to the presence of this term the response will go to infinite, but in actual case our response of our system is bounded. So, the response of the system is bounded. So, these terms will not be there. So, as these terms will not be there; so we have to eliminate these term. So, eliminating the secular term, so you can see this term can be eliminated. So, already all the other terms are nonzero. So, it will be eliminated if omega 1 is 0.

So, this way the unknown omega 1 so you just see we have omega 0 is only known to us this omega 1 omega 2 all those terms are not known to us. So, by eliminating the secular term as we have explained before; so we can get omega 1 omega 2 and other terms and proceeding that way; so we can get the solution.

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To eliminate the secular term from x_3 we must put

$$\omega_2 = \frac{\left(9\alpha_3\omega_0^2 - 10\alpha_2^2\right)a^2}{24\omega_0^3}$$

$$x = \varepsilon a\cos\left(\omega t + \beta\right) - \frac{\varepsilon^2 a^2 \alpha_2}{2\alpha_1}\left[1 - \frac{1}{3}\cos\left(2\omega t + 2\beta\right)\right] + O\left(\varepsilon^3\right)$$

$$\omega = \sqrt{\alpha_1}\left[1 + \frac{9\alpha_3\alpha_1 - 10\alpha_2^2}{24\alpha_1^2}\varepsilon^2 a^2\right] + O\left(\varepsilon^3\right)$$

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$$\ddot{u}+u+0.1x^{3}=0 \qquad x=0.001\,\text{m and } \dot{x}=0.1\,\text{m/s.}$$

Solution: Here $\omega_0^2=1, \alpha_2=0, \alpha_3=1$ and ε=0.1

Substituting these parameters in equation (3.2.15),

$$\omega=\omega_0\left[1+\frac{9\alpha_3\omega_0^2-10\alpha_2^2}{24\omega_0^4}\varepsilon^2a^2\right]=1\left[1+\frac{9-10\times 0}{24}(0.1)^2a^2\right]=\left[1+\frac{3}{800}a^2\right]$$

Also, $x=\varepsilon a\cos(\omega t+\beta)-\dfrac{\varepsilon^2a^2\alpha_2}{2\omega_0^2}\left[1-\dfrac{1}{3}\cos(2\omega t+2\beta)\right]+O(\varepsilon^3)$

Now from initial condition

$$0.001=0.1a\cos\beta-\left(\frac{0.01a^2\times 0}{2}\right)\left[1-\frac{1}{3}\cos 2\beta\right]=0.1a\cos\beta$$

$$0.1=-0.1a\omega\sin\beta-\left(\frac{0.01a^2\omega\times 0}{3}\right)\sin 2\beta=-0.1a\omega\sin\beta$$

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So, we have taken this example last class and we have seen the response of the system ok; so doffing equation and how to solve.

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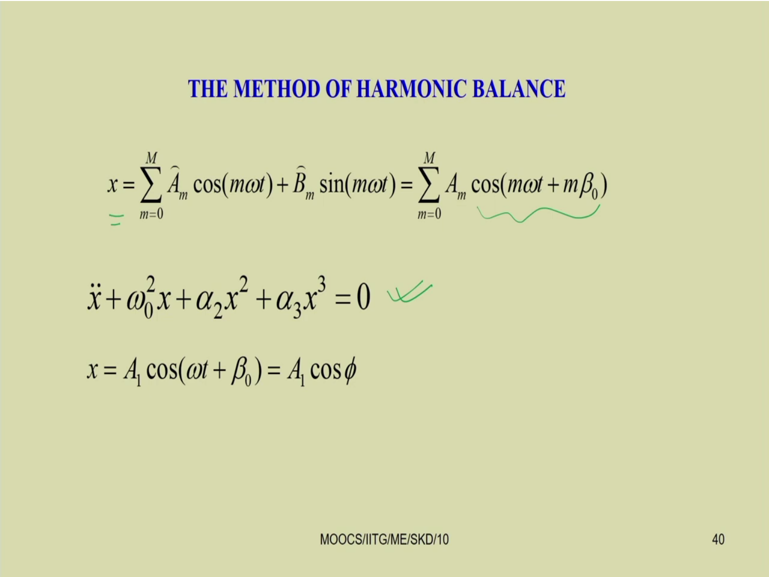

So, harmonic balance method also we have studied. So, in harmonic balance methods we are putting different harmonics. So, for example, we are writing x equal to A m cos m omega t plus B m sin m omega t.

So, we can substitute these m equal to 0, 1, 2 all these terms we can substitute and we can write this equation or this instead of writing in cos and sin one can substitute only cos also. So, here A m cos m omega t plus m beta 0 then so let if you are taking this governing equation so we can; so that by you can take a single term, two terms or multiple terms.

So, as you go on taking more and more terms, so the analysis will be more and more complicated, but by using this developed computational schemes or symbolic software's, so you can easily solve these problems. So, now, you substitute with many higher order terms also.

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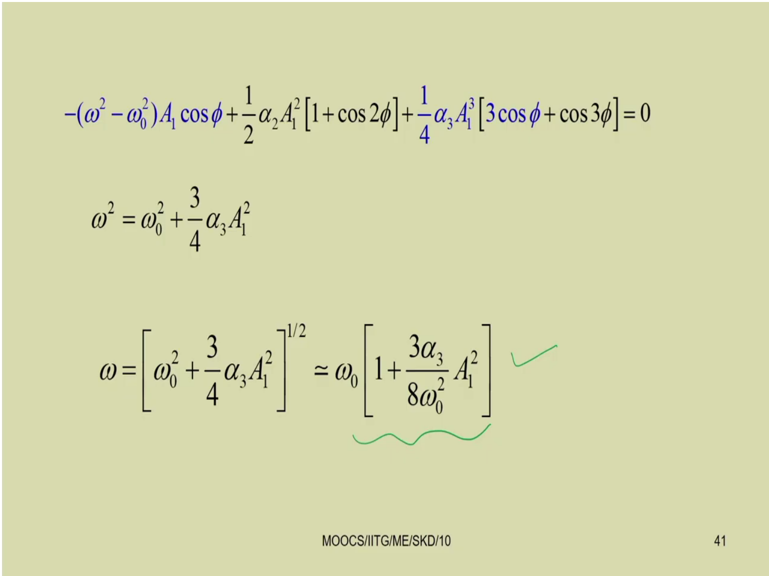

So, for example, you can substitute this x 1 equal to x 1 equal to A cos omega t plus beta 0 or you can write this thing as A 1 cos phi only single term. So, by putting single term you can see that omega becomes omega 0 plus 3 by 3 alpha 3 by 8 omega 0 square A 1 square.

