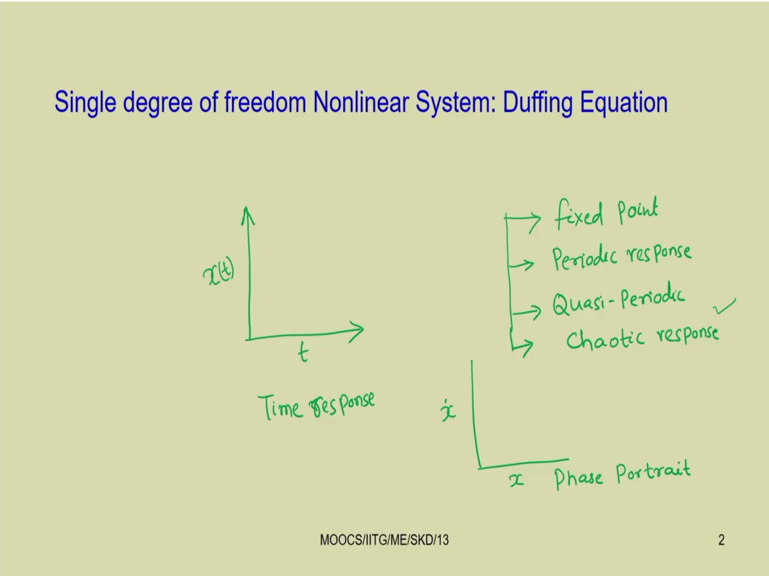
Nonlinear Vibration
Prof. Santosha Kumar Dwivedy
Department of Mechanical Engineering
Indian Institute of Technology, Guwahati

Lecture - 13
Free vibration of undamped and damped SDOF systems with quadratic and cubic nonlinearity

So, welcome to today class of Nonlinear Vibration. Today, we are going to start module 4 of this Nonlinear Vibration course.

(Refer Slide Time: 00:39)



So, in this module we are going to study regarding the Single degree of freedom Nonlinear Systems. So, in single degree of freedom system, so, we will study regarding the free vibration force vibration. So, in case of force vibrations we will study if the system is weakly non-linear or the forcing is weak or the forcing is strong.

So, strong forcing and weak forcing both the two types of things we will study. And, particularly we are going to take the example of duffing equation and some other types of equations we are which are used in this single degree of freedom non-linear systems. So, already we are familiar with different type of response.

For example, so, we know this equation have a fixed point response, periodic response, quasi-periodic response and chaotic response. So, four different types of response we know. So, first one is the fixed point response. Second one that is the periodic response, the response of the systems may be periodic.

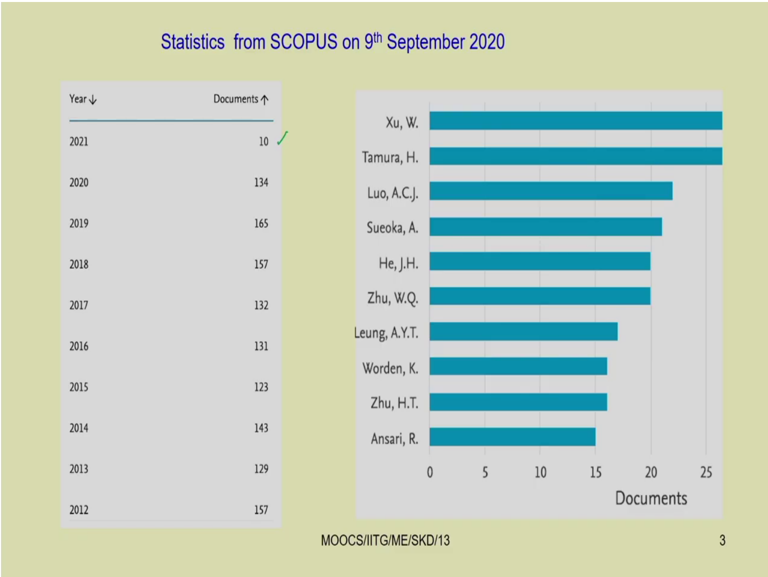
And, then quasi-periodic response – quasi-periodic is also known as a periodic quasi periodic response. And finally, we know the response may be chaotic the response may be chaotic and this response actually we can characterize by using this time response plot.

So, that means, so, we can plot this response for example, the response is x. So, x t versus t, so, this is the time response or we may plot the phase portrait. So, in case of phase portrait or state space, so, in case of phase portrait generally we plot x versus x dot that is the displacement versus velocity.

So, using this displacement versus phase velocity we can tell these are the phase portrait also we may plot the Poincare section. So, by using this Poincare section also we can characterize. So, in this module, I will tell you how we can derive or find the Poincare section.

Also how we can characterize particularly to characterize this chaotic response we may required some other characterization for example, Lyapunov exponent. So, all these types of response and how to characterize those things also we will study while studying this single degree of freedom system.

(Refer Slide Time: 03:49)



So, already we are familiar with the equation and solution method. For example, so, we know we can solve these equations by using these numerical methods or we may go for this perturbation analysis. So, for the duffing equation so, today I checked the literature in this SCOPUS and you can find. So, there more than 3400 papers found in SCOPUS related to duffing oscillators only.

So, in 2010 you can find 10 publications; 20 around 134 publication; 19 – 165 publication; 18 – 157. So, more than 100 publications are you can find in each year from 2012 to last year itself. So, this year also in 2020 these 10 is in 21. Actually the now it is November 20, but already in 21, 10 paper have been those will be published in 21 – 10 papers have already been published related to duffing oscillator.

So, the authors most prominent authors who have worked in this field are W Xu, H Tamura, A C J Luo, A Sueoka, J H He, W Q Zhu, A Y T Leung, K Worden, H T Zhu and R Ansari. So, you can find many other authors are also working in this field. So, as more than 3000 papers are just simply on duffing oscillator is found.

So, you may be knowing so, how useful the study is. So, there are several studies available on these things some of the prominent papers or some of the paper or in these last two years which I have marked, so, you can see in these slides.