So, actually this is not matching what we have found by using this lp method. So, by taking a single term we have seen the accuracy of the response is not so good. So, we have to take more terms.

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$$x = A_0 + A_1 \cos\phi$$

$$\left[\omega_0^2 A_0 + \alpha_2 A_0^2 + \frac{1}{2}\alpha_2 A_1^2 + \alpha_3 A_0^3 + \frac{3}{2}\alpha_3 A_0 A_1^2\right]$$

$$+\left[-\left(\omega^2 - \omega_0^2\right)A_1 + 2\alpha_2 A_0 A_1 + 3\alpha_3 A_0^2 A_1 + \frac{3}{4}\alpha_3 A_1^3\right]\cos\phi$$

$$+\left[\frac{1}{2}\alpha_2 A_1^2 + \frac{3}{2}\alpha_3 A_0 A_1^2\right]\cos 2\phi + \frac{1}{4}\alpha_3 A_1^3 \cos 3\phi = 0$$

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So, if we are taking 2 terms for example, A 0 plus A 1 cos phi.

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$$\omega_0^2 A_0 + \alpha_2 A_0^2 + \frac{1}{2}\alpha_2 A_1^2 + \alpha_3 A_0^3 + \frac{3}{2}\alpha_3 A_0 A_1^2 = 0$$

$$-\left(\omega^2 - \omega_0^2\right) + 2\alpha_2 A_0 + 3\alpha_3 A_0^2 + \frac{3}{4}\alpha_3 A_1^2 = 0$$

$$A_0 = \left[-\frac{1}{2}\frac{\alpha_2}{\omega_0^2} A_1^2 + O(A_1^4)\right] \qquad \omega^2 = \omega_0^2 + \left(\frac{3}{4}\alpha_3 - \frac{\alpha_2^2}{\omega_0^2}\right) A_1^2$$

$$\omega = \omega_0 \left[1 + \frac{3\alpha_3\omega_0^2 - 4\alpha_2^2}{8\omega_0^4} A_1^2\right]$$

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And we have seen omega equal to; so this is the expression for omega we have obtained. So, you have seen omega equal to omega 0 plus 3 alpha 3 omega 0 square minus 4 alpha 2 square by 8 omega 4 A square. So, this way by taking a number of terms so we can. So, that it is matching or closely matching with what solutions we have obtained by using this Lindstedt Poincare method.

So, we have to take more number of terms. So, judiciously we have to have a trade off between the number of terms and the or one has to do the convergence analysis to see how many terms we can take to get the actual solution or actual solution of the system. So, we have solved some examples also and we have found the solution.

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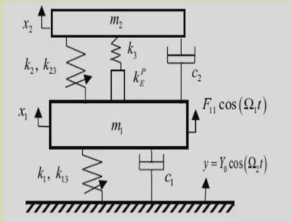
$$F_c = k_r (x_1 + \delta_0 - x_2)$$

$$\delta_0 = n d_{33} V$$

$$k_r = \left(k_3 k_p^E\right) / \left(k_3 + k_p^E\right)$$

$$V = k_c (\ddot{x}_1 (t - \tau)).$$



$$m_1\ddot{x}_1 + c_1(\dot{x}_1 - \dot{y}) + c_2(\dot{x}_1 - \dot{x}_2) + k_1(x_1 - y) + k_{13}(x_1 - y)^3 + k_2(x_1 - x_2) + k_{23}(x_1 - x_2)^3 = F_{11}\cos(\Omega_1 t) - F_c$$

$$F_c = k_r(x_1 - x_2 + n d_{33} k_c \ddot{x}_1 (t - \tau))$$

$$\tau_1 = \omega_1 t \text{ where } \omega_1 = \sqrt{k_1/m_1}.$$

$$m_2\ddot{x}_2 + c_2(\dot{x}_2 - \dot{x}_1) + k_2(x_2 - x_1) + k_{23}(x_2 - x_1)^3 = F_c$$

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$$
\begin{aligned}
&\ddot{u}_1 + 2\xi_1\dot{u}_1 - 2\xi_2\dot{u}_2 + u_1 + \alpha_{13c}u_1^3 \\
&\quad - (\alpha + \alpha_r)\,u_2 - \beta u_2^3 \\
&\quad = F_1\cos\Omega\tau_1 + Y\cos(\Omega\tau_1 - \gamma) \\
&\quad + \alpha_{13c}\,(Y\cos(\Omega\tau_1 - \gamma))^3 \\
&\quad + 3\alpha_{13c}\begin{pmatrix} u_1^2 Y\cos(\Omega\tau_1 - \gamma) - \\ u_1\,(Y\cos(\Omega\tau_1 - \gamma))^2 \end{pmatrix} \\
&\quad - F_{c1}\ddot{u}_1\,(\tau_1 - \tau) \\
&\mu\ddot{u}_2 + 2\xi_2\dot{u}_2 + (\alpha + \alpha_r)\,u_2 + \beta u_2^3 \\
&\quad = F_{c1}\ddot{u}_1\,(\tau_1 - \tau) - \mu\ddot{u}_1
\end{aligned}
$$

$$
\begin{aligned}
&u_1 = x_1/x_0,\ u_2 = (x_2 - x_1)/x_0, \\
&\mu = \frac{m_2}{m_1},\ \xi_1 = \frac{c_1}{2m_1\omega_1}, \\
&\xi_2 = \frac{c_2}{2m_1\omega_1},\ \alpha = \frac{k_2}{k_1}, \\
&\alpha_r = \frac{k_r}{k_1}, \\
&Y = Y_0/x_0,\ \alpha_{13} = \frac{k_{13}x_0^2}{k_1}, \\
&\beta = \frac{k_{23}x_0^2}{k_1},\ F_1 = \frac{F_{11}}{m_1\omega_1^2 x_0}, \\
&F_{c1} = \alpha_r k_c n d_{33},\ \Omega = \frac{\Omega_1}{\omega_1},
\end{aligned}
$$

$$\frac{\Omega_2}{\omega_1} = \Omega - \gamma,\ \gamma = \text{phase},\ x_0 = \text{ reference length}$$

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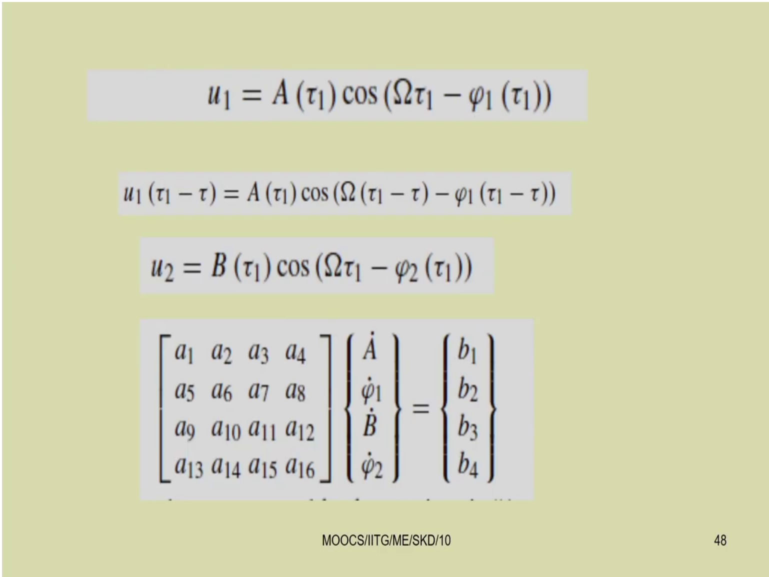

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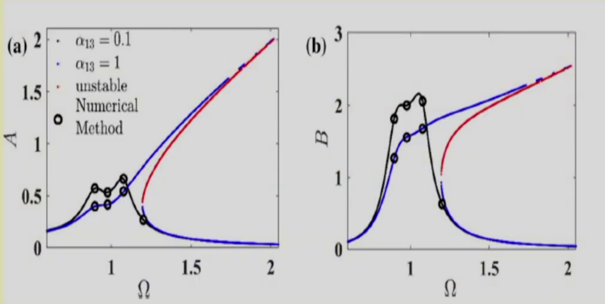

S. Mohanty, S. K. Dwivedy

Nonlinear dynamics of piezoelectric-based active nonlinear vibration absorber using time delay acceleration feedback

Nonlinear Dyn (2019) 98:1465–1490
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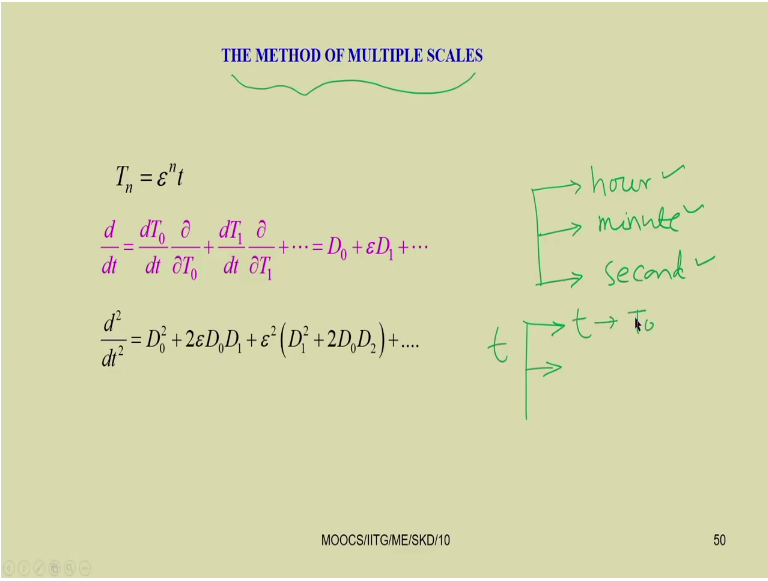

So, today class particularly we are interested to know regarding another method which is known as method of Multiple Scales. So, here we are going to use multiple time scales. For example so the time scale we can divide into like our watch we can divide the time into hour hand, then this is minutes and then seconds. Like in our watch we have this hour minute and second similarly the time t we can be divide into different time scales. For example we can divide into a slowly varying time and very fastly varying time. So, t or we can put this is T 0.

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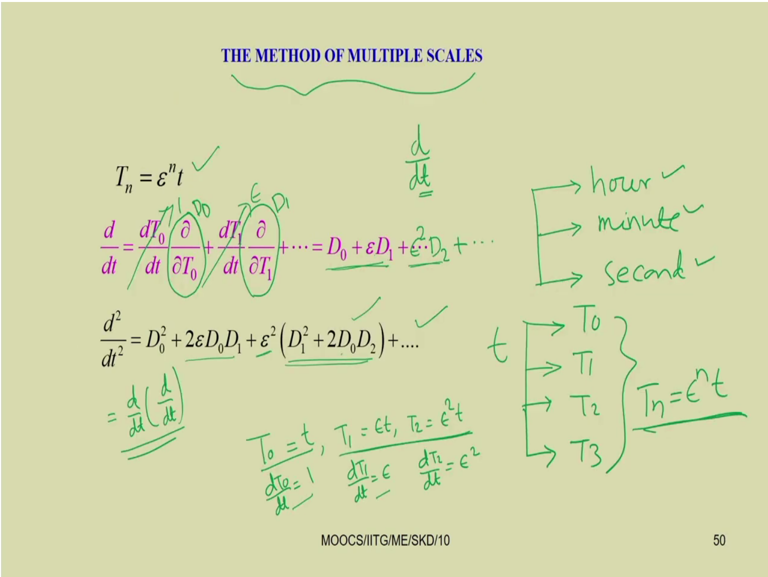

We can take a term T0, T 1, T 2; so we can divide into different time term that it T 0, T 1, T 2, T 3 that way you can divide or you can go on writing this way or you can use this thing T n equal to epsilon n t. So, epsilon is the bookkeeping parameters. So, the nth time scale term will be equal to epsilon to the power n t.

For example, then T 0; so T 0 will be equal to so T 0 equal to epsilon to the power 0 T. So, that is equal to t epsilon to the power 0 equal to 1. So, T 0 equal to t then T 1 becomes. So, T 1 becomes epsilon t then T 2 becomes epsilon square t. So, this way we can take different timescales and we can we can find we can find the response of the system.

So, when we are taking different time scales. So, that is the possibility that we are not neglecting these higher order terms and we are getting a better and better solution. So, by taking these T n equal to epsilon to the power n t we got different time scales. Now taking those timescales now we have to solve the differential equation. So, in this case this d by dt can be written by using this chain rule. So, we can write this d by dt equal to d.