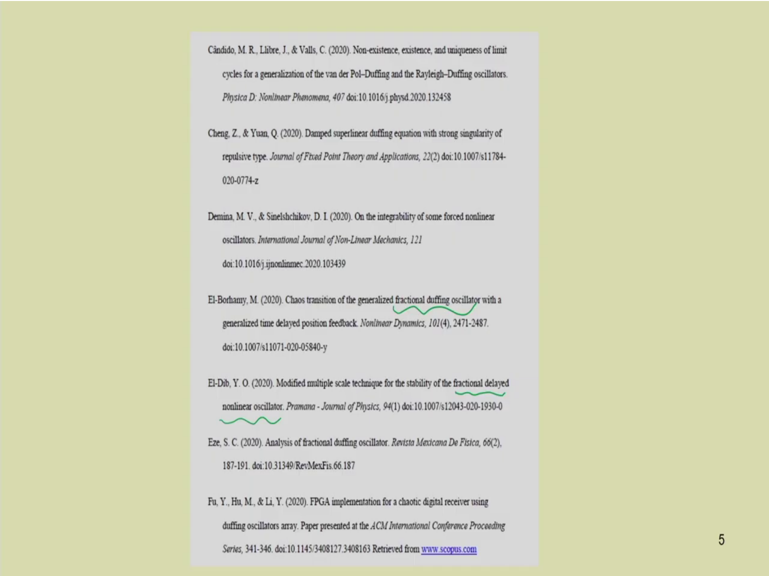
(Refer Slide Time: 05:39)

So, for example, this residual series representation algorithm for solving fuzzy duffing oscillator, so, this duffing equation now has been modified in many different way for example, this fuzzy duffing oscillator you can find here. So, the similarly this duffing oscillator is used for this energy harvester purpose also you can see in this paper piezoelectric duffing energy harvester.

So, one can model a cantilever beam as a cantilever beam with this follower load also as a duffing oscillator which is studied in this paper. So, then a simple cubication method for approximate solution of non-linear Hamiltonian oscillators so, here also the system is considered a that of a duffing equation.

So, you may study this continuous piecewise linearization method for approximate periodic solution of a relativistic oscillator. And then a generalization of the S-function method applied to a Duffing-Van der Pol forced oscillator. So, along with the duffing oscillator you may add Van der Pol oscillator, you can add Mathieu oscillator Mathieu equations.

(Refer Slide Time: 07:13)



Cândido, M. R., Llibre, J., & Valls, C. (2020). Non-existence, existence, and uniqueness of limit cycles for a generalization of the van der Pol–Duffing and the Rayleigh–Duffing oscillators. *Physica D: Nonlinear Phenomena, 407* doi:10.1016/j.physd.2020.132458

Cheng, Z., & Yuan, Q. (2020). Damped superlinear duffing equation with strong singularity of repulsive type. *Journal of Fixed Point Theory and Applications, 22*(2) doi:10.1007/s11784-020-0774-z

Demina, M. V., & Sinelshchikov, D. I. (2020). On the integrability of some forced nonlinear oscillators. *International Journal of Non-Linear Mechanics, 121* doi:10.1016/j.ijnonlinmec.2020.103439

El-Borhamy, M. (2020). Chaos transition of the generalized fractional duffing oscillator with a generalized time delayed position feedback. *Nonlinear Dynamics, 101*(4), 2471-2487. doi:10.1007/s11071-020-05840-y

El-Dib, Y. O. (2020). Modified multiple scale technique for the stability of the fractional delayed nonlinear oscillator. *Pramana - Journal of Physics, 94*(1) doi:10.1007/s12043-020-1930-0

Eze, S. C. (2020). Analysis of fractional duffing oscillator. *Revista Mexicana De Fisica, 66*(2), 187-191. doi:10.31349/RevMexFis.66.187

Fu, Y., Hu, M., & Li, Y. (2020). FPGA implementation for a chaotic digital receiver using duffing oscillators array. Paper presented at the *ACM International Conference Proceeding Series*, 341-346. doi:10.1145/3408127.3408163 Retrieved from www.scopus.com

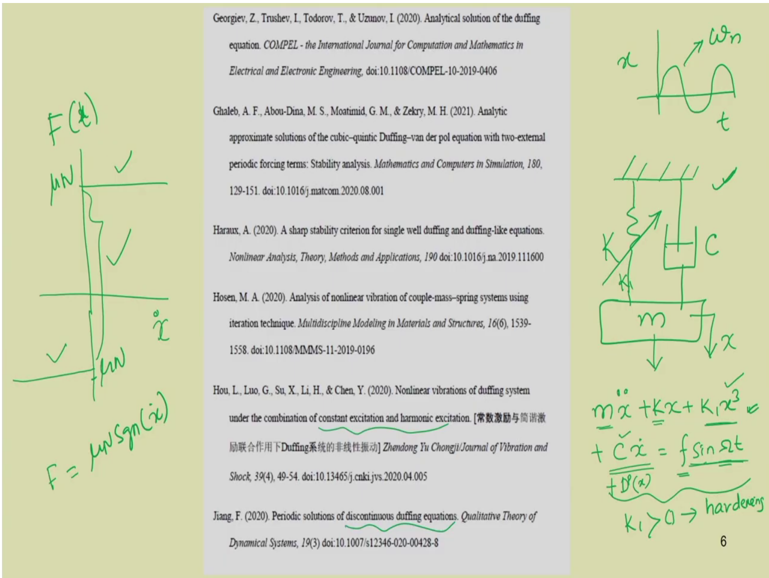
So, in recent times, so, you can find a number of paper along with the duffing oscillator also. Similarly, you can see these paper non-existence, existence and uniqueness of limits cycles for generalization of Van der Pol-Duffing and Rayleigh-Duffing oscillator. So, you already you are familiar with this Van der Pol oscillator where one can get the limit cycle.

So, one can add this duffing oscillator along with duffing with this Van der Pol equation so that one can get this Van der Pol-duffing equations also. Similarly, Rayleigh-duffing oscillator also one can get. Similarly this damped super linear duffing equation with strong singularity for repulsive type. So, this is very recent paper. Similarly, on the integrability of some forced non-linear oscillation, then chaos transition of the generalized fractional duffing oscillator.

So, this is another interesting topic which is advanced topic related to the simplified duffing oscillator that is your fractional order. So, in duffing oscillator, so, some fractional order derivative is added to make it fractional duffing oscillator. Also one can add the time delay in this duffing type of equation to get this delayed oscillator in case of the duffing equation also.

So, this paper you can see modified multiple scale technique for the stability of the fractional delayed; so, here both fractional and again delayed non-linear oscillator; so, then, analysis of fractional duffing oscillator. FPGA implementation for a chaotic digital receiver using duffing oscillator array.

(Refer Slide Time: 09:07)



So, these duffing oscillator equations can be applied to almost all the fields of engineering and science. In biological systems also you can apply or model some of the systems as a duffing oscillator. A simple spring mass system so, to realise a duffing oscillator so, you can take a simple spring mass system where. So, this is a simple spring mass system and if we are considering the spring to be non-linear, then this equation can be written.

So, for example, this is x you are writing. So, this spring you are considering to be non-linear then this equation can be retained in this form mx double dot plus K x plus let me write K and K 1. So, plus K 1 x cube. So, if I am taking cubic out of nonlinearity and if a force is applied to the system. So, for the damping you can write C x dot equal to f sin omega t, where f is the amplitude of the forcing omega is the frequency of the forcing.