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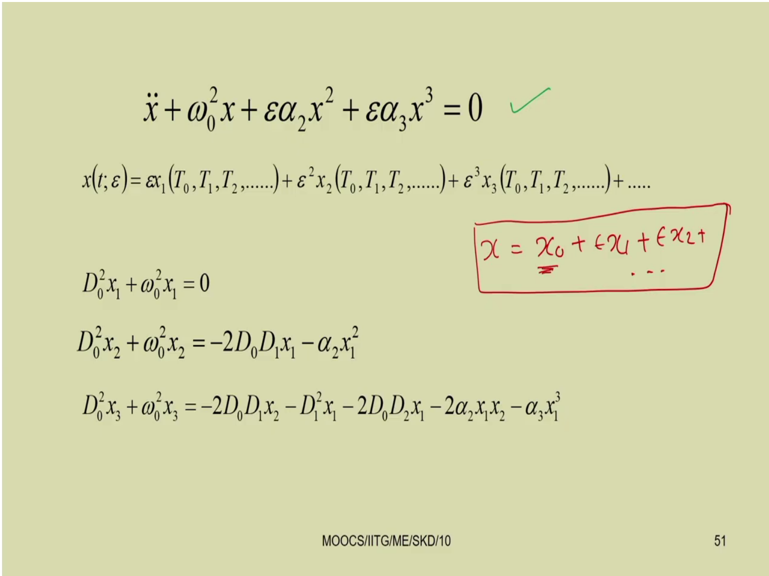

So, it can be d by dt can be written for example, we have d by dt. So, we can expand that thing by using this t equal to T 0, T 1, T 2; so different time scales we can use. So, it can be written as d T 0 by d t into del by del T 0. And these plus del T 1 by del t del by del t 1. So, taking this del by del T 0 so you just see dt 0 by dt dt 0 by dt equal to so d T 0 by d t you have already seen so this is equal to 1. Similarly d T 1 by d t so this is equal to epsilon similarly d T 2 by dt equal to epsilon square.

So, if we substitute in this equation so we are getting. So, by taking these so this is these tends to 1. So, these tends to epsilon. So, if you take the higher order terms then it will be epsilon square. So, by taking these del by del T 0 equal to D 1 and del by del T 1 equal to. So, this is D 0, so this is D 1.

So, we can write this thing equal to D 0 plus epsilon D 1. So, if you want to write the higher order term then it will be epsilon square D 2. So, this way you can go on expanding this thing. So, this time so this time derivative the derivative term can be written d by dt equal to epsilon 0 plus epsilon D 1 plus epsilon square D 2.

Now, this double derivative you can write; so D square by dt square is nothing, but so d by dt of d by dt. So, this way if you expand this thing; so if you are taking only two terms you can write this way; so it will be D 0 square. So, d by dt of D 0 plus epsilon D 1 plus epsilon square D 2 that way you can write. So, if you apply that thing then this becomes D 0 square plus 2 epsilon D 0, D 1 plus epsilon square D 1 square plus 2 D 0, D 2

So, if you are taking only two terms then you are getting this one; so you just expand these things. So, this is D 0 square plus 2 epsilon D 0 D 1 plus epsilon square; so epsilon square D 1. So, you just see; so you have taken another term if you take this D 0 plus epsilon D 1 plus epsilon square D 2 and square of this thing. If you take then this will come to D 0 square if you are keeping up to order of epsilon square then this is the term will be there.

So, D 1 square plus this additionally 2 D 0 D 2 because epsilon square D 2 into D 2 a plus b plus c a plus b plus c whole square equal to a square plus b square plus c square plus 2 a b plus 2 b c plus 2 c a. So, if you use that form then this d square by dt square will be equal to D 0 square plus 2 D 0 2 epsilon D 0 D 1 plus epsilon square D 1 square.

Then if you take the square of this thing then this becomes epsilon to the power 4. So, you can neglect this term then 2 D 0 D 1. So, 2 D 0 into 2 2 a b 2 D 0 into epsilon D 1; so this is the term already we have written then 2 b c. So, if you take these and these terms this is v this is c 2 b into c that is epsilon order of epsilon cube. So, you need not have to take the term then 2 ca 2 into D 0 into epsilon square D 2.

So, if you are taking of two order epsilon square then this term will come into p T n equal to epsilon n t. So, you have different time scales these are T 0, T 1, T 2 that way then this time derivative d by dt equal to d T 0 by dt into del y del T 0 plus d T 1 by dt into del by del T 1 plus d T 2 by dt into del by del t 2. So, that way you can write.

And taking this del y del T 0 as D 0 del by d del T 1 as D 1 del by del T 2 as D 2 and knowing that these dt 0 by dt equal to 1 dt 1 by dt equal to epsilon and dt 2 by dt equal to epsilon square. So, we can write down this equation. So, this d square by dt square equal to D 0 square plus 2 epsilon D 0 D 1 plus epsilon square into D 1 square plus 2 D 0, D 2.

So, now let us take this example; so taking these terms. So, we can see; so now, we can find these things. So, let us take this example that is x double dot plus omega 0 square x plus epsilon alpha 2 x square plus epsilon alpha 3 x cube. So, this is a Duffing equation with both quadratic and cubic nonlinearity. This is that for a free vibration system a system freely vibrating with without damping and having this quadratic and cubic nonlinearity can be written in this form.

So, now let us see how we can find the response of the system. So, in this case like the previous straightforward or Lindstedt Poincare method. So, we can expand this t epsilon equal to epsilon x 1, T 0, T 1, T 2 plus epsilon square x 2 T 0, T 1, T 2 then epsilon cube. So, epsilon cube T 0, T 1, T 2. So, by expanding that way; so expanding this x by using different timescales that is T 0, T 1, T 2.

So, we can write this x t epsilon equal to epsilon x 1 T 0 T 1 T 2 plus epsilon square x 2 T 0, T 1, T 2 epsilon cube x 3 ,T 0, T 1, T 3. So, that way you can write sometimes we may write also by using these x 0 plus epsilon x 1. So, this sometimes you can write this x also in this way x 0 plus epsilon x 1 epsilon x 1 plus epsilon epsilon x 2.

So, this way also you can write particularly this will be useful when you are taking a forced vibration case where you have this some displacement initial displacement will be the there. So, x 0 terms will be there x 0 will be the major term in those cases. So, now by using this expression that is; instead of taking this way in this case we have taken this x equal to epsilon x 1 plus epsilon x 2.

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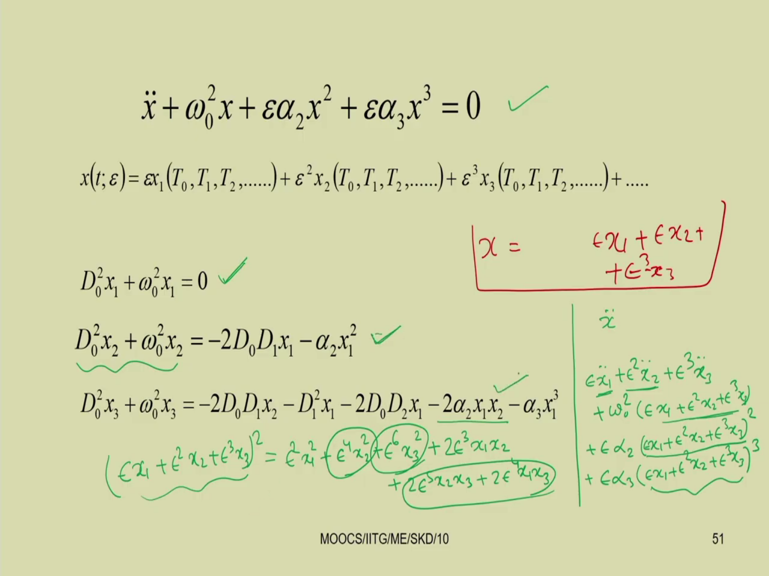

Plus epsilon q x 3 and substituting in this original equation; so we have seen that or we can write with different order of epsilon. So, now, by writing with different order of epsilon we can. So, let us write using different order of epsilon. So, let me write down here.

So, for x double dot for x double dot term I can write this is equal to epsilon x 1 double dot plus epsilon square x 2 double dot plus epsilon cube x 3 double dot. So, plus omega 0 square x; so for x I will write epsilon x 1 plus epsilon square x 2 plus epsilon cube x 3. Similarly, plus epsilon alpha 2 x square. So, for x square x 1 epsilon x 1 plus epsilon square x 2 plus epsilon cube x 3 whole square plus.

Then this epsilon alpha 3 into x cube 4 x cube again I have to write this epsilon x 1 plus epsilon square x 2 plus epsilon cube x 3 to the power 3. You just see as you go on increasing these thing these complexity increases. For example, the square you can though you can find the square of this is a plus b plus c whole square you can use this a square plus b square plus c square plus 2 a b plus 2 bc plus 2 ca.

But while expanding these thing this cubic order you may face some problem manually if you are doing. So, for that purpose; so you can use the symbolic software either in MATLAB or in mathematica and you can solve this thing. Or you can collect the term collect the term of epsilon and write down these equation. So, here this double dot x 1 double dot. So, for x 1 double dot you have to substitute.

So, already we know that double dot this d square by d tau dt square. So, this is equal to D 0 square plus 2 epsilon D 0, D 1 plus epsilon square D 1 square plus 2 D 0, D 2. So, you have to substitute these thing in this equation. So, substitute these in this equation and then order of the epsilon order of epsilon square and order of epsilon cube. So, by putting order of epsilon; so you got this equation. So, that is D 0 square x 1 plus omega 0 square x 1 equal to 0.

Similarly, by taking this order of epsilon; epsilon square. So, you just see order of epsilon square what are the terms will be there. So, this is the; this is the order of epsilon square x square into epsilon square into x 2 double dot for x 2 double dot I will put this D 0 square plus 2 epsilon D 0 D 1 plus so the expression what I have written before.

So, by substituting these in this equation; so you can conveniently write of the order of epsilon square keeping this x 2 term in the left hand side and other terms in the right hand side. So, it will be equal to D 0 square x 2 plus omega 0 square x 2 equal to minus 2 D 0 D 1 x 1 minus alpha 2 x 1 square. So, already these expression for x 1 is known to you.

So, that x 1 can be written either by using the sin cosine or by using these exponential functions. So, we will see it just after this thing then order of epsilon cube you can write you just see while expanding this thing while expanding these with order of epsilon cube. So, you have to keep up to. So, for example, this square term when we are writing the square term.

So, this becomes first term epsilon square x 1. So, let me expand these term only. So, that it will clear to you. So, epsilon x 1 plus epsilon square x 2 plus epsilon cube x 3. So, you want to expand this square term and. So, this becomes epsilon square x 1 square plus epsilon to the power 4 x 2 square plus epsilon to the power 6 x 3 square plus. So, a square plus b square plus c square plus 2 a b; so 2 into first and second term.

So, this becomes epsilon cube x 1 x 2 x 1 x 2 then plus 2 bc. So, 2 into epsilon to the power 5 x 2 x 3 then plus 2 epsilon to ca to epsilon to the power 4 x 1 x 3. So, out of these thing, so we have to keep up to cubic order. So, other things other terms we need not have to write or we can neglect it easily. So, while expanding these thing; so if you can take care then from the beginning itself you can neglect those terms.

For example, you need not have to consider this term you need not have to consider this epsilon to the power 6 you need not have to consider these two terms also. So, the expansion of these things square of this thing will contain only epsilon square x 1 square plus 2 epsilon cube x 1 x 2. So, other terms will not be there. So, it is multiplied with this alpha 2.

So, you can see this term only. So, that is it contains minus 2 alpha 2 x 1 x 2. So, this way you can go on expanding and neglect the higher order terms. And you write the equation; so this is of the order of epsilon this is of the order of epsilon square and this is of the order of epsilon cube. Now, we can write the solution of the first equation.

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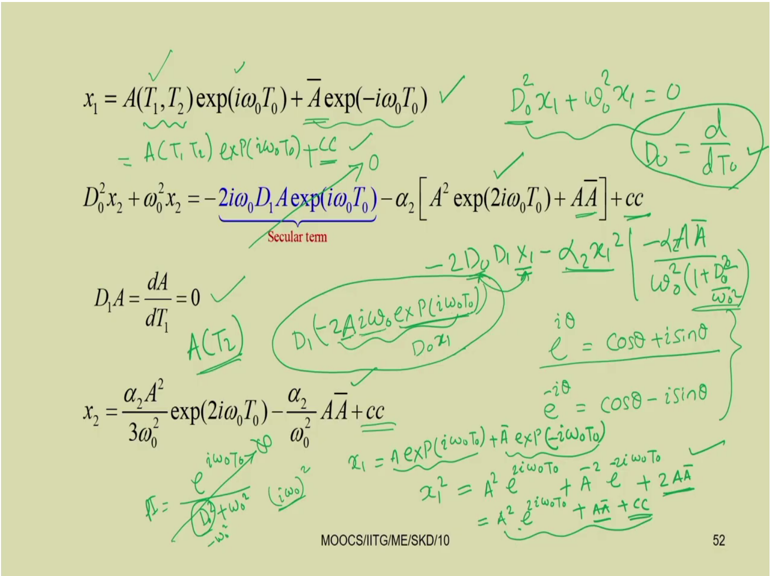

So, that is D 0 square D 0 square x 1 the equation is D 0 square x 1 plus D 0 square x 1 plus omega 0 square x 1 equal to 0. So, already you know the solution of these thing; I have explained several times that this is the equation of a simple harmonic motion. So, where your x 1 will be equal to A sin omega 0 t plus beta.