And, here you just see the term K 1 x cube so, these represent the non-linearity cubic non-linear and this is the forced duffing equation. So, if the forcing f equal to 0 then this is the duffing equation for a free vibration type. So, if the damping term is also neglected if the damping factor C is taken to be 0 so, then this equation reduced to mx double dot plus K x plus K 1 x cube that is a undamped duffing oscillator.

So, this undamped duffing oscillator mx double dot plus K x plus K 1 x cube. Here so, if this K 1 is greater than 0 so, if K 1 greater than 0 so, this is hardening type that is positive, then this is hardening the spring is said to be hard spring and if K 1 less than 0 the spring is said to be soft spring.

So, you can have a soft type of spring. So, if K 1 is negative and if K 1 is positive then you can have hardening type of spring or hard spring. So, this duffing equation can be modified in many different way. So, for example, this damping term if we can add that of a Van der Pol type then so, it will be known as Van der Pol duffing equation.

Similarly, if a fractional order term will be added. So, for example, I will add another term that is D P x. So, here the P-th order fractional order derivative if we are adding then this will be a fractional order duffing equation. So, this way many different type of oscillators can be generated out of the simple system.

So, here the forcing also may be divided into or can be taken as a simple harmonic forcing or it can be taken in many different way, two frequencies can be taken or multiple frequencies can be taken or stochastic type of forcing can be given also to the system.

So, one can analyze the simplest type of spring mass damper system or write down this equation of a simple spring mass damper system in many different ways. Also you may note that by taking different value of this m K K 1 c and f and omega. So, different type of response can be generated.

So, already you are familiar when these K 1 c and f are 0 so, the system is that of a linear undamped system and in case of linear undamped system the response is. So, already you are familiar with the response the response will be only sinusoidal or sinusoidal with a frequency of omega n with a frequency of omega n.

So, if you plot x versus t so, this will be sinusoidal with frequency of omega n where omega n equal to root over K by m. Now, by adding this damping so, the system can be modelled as a under damped over damped or critically damped system and one can study the response of the system.

So, for example, in case of under damped system so, the response will be oscillatory; in case of critically damped it will take minimum time to reach the equilibrium position and in case of over damped system so, it may not reach that critical value or the system will be over damped depending on the value of C.

Now, by putting this non-linearity, so, today class we are going to see how the effect will be there if we take different value of these m K K 1 c and f. So, these are the recent papers which are you may go through. For example, this last paper this paper also periodic solution of discontinuous duffing equation. So, you can have a continuous. So, this equation you have seen.

So, you can have a discontinuous duffing equation means discontinuous type of forcing the forcing if. So, for example, you just take the forcing that of a frictional. So, if you have a friction acting on the system so, in that case or if you are taking a forcing so, if you remember the coulomb type of. So, this is your minus mu N. So, this is mu N.

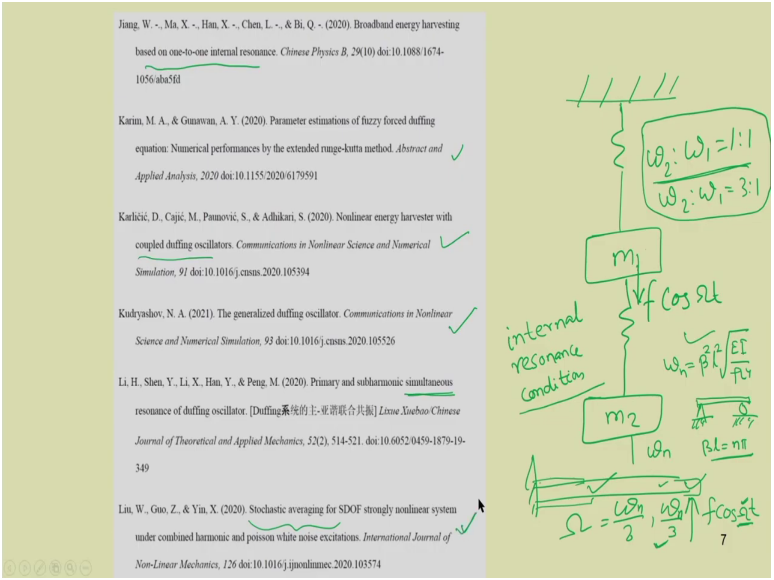
So, coulomb damping if you remember the coulomb damping then this forcing you can write. So, these F t the forcing you can write this forcing F equal to so, F equal to mu N signum x dot. So, here so, when x dot greater than 0, so, it takes a value of mu N.

So, if x dot less than 0, it takes a value of minus mu N so, when x dot equal to 0 so, it jumps from minus mu N 2 plus mu N. So, the system is not linear in this case. So, there is a jump from minus mu N to plus mu N. Similarly, so, if you can take a system with different type of stops then also the response or the forcing may not be continuous.

It may be piecewise continuous or discontinuous it may be discontinuous like here it is discontinuous it jumps from minus mu N to plus mu N. Similarly, you can design many systems or there are several physical systems so, where the forcing may not be continuous, so, it may be discontinuous.

So, here, another paper – non-linear vibration of duffing system under combination of a constant excitation and harmonic excitation so, you just see. So, there is two different type of excitation is given. So, one is constant term and another is harmonic excitation.

(Refer Slide Time: 17:09)



So, we can see many other papers also, some more papers are here. So, broadband energy harvesting based on one-to-one internal resonance conditions. So, all of you now by this time you know what is external excitation. So, if you apply external force to the system that is external excitation.

So, many systems will have for example, let me take a two-degree of freedom system this is one spring and mass, let me take another spring and mass. So, this is m 1 and m 2 m 1 and m 2 or let me take a cantilever beam or a that is a continuous system. So, in this system lead this the forcing applied here. So, this is the forcing force applied here.

So, let the force equal to f cos omega t. So, as this is a two degrees of freedom system so, you know, so, it has two frequencies two natural frequencies similarly in these continuous systems, so it will have infinite number of natural frequency. So, for example, so, if we take the simply supported beam simply supported – so, one side it is one side it is hinged and other side it is a roller support. So, this is simply supported beam.

So, in this case of simply supported beam so, you know the formula for this thing this can be modelled as a Euler Bernoulli beam where this omega n can be retained as beta square l square root over EI by rho L 4th E I by rho L 4th, where this beta l equal to so, where beta l equal to n pi so, for the simply supported beam.

So, for n equal to 1, so, this becomes beta l equal to pi. So, the for first natural frequency becomes pi square root over EI by rho L 4th. Similarly, for omega 2; so, this is 4 pi square and into EI root over EI by rho L 4th. So, that way we can get different natural frequencies. So, the system has infinite number of natural frequency if the system is a continuous system.