So, or you can write by using this exponential function. So, by using exponential function the solution can be written A we just see this solution A; so you can write the solution in this form. So, where a should not be a constant; so it should not be; so this as a D 0 is d by dt 0. So, what is d 0? So, you can note that D 0 equal to d by d T 0.

So, as it is function of d y; so it is d 0. So, the constants should not be a function of T 1 T 0. So, that is why it is written A T1, T 2. So, as A is a constant which is a constant for T0. So, for timescale T 0 it is a constant, but it can be a function of T 1 and T 2. So, this x 1 is written equal to a T 1, T 2 e to the power i omega 0 T 0 plus a bar e to the power minus i omega 0 t 0

So, you can note this A bar. So, this A is a complex number. So A bar is its complex conjugate. So, A is a complex number and A bar is its complex conjugate. So, already you know e to the power i theta e to the power i theta can be written as cos theta plus i sin theta i sin theta that is why you just see. So, already in the free ok. So, e to the power i theta you can write this way similarly e to the power minus i theta is nothing, but cos theta minus i sin theta.

So, either you can write by using these sin or cosine function or by using this exponential function. So, this x 1 is written a T 1, T 2 as it is not a function of t 0. So, it is written a T 1 T 2 e to the power i omega 0 T 0 plus A bar. So, what is A bar? A bar is complex conjugate of A A bar is complex conjugate of AA bar e to the power minus i omega 0 T 0.

Actually these this term is complex conjugate of the first term. So, sometimes instead of writing these two terms. So, sometimes it is written the first term plus cc where cc is its complex conjugate of the preceding term complex conjugate of the preceding term. So, sometimes it is written this way also. So, this is equal to a T 1, T 2 e to the power i omega 0 T 0 plus cc.

So, here the cc represent the complex conjugate of the first term complex conjugate of the previous term preceding terms. So, if a more number of terms are there then the complex conjugate of all other terms either you write this way or you write this way. So, now, substituting this expression for; x 1 in the previous term.

So, we can write. So, you can note the previous term is; so the previous term is minus 2 D 0, D 1 x 1 minus alpha 2 x 1 square minus 2 D 0, D 1 x 1 minus 2 D 0 d 1. So, minus 2 D 0, D 1 x 1 minus. So, you can see that thing again. So, so that is minus alpha 2 x 1 square minus alpha 2 x 1 square. So, already you know x 1 equal to a e to the power i omega 0 t 0. So, we just see you can have D 0 x 1; so D 0 x 1. So, these expansions you should know.

So, this becomes minus 2 for D 0 I can write d 0. So, as A constant so if you take out D 0. So, this becomes A and D 0 of e to the power i omega T 0 is nothing but this is i omega 0 i omega 0 then e to the power e to the power i omega 0 t 0. So, then so D 1 of these things. So, you have taken this D 0 x 1. So, this is this part is D 0 x 1 D 0 x 1.

So, D 0 x 1 become; so x 1 equal to A e to the power i omega 0 t 0. So, for that thing you have written; so this is there is a minus sign. So, minus 2 A then e to the power i omega 0 t 0; so derivative of i omega 0 T 0 equal to i omega 0. So, this i omega 0 term is here. So, now, you can have these only for the first term I am doing only for the first term.

So, you can write cc plus cc. So, that will give you the complex conjugate. Now, D 1 of these things will be equal to you just see when we are writing D 1, D 1 is d by dt 1. So, this e to the power i omega 0 T 0 is not a function of T 1. So, this becomes constant, but A is a function of T 1.

So, then in that case; so this becomes minus 2 i omega 0 D 1 A. So, this becomes D 1 A e to the power i omega 0 t 0. So, this way you can expand this thing. So, these terms should be clear to you otherwise you cannot solve this problem. So, first you have to do first find this D 0 x 1 as you know D 0 x 1 A is a function of T 1, T 2 not a function of t 0. So, this is constant for A D 0.

So, first you do this D 0 x 1. So, for that thing this i omega will come out from this one and then you differentiate with respect to D 1. So, for D 1 this e to the power i omega 0 T 0 is a constant. So, that is why it becomes minus 2 i omega 0 D 1 e to the power i omega 0 T 0 then for this alpha 2 x 1 square already I told you. So, how to expand this; x 1 square? So, x 1 equal to A e to the power i as x 1 equal to A e to the power e to the power i omega 0 T 0 plus a bar a bar e to the power i omega 0 T 0 a bar e to the power i; i omega 0 T 0.

So, square of these things. So, x 1 square will contain this A square e to the power is A. So, A square of these thing square of these thing becomes square of these becomes a square e to the power 2 i omega 0 T 0 plus a bar square e to the power 2 i omega 0 T 0 plus 2. So, that term you just see. So, so it becomes so this x 1 square becomes now I am write going to write x 1 square.

So, here you used to make mistake. So, this x 1 square becomes a square e to the power. So, then this becomes 2 i omega 0 T 0 plus A bar square e to the power 2 i omega 0 T 0 plus 2 AA bar you just see to AA bar. So, e to the power i omega 0 T 0 into e to the power. So, this is minus minus i omega 0 t 0; so this becomes 1. So, then this becomes so here you have a minus sign ok. So, this becomes a square e to the power 2 i omega 0 T 0 plus a bar square e to the power minus 2 i omega 0 T 0 plus 2 AA 1 bar

So, either you can write completely this full expression or you can write this equal to A square e to the power 2 i omega 0 T 0 plus AA bar plus cc; cc becomes this complex conjugate of the preceding term complex conjugate of the preceding term a square this becomes a bar square, then e to the power 2 i omega 0 T 0. Complex conjugate becomes minus 2 i omega 0 T 0, then a bar complex conjugate becomes A bar A which is same as AA bar. So, these 2 AA bar can be written as AA bar plus AA bar. So, the complex so this x 1 square you can write in this form by using only two term plus cc; cc becomes complex conjugate of the preceding term.

So, now this way by expanding; so you can write this D 0 square x 2 plus omega 0 square x 2 equal to minus 2 i omega 0 D 0 A e to the power i omega 0 T 0 minus alpha 2 a square e to the power 2 i omega 0 T 0 plus AA bar plus cc. So, you just check these term that is terms with e to the power i omega 0 t 0. So, terms with e e i omega 0 T 0.

So, the particular integral will becomes. So, if have a term e to the power. So, you have a term e to the power e to the power i omega 0 T 0 in the right hand side. So, particular integral will be D 0 square divided by D 0 square plus omega 0 square So, the particular integral e 2 e 2 i omega 0 t 0. So, the particular integral of this thing will be this here we have to substitute this D 0 square by i omega 0 square.

So, the substitute this D 0 by so D 0 by i omega 0 square so i omega 0. So, i is root over minus 1; so i omega 0 square becomes minus omega 0 square. So, this becomes minus omega

0 square. So, minus omega 0 square plus omega 0 square equal to 0. So, the denominator becomes 0 and the particular integral tends to infinite.

So, as the denominator part is 0. So, the particular integral tends to infinite. So, so the presence of the term with i omega 0 T 0 tends in the system to infinite or unbounded condition, but in actual case as the system response is bounded. So, this is a free vibration only. So, in that free vibration so as the system response is bounded, so these term must be eliminated. So, these terms are known this term is known as secular term. So, the secular term must be eliminated to get the result.

So, how it can be eliminated? So, for example, in this case omega 0 is a constant. So, then this e to the power i omega 0 this exponential function cannot be 0. So, it can only be 0 if these D 1 a equal to 0, but what is D 1 A. So, D 1 A equal to d A by dT 1 D 1 A equal to d A by dT 1. So, we observe that d A by dT 1 equal to 0. So, A is a constant.

So, as A is a constant; so previously we have seen A is not a function of A 0. So, from these thing we know a is also not a function of T 1 as it should be a constant with respect to T 1 as d A by dT 1 equal to 0; so A is a constant. So, that constant with respect to T 0 and T 1 previously we have seen A is not a function of T 0.

Now, we have seen A is not a function of T 1. So, a must be a function of T 2 or higher order terms so then so we can write instead of writing previously we have written A T 2 T 1 t 2. So, now, we have seen A is a function. So, A is a function of T 2 only. So, if you are taking up to T 2. So, then A is a function of T 2.

So, now so these term has gone. So, as this term has gone so we can find the particular integral D 0 square x 2 plus omega 0 square x 2 equal to minus alpha 2 A square e to the power i 2 i omega 0 T 0 plus AA bar. So, particular integral will be divided by D 0 square omega 0 square and. So, in place of for this part.

So, for this part by substituting for D 0 equal to 2 i omega 0 you can substitute and this is A constant term. So, for this thing also we can find the particular integral. So, this will be AA bar by D 0 square plus omega 0 square taking this omega 0 square out. So, this becomes 1 plus D 0 by omega square into AA bar. So, the solution will becomes AA bar by omega 0 square because the D 0 square AA bar become 0.

So, the solution of this thing alpha 2 A square. So, the solution of D 0 square x 2 plus omega 0 x 2 equal to minus alpha 2 A square e to the power i 2 i omega 0 T 0 plus AA bar becomes alpha 2 A square this alpha 2 A square by; so you just see. So, this is 2 i omega 0 whole square this becomes minus minus this becomes minus 4 omega 0 square plus omega 0 square. So, this becomes minus 3 omega 0 square and outside you have a minus sign; so minus minus plus.

So, that is why this becomes alpha 2 A square by 3 omega 0 square e to the power. So, 2 i omega 0 t 0 so plus. So, this you have this minus alpha 2. So, already I told you how you can find for this constant. So, this becomes particular integral becomes AA bar alpha 2 minus alpha 2 AA bar by; so you have taken omega 0 square common.

So, to take omega 0 square common; so then this becomes 1 plus D 0. So, this becomes D 0 square by omega 0 square. So, you can take this thing to the numerator and expand that thing by using binomial theorem. So, this becomes minus omega 0 square or ok. So, only this part you take to the top. So, then this becomes 1 minus D 0 square omega 0 square into minus alpha 2 AA bar.

So, first term; so when we are taking 1; so this becomes minus alpha 2 AA bar by omega 0 square. So, when we are taking D 0 square. So, D 0 square of AA 1 bar. So, this becomes 0. So, only term remaining is minus alpha 2 omega 0 square AA bar. So, we need not have to complete the whole thing and we can write by writing these two terms.

So, as the for the complex conjugate similar terms will get, so you can write plus cc. So, this becomes alpha 2. So, x 2 becomes alpha 2 A square by 3 omega 0 square e to the power 2 i omega 0 T 0 minus alpha 2 omega 0 square AA bar plus cc; so this way.

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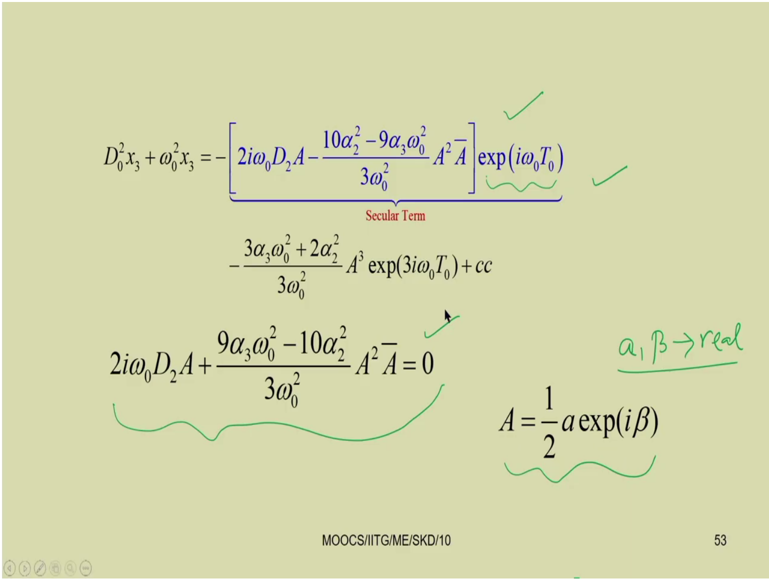

So, now, you have got the expression for x 2 already we have this expression for x 1. So, now, substituting this expression for x 2 and x 1 in the third equation that is order of epsilon square we can write this equation equal in this form; D 0 square x 3 plus omega 0 square x 3 equal to minus 2 y omega 0 D 2 a minus 10 alpha 2 square minus 9 alpha 3 omega 0 square by 3 omega 0 square A square A bar e to the power i omega 0.

So, already I told you due to the presence of the term e to the power i omega 0 T 0. So, this must be a secular term because the particular integral of this term tends to infinite. Because this due to presence of these i omega 0 T 0. So, the particular solution becomes this whole term divided by D 0 square plus omega square omega 0 square and you have to substitute this D 0 equal to i omega 0 for finding this particular integral.

So, in that case so this term becomes the secular term. So, this D 0 square x 3 plus omega 0 square x 3 equal to the secular term plus this additional term that is minus 3 alpha 3 omega 0 square plus 2 alpha 2 square by 3 omega 0 square a cube e to the power 3 i omega 0 T 0 plus cc. So, now we have to eliminate the secular term.

So, actually by eliminating the secular term; so we will get; we will get the required expression for our response. So, now, by eliminating the secular terms; that means, we have to substitute this equal to 0. So, here as already I told you A is a complex number.