But, for these two degrees of freedom system so, it has two natural frequency. Sometimes if the natural frequency are in integer relationship that is omega 1 or omega 2 is to omega 1. So, if it is equal to 1 is to 1 in case of tune vibration observer actually this condition is maintained that is omega 2 is nearly equal to omega 1.

And, it is kept near to the excitation frequencies, so that the primary system the motion of the primary system is completely observed, but it can be taken in different ratio also. So, for example, omega 2 is to omega 1 can be taken 3 is to 1 or it can be taken in different order.

So, in these type of cases when the relation is integer type then generally energy transfer. So, when you are exciting one mode, the other mode got excited due to internal resonance. So, this is the condition for internal resonance; that means, the natural frequencies must be in integer relationship.

So, if they are in integer relationship sometimes due to interaction due to this non-linear forcing or non-linear terms. So, some energy get transferred from the excited mode to the other modes. So, that is why the energy transfer takes place between the modes. So, this giving gives rise to different type of resonance and that type of resonance conditions are known as internal resonance condition internal resonance condition.

So, particularly in case of continuous systems as there are an infinite number of natural frequencies many times some of the natural frequency are in integer relationship. Also by providing different boundary conditions or the support conditions or by making the structure in different way or the configuration topologically one can make the structure in such a way that one may get these internal resonance conditions.

So, there are several papers available for example, in this paper the authors have taken 1 is to 1 internal resonance condition. So, you can see this next paper that is parameter estimation of fuzzy forced duffing equation: numerical performances by extended Runge-Kutta method.

So, you just see you can use already we have discussed that by numerical methods you can solve these equations. So, Runge-Kutta method can easily be used for finding fourth order fifth order Runge-Kutta method can be easily used to find the response of the system by numerically solving this first order differential equation. So, if you have a second order differential equation, first you convert that thing to a set of first order differential equation and then apply this Runge-Kutta method to find the solution.

Non-linear energy harvester with coupled duffing oscillator, so in this paper this for a energy harvester this duffing equation is used. So, here it is written coupled duffing oscillator; that means, the. So, in case of the energy harvester so, two things are there. So, one is for example, this cantilever beam can be used as a energy harvester.

So, where this piezoelectric one can put these piezoelectric paths and so, the motion of the cantilever beam will give rise to a forcing or strain in the piezoelectric paths which give rise to the voltage in the system.

So, you have two different equations one equation for the displacement of the cantilever beam and second equation is the voltage equation. So, there is a coupling if there is coupling between these voltage equation and these beam equation so, then it is a coupled duffing oscillator. So, in this case we are getting a coupled duffing oscillator.

So, the generalized duffing oscillators so, you can see this paper a generalized communication in non-linear science and numerical simulation. So, how this duffing oscillator generalized duffing oscillator is solved those things have been discussed in that paper. So, also you can see some paper like these the primary and sub harmonic simultaneous resonance duffing oscillator.

So, primary resonance occur when your external excitation frequency for example, these external excitation frequency f cos omega t. So, when this omega is near to so when omega is near to the natural frequency of the system. For example, in this case you have n number of natural frequency and when the external forcing external frequency equal to the any of the natural frequency, then you can get the primary resonance.

Then, so, two particularly in case of the non-linear systems two more different type of resonance conditions you may see one is the supercritical and other one is the subcritical resonance condition. In supercritical resonance condition so, this omega will be equal to omega some fraction of the natural frequency and for example, this omega may be equal to omega n by 2 or omega n by 3 omega n by 3. So, in these cases, so it will be super critical or super harmonic resonance conditions.

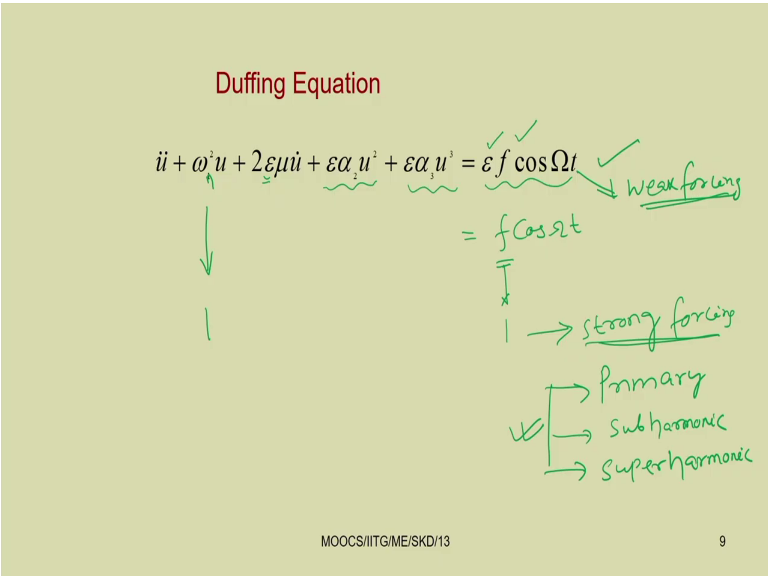
Similarly, this omega may be equal to 3 times omega n or 2 times omega n depending on if the system is having cubic nonlinearity or quadratic non-linearity, in that case you can have these sub critical resonance conditions. So, particularly in case of these non-linear systems we will be interested to study. So, all these resonance conditions sometimes two or more resonance conditions can occur simultaneously.

So, that is why you can see the simultaneous resonance conditions written here primary. So, both primary and sub harmonic resonance conditions are occurring at the same time. So, that is why in this paper so, these primary and sub harmonic resonance condition simultaneous resonance conditions have been discussed.

Similarly, one can apply these stochastic type forcing. So, in stochastic type of forcing one may apply these stochastic averaging for single degree of freedom strongly non-linear system under combined harmonic and poissons white noise excitation, this is discussed in this paper.

So, there are several papers available, but I have shown some sample papers which are published in these 2000 and going to published in 2020 and 2021 so, only these paper in 2020 and 21 have been shown. So, you can go through some recent papers also which are published in different journals related to non-linear vibration ok.

(Refer Slide Time: 27:15)



So, let us see the duffing equation. So, the duffing equation can be written in its generalized form you can write this u double dot plus omega 1 square u or omega generally it is written omega n square u that is natural frequency square of the natural frequency into u plus 2 epsilon mu u dot. So, if we are assuming the damping generally the damping is 1 or 2 order lower than that of the linear term. That is why this bookkeeping parameter epsilon is used here.

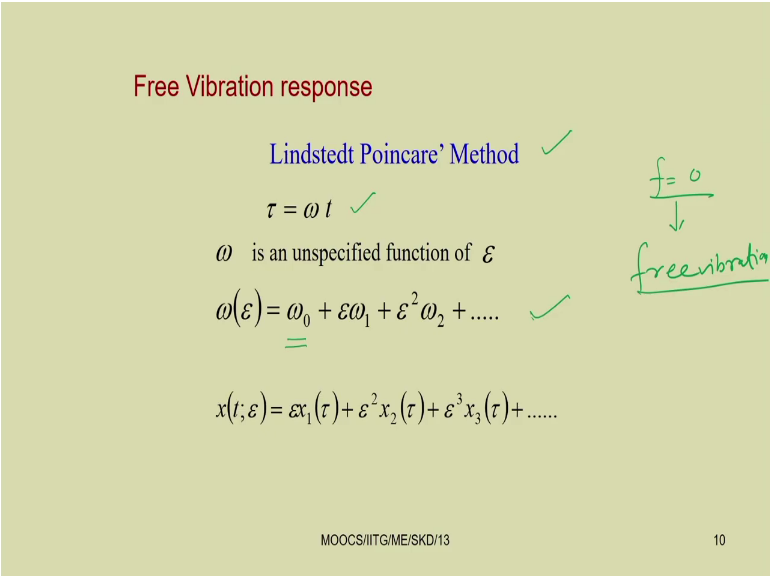
Then this quadratic non-linearity you may put a quadratic non-linearity epsilon alpha 2 u square then a cubic nonlinearity epsilon alpha 3 u cube and then the forcing term. So, if we are assuming the forcing to be weak, then this epsilon term can be used here. So, if it is not weak so, in that case it can be written as f cos omega t.

So, in this case the forcing term for example, if your omega n so, for example, if omega n equal to 1. So, this forcing f also the magnitude of if this omega n square value is 1, then these f also should have a value near to 1. So, in that case we can tell the system is strongly non-linear.

Strong forcing and if you are using this epsilon, then the system is weak forcing, the forcing term is weak. So, later will see that when the forcing is very strong so, in that case we can have. So, this already what I told you that is primary resonance condition, then we may have the secondary resonance conditions like sub harmonic and super harmonic resonance conditions.

So, all these type of resonance conditions we will see, so if the system is will have these strong forcing. So, initially, we will see the free vibration that is when f equal to 0. So, already we have seen these examples when we solved or we have taken the application with different perturbation methods. So, you can solve this numerically or you can use these perturbation methods.

(Refer Slide Time: 29:57)



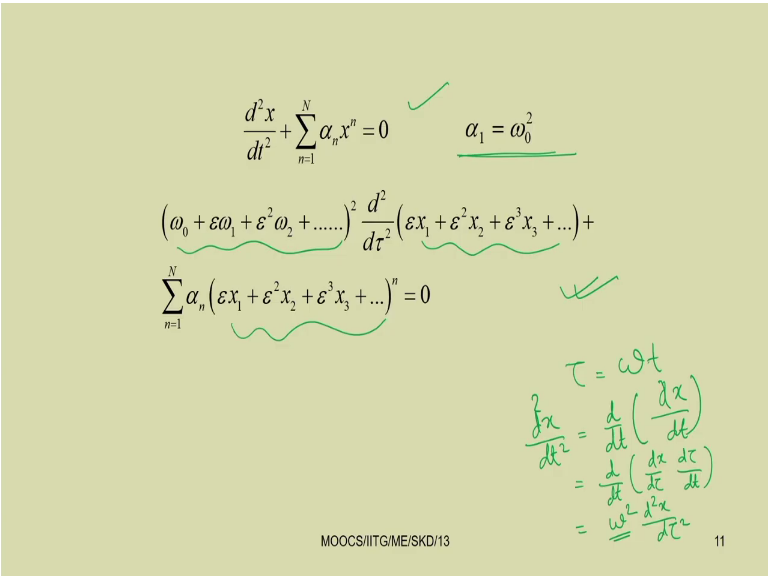
So, let us see already we have seen this thing, but we can revise this part by using this Lindstedt Poincare method you can use the modified Lindstedt Poincare method also or other type of methods like averaging method, method of multiple scales to find the solution. So, when this f equal to 0. So, when f equal to 0 that is the free vibration part free vibration.

So, in case of free vibration so, we will see, so, in case of Lindstedt Poincare method, so, generally we write the equation in this form that is tau we take tau equal to omega t and omega is an unspecified function of epsilon and this omega we are taking equal to omega 0 plus epsilon omega 1 plus epsilon square omega 2.

So, till now this omega neither omega nor this only omega 0 is known to you, omega 0 equal to root over K by m other terms are not known to us. So, the solutions in Lindstedt Poincare method we will take these omega this way and then this x t equal to epsilon x 1 plus epsilon square x 2 plus epsilon cube x 3 and if you have this forcing term, then you can add this x 0 term also.

So, if there is forcing then you can add this x 0 term because in case of forcing so, the response will be non-trivial. So, generally, it will be non-trivial response. So, some response amplitude will be there, but if there is no if there is no forcing continuous forcing to the system the response due to the presence of damping or non-linearity it may come to the trivial state; that means, finally, it may come to 0.

(Refer Slide Time: 31:23)

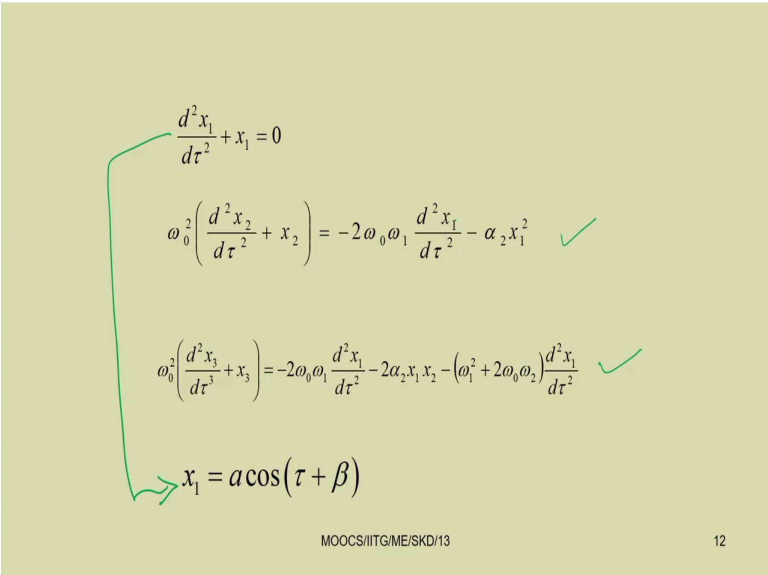


So, if you are taking this equation in this form that is d square x by dt square plus alpha 1 alpha 1 x plus alpha 2x square plus alpha 3x cube up to cubic order if you are taking, then this alpha one equal to omega 0 square. So, now, by substituting that previous equation so, we can write this equation in this form that is omega 0.

So, first what we have to do? So, first we have put this tau equal to omega. So, you just see we have put tau equal to omega t, omega t. So, d square x by dt square d square x by dt square equal to d by dt. So, already we did this thing and again, let me revise these things. So, dx by dt; so, this is equal to d by dt of this is d by d tau into so, dx by d tau into d tau by dt. So, this way this term is omega.

Again, if you are doing this thing, then omega square so, this will come to omega square d square x by d tau square. So, this omega square term will be multiplied so, which is written here. So, this is again this omega can be written as omega 0 plus epsilon omega 1 plus epsilon square omega 2 this whole square d square by d tau square and for x you can substitute this thing and for this one. So, you can substitute for x n you can substitute this.

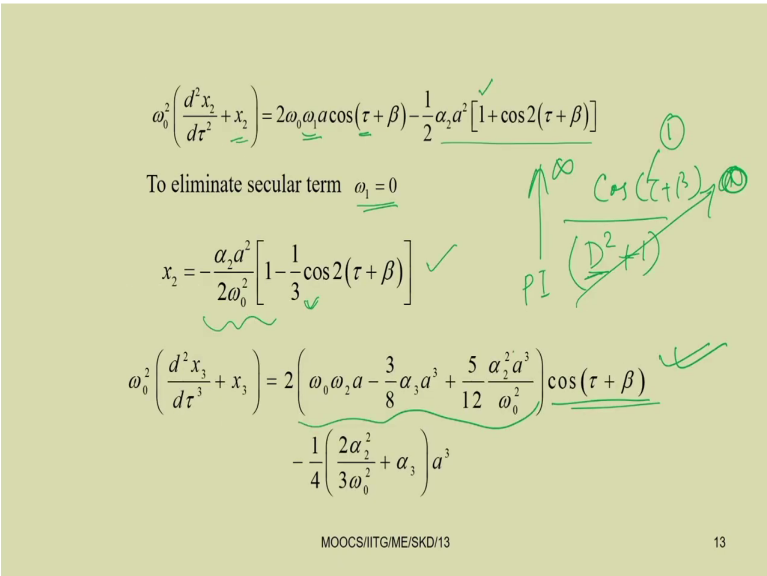
(Refer Slide Time: 33:37)



Now, by separating of the different order of epsilon so, you can write these d square x 1 by d tau square plus x 1 equal to 0. Similarly, plus omega square into d square x 2 by d tau square plus x 2 equal to minus 2 omega 0 omega 1 d square x 1 by d tau square minus alpha 2 x 1 square.

And, similarly the third equation can be written in this way and the solution of the first equation. So, you know can be written this x 1 equal to a cos tau plus beta. Then substituting this x 1 in this equation you can find the expression for x 2 and substituting.

(Refer Slide Time: 34:15)



So, here you can see when you are substituting that equation so, there will be some terms. So, the solution of which will leads to the response to infinite, but actual case the response is bounded that is why. So, those terms are known as secular terms and one must eliminate those secular terms, so that one can get a bounded solution.

So, in this case, so, you can see this omega 0 square into d square x 2 by d tau square plus x 2 equal to 2 omega 0 omega 1 a cos tau plus beta. So, here the coefficient of tau equal to 1 and the coefficient of x 2 equal to 1. So, that is why, so, if you take the particular solution of this one so, in the denominator; so, denominator your D is becoming D square plus 1 as in the denominator D square plus 1 and you have a cos term here cos tau plus beta.

So, as the coefficient is 1 here coefficient is 1 here so, you have to substitute this D square equal to minus 1. D square will be equal to so, this omega. So, this is omega t. So, it will be replaced by minus omega square. So, minus 1 plus 1, this tends to infinite. So, this tends to so, this part tends to 0. So, as this part tends to 0, the whole term this whole term tends to infinite.

So, these leads to, this particular integral tends to infinite. So, the solution so, this term is a secular term and it must be eliminated. So, as you know this cos term has a plus maximum value of 1 and so, it cannot be 0. So, and omega 0 is not 0. So, a equal to 0 will lead to trivial solution. So, the so, if you required a non-trivial solution here omega one must be equal to 0. So, by making omega 1 equal to 0, only we can eliminate the secular term. So, this way the term omega 1 we have determined.

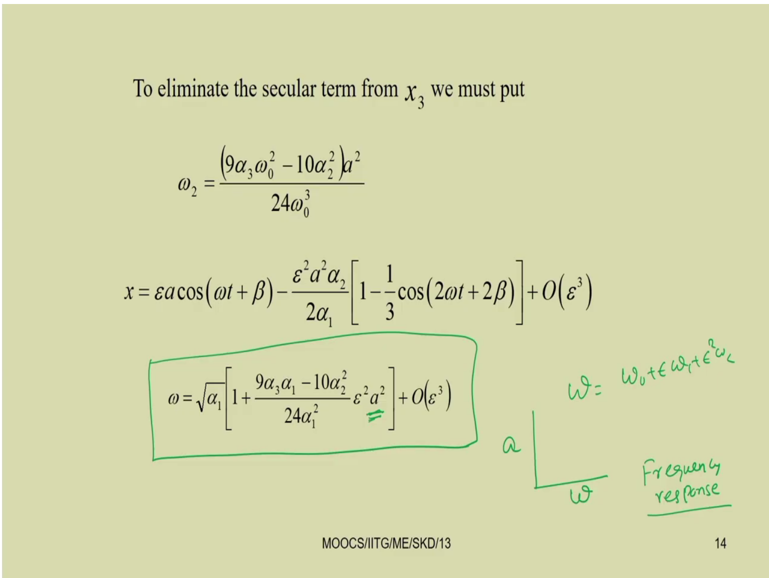
So, now by putting this omega 1 equal to 0, so, we can write the solution of x 2. So, the in the solution of x 2, so, the particular solution only you can write. So, particular solution will contain by putting this omega 1 equal to 0. So, you get this particular solution that is x 2 equal to this.

So, how to get this particular solution? So, this term has already gone. So, this term divided by D square plus 1 into omega 0 square. So, for D square, so, you just see it has two parts. So, this is the constant part and this is the cos part cos 2 tau omega t. So, in case of cos 2 tau, the coefficient is 2. So, you can substitute this D square by minus 4. So, minus 4 plus 1, so, this becomes minus 3. So, this is the term you got.

And, for the first part; so, you can take these D square plus 1 to the numerator with mine to the power minus 1. So, this becomes 1 minus 1 minus D square 1 minus D square of these term whole term. So, in that case differentiation of a constant so, only the constant term will be there; the other part will be 0. So, you can get only these term that is minus alpha to s square by 2 omega 0 square.

So, the x 2 can be written in this form. So, now, by substituting this x 1 and x 2 in the third equation, so, you can get this one. So, here again you have to check the terms which are secular term. So, the coefficient of this cos omega the cos t tau plus beta is the secular term; so these terms in the secular term.

(Refer Slide Time: 38:25)



So, now by eliminating the secular term, so, you can find so, this term can be eliminated. So, if this is equal to 0, so, from these things or you can get omega 2. So, now after getting this omega 1 and omega 2, and keeping these omega equal to up to that thing that is omega 0 plus epsilon omega 1 plus epsilon square omega 2 so, which we have assumed in the beginning, so, omega equal to omega 0 plus epsilon omega 1 plus epsilon square omega 2. So, now we can get the solution. So, that is omega equal to this.

So, here you can note or this is an assignment to you so, you can plot. So, you can see this omega is a function of a omega is a function of a. So, unlike in case of linear system where omega is not function of the response amplitude, in case of the non-linear system the frequency of oscillation the or the frequency of the response depend on the amplitude of excitation amplitude of the response, it depends on the amplitude of the response.

So, you can plot the relation between or you can plot this a versus omega a versus omega and. So, from this equation so, by using this equation you can plot a versus omega which is known as the frequency response plot frequency response ok. So, this way you can find how the frequency is related to the response amplitude.

(Refer Slide Time: 40:15)

$$\ddot{u}+u+0.1x^3=0 \qquad x=0.001\,\text{m and } \dot{x}=0.1\,\text{m/s}.$$

Solution: Here $\omega_0^2=1, \alpha_2=0, \alpha_3=1$ and ε=0.1

Substituting these parameters in equation (3.2.15),

$$\omega=\omega_0\left[1+\frac{9\alpha_3\omega_0^2-10\alpha_2^2}{24\omega_0^4}\varepsilon^2a^2\right]=1\left[1+\frac{9-10\times 0}{24}(0.1)^2a^2\right]=\left[1+\frac{3}{800}a^2\right]$$

Also, $x=\varepsilon a\cos(\omega t+\beta)-\dfrac{\varepsilon^2a^2\alpha_2}{2\omega_0^2}\left[1-\dfrac{1}{3}\cos(2\omega t+2\beta)\right]+O(\varepsilon^3)$

Now from initial condition

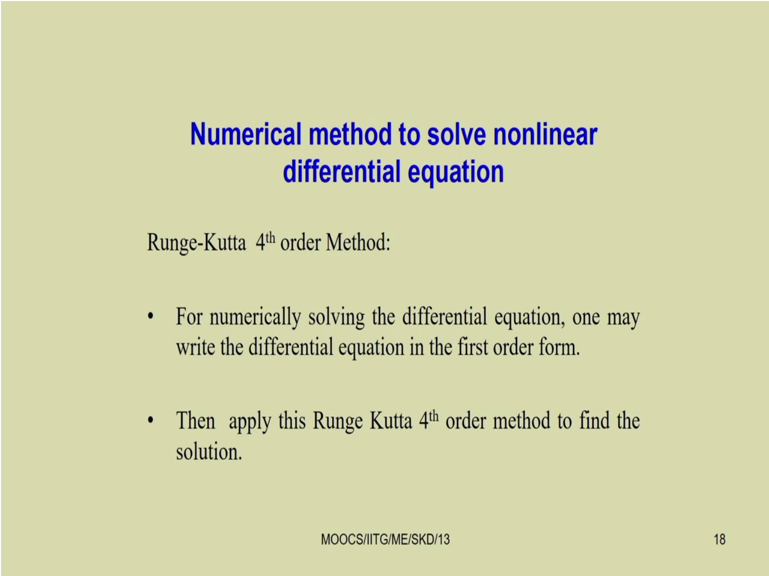
$$0.001=0.1a\cos\beta-\left(\frac{0.01a^2\times 0}{2}\right)\left[1-\frac{1}{3}\cos 2\beta\right]=0.1a\cos\beta$$

$$0.1=-0.1a\omega\sin\beta-\left(\frac{0.01a^2\omega\times 0}{3}\right)\sin 2\beta=-0.1a\omega\sin\beta$$

15

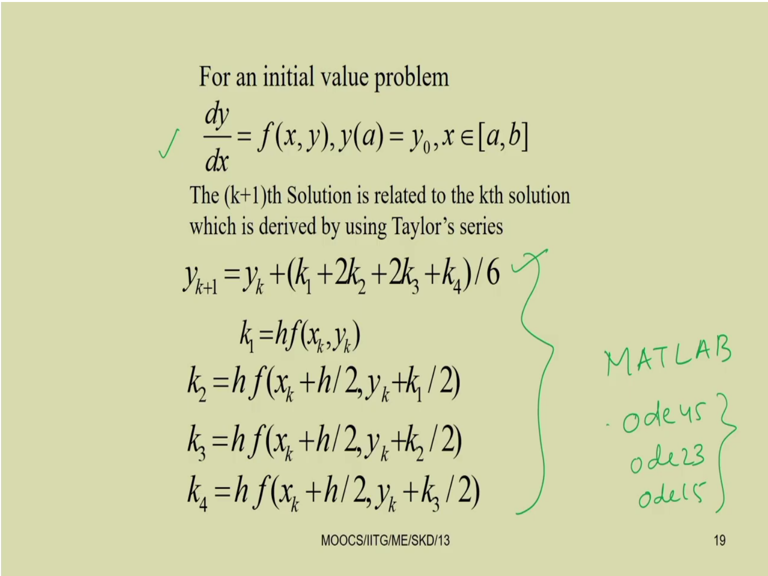
And, this is the free vibration response of the system.

(Refer Slide Time: 40:19)



So, already we have discussed regarding this Runge-Kutta method, 4th order Runge-Kutta method.

(Refer Slide Time: 40:25)

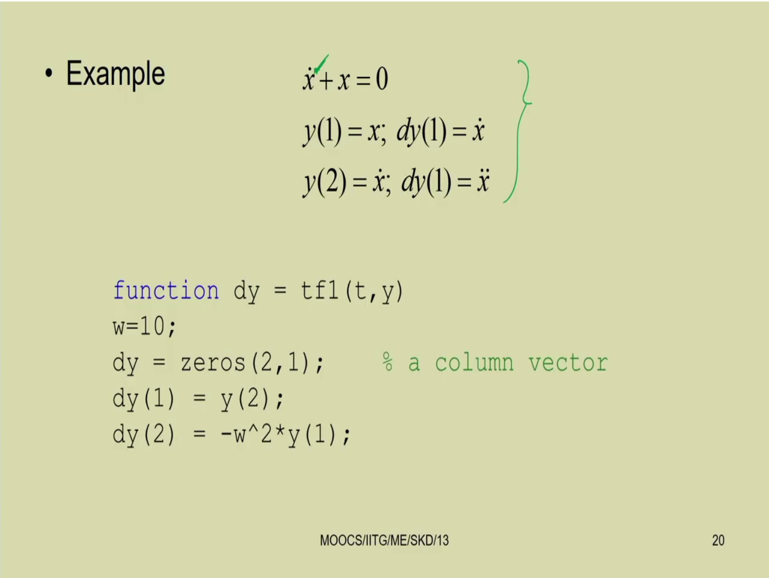


So, in this case, so, these equations, so, you can write your own equation. For example, so, if you have been given a differential equation first order differential equation in this form that is dy by dx equal to f x, y and the initial conditions are given to you. So, you can find by using these 4 equations or using these equation with the 4 coefficients k 1, k 2, k 3, k 4 where h is the step size by taking different step size.

So, you can find the y k plus 1 in terms of y k. So, that means, you can find the response of the system by using this method. So, you can write your own code by using these equations or you can use different functions available in different softwares. For in for example, in MATLAB so, in case of MATLAB so, you have these ode 45, ode 23, ode 15 so, different methods are there for different types of equations.

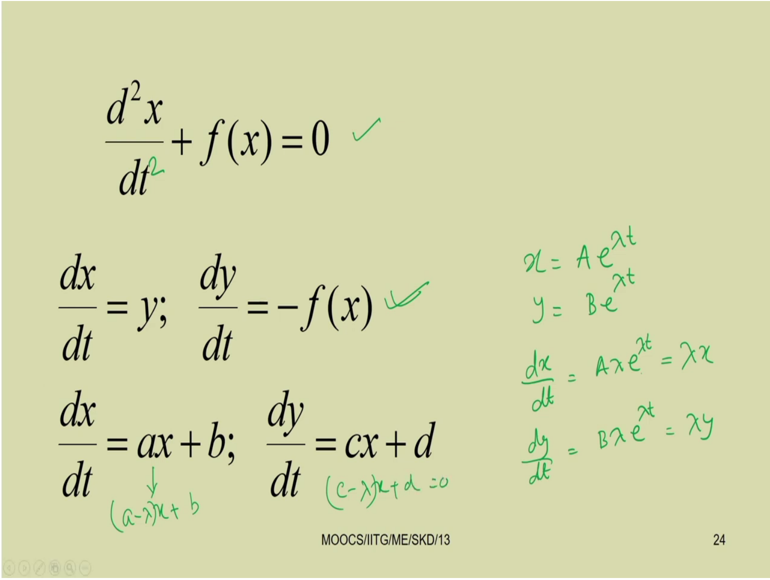
Sometimes the equation may be stiff, so, sometimes the equations may be of different types. So, depending on your equations so, you can use different type of function. Similarly, if I have a delay differential equation, so, you can use this dde instead of ode. So, you can use this dde delay differential equations. So, later we will see one example of delay differential equation in case of the duffing equations also.

(Refer Slide Time: 42:09)



So, if you have a equation like this for example, x dot plus x equal to. So, you can take for example, x double dot plus x equal to 0. So, you can take y 1 equal to x and y 2 equal to y 1 equal to. So, depending on the equation, so, this is a first order equation, so, you can dy 1 equal to x double dot. So, you can find; so, you can write two differential equation and you can solve it. So, these are the response already we have seen this thing.

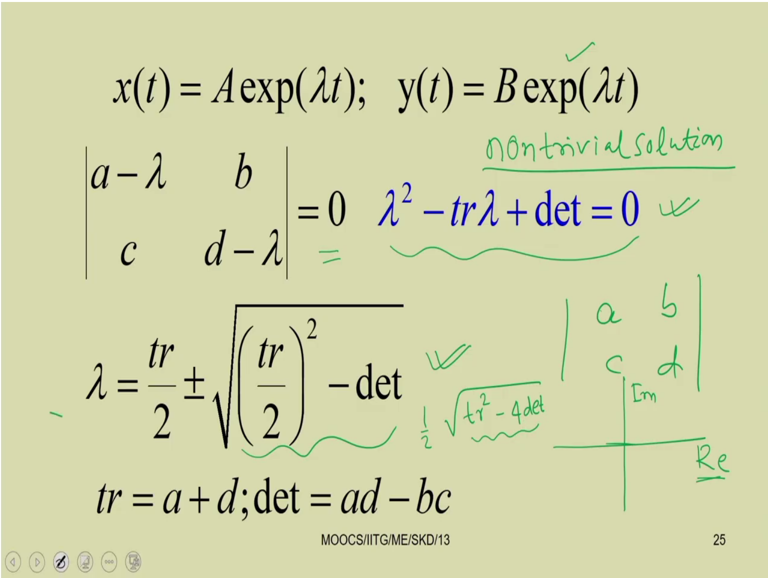
(Refer Slide Time: 42:41)



So, now let us see. So, if you have any generalized equation d square x by dt square d t square plus f x equal to 0. So, always you can write this equation by using two first order equation. For example, if we are taking these dx by dt equal to y, then this dy by dt will be equal to d square x by dt square which is nothing but minus f x.

So, this way you can you have written two first order equation for a given second order equation. So, now if you linearize these equations or linear form if these equations can be written, so, you can write this dx by dt equal to ax plus b. Similarly, this dy by dt equal to cx plus d.

(Refer Slide Time: 43:35)



So, if you assume the solution of x for example, let us assume the solution x t equal to a to the power lambda t and y t equal to B e to the power lambda t. Then these two equation this dx by dt will be equal to a lambda. So, you can write this let x equal to A e to the power lambda t and y equal to B e to the power lambda t.

So, in this case dx by dt will be equal to A lambda e to the power lambda t and similarly dy by dt, so, this will be nothing, but this is B lambda e to the power lambda t. So, but so, this is equal to A e to the power lambda t equal to x. So, this becomes lambda x. So, this becomes lambda y.

So, in this case so, if it is dx by dt so, for dx by dt we can write this is lambda x equal to ax plus b or a minus lambda into x plus b equal to 0. So, this equation reduced to a minus lambda into x plus b equal to 0. Similarly, here also we can write this b. So, this is c minus lambda x plus d equal to 0. So, this way we can write these two equation.

So, then we can write so, or we can write these in matrix form that is a minus lambda b c d minus lambda so, this must be equal to 0. So, for non-trivial solution; for non-trivial if x naught equal to 0, so, that is non-trivial solution. So, for non-trivial solution so, for non-trivial solution so, when that is x naught