So, as A is a complex number it can be written in this form in its polar form that is A equal to half a e to the power i beta; where both a and beta are real number. So, A and beta are real number so we can write this complex number a by its polar form A equal to half a e to the power i beta by substituted in this equation and separating the real and imaginary parts.

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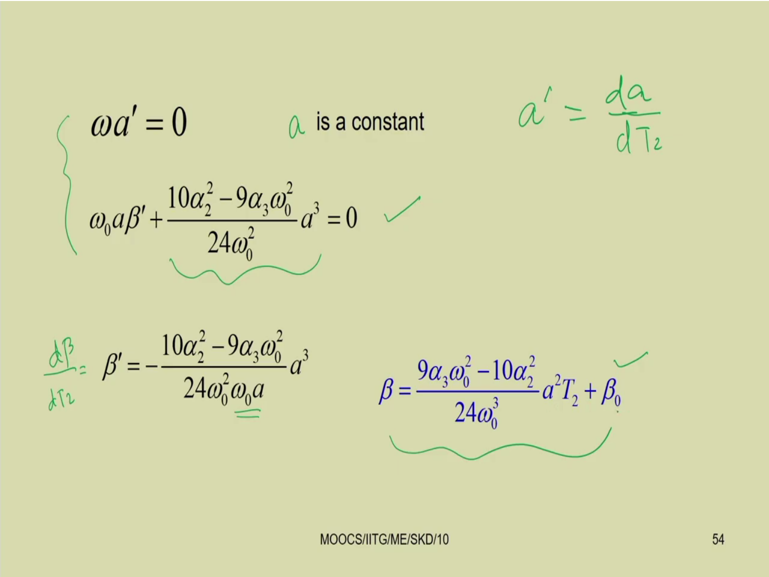

So, we can substitute this thing in this equation and equate the real and imaginary part to 0; so we get these two equations. So, in the first equation we get this omega a dash equal to 0. So, you just see a dash this dash is a differentiation with respect to T 2. So, this is equal to d a by dt 2. Because already we know this a is not a function of T 0 and T 1 and their function of T 2. So, d a by dT 2 equal to 0 so; that means, a is a constant. So, which is not a function of T 2 also; so A is a constant we got.

So, now from the second equation omega 0 a beta dash plus 10 alpha 2 square minus 9 alpha 3 omega 0 square by 24 omega 0 square a cube equal to 0. So, you just see so omega 0 a beta dash equal to so these whole terms we can take into right hand side, but this is not a function of T 2.

So, already we have seen this a is not a function of T 2. So, that is why this is a constant term. So, this beta dash so we can get beta dash equal to this divide by it this 24 omega 0 square then this omega 0 into a we have divided this part. So, we can write this way. Then as this is d beta so this is d; so this is d beta by dT 2 equal to this beta dash equal to this.

So, beta will be equal to those whole thing into whole thing into T 2 plus beta 0 another constant of integration we can substitute; so beta can be written in this form. So, we got a is a constant and beta equal to 9 alpha 3 omega 0 square minus 10 alpha 2 square by 24 omega 0 cube a square T 2 plus beta 0.

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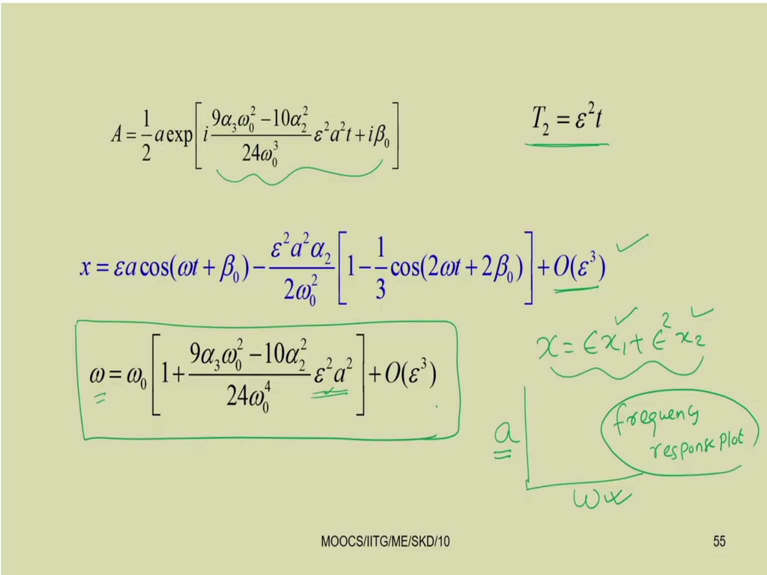

So, now by substituting this expression for a and beta 0. So, we can write down; so a equal to half a e to the power i beta. So, A equal to half a e to the power i so this is beta. So, this is the expression for beta; so we have written this expression for beta. So, so here you just note that T 2 equal to epsilon square t. So, we have substituted T 2 equal to epsilon square t.

So, we have written as we have written our x equal to x equal to epsilon x 1 plus epsilon square x 2 let us keep only up to these two term then by substituting these expression for x 1 and x 2. So, we can write this expression for x equal to this. So, x equal to epsilon A cos omega t plus beta 0.

So, minus epsilon square a square alpha 2; 2 omega 0 square 1 minus 1 3rd cos 2 omega t plus 2 beta 0 plus order of; so this is order of epsilon q. So, we have neglected the higher order term that is order of epsilon cube.

So, from here so easily you can observe that these omega 0 that the frequency of the response can be written in this form. So, frequency of the response omega equal to omega 0 into 1 plus 9 alpha 3 omega 0 square minus 10 alpha 2 square by 24 omega 0 to the power 4 epsilon square A square. So, you can see the frequency is a function of displacement or the displacement is changing the frequency will change unlike in linear system.

So where the frequency is independent of the displacement. So, here it depends on the displacement. So, you just see the in linear case only you will have the first term that is omega equal to omega 0. So, here you have omega equal to omega 0 into 1 plus. So, these 9 alpha 3 omega 0 square minus 10 alpha 2 square by 24 omega 0 4th epsilon square a square.

So, this way one can find the response; so can plot these response that is A versus omega to find the frequency response plot. The plot what we will get; so that is known as frequency response plot. So, here knowing so for a different value of omega we can find different value of A or for different value of A we can get different value of omega also. So, this way we can find the expression for different response.

So, with this we know how to solve this non-linear differential equation by using method of multiple scale. So, next class we will take two more examples to use this method of multiple scale for a force vibrating system and parametrically excited systems. And later we will see many other methods for example, this method of averaging to solve this non-linear equation motion.

Thank you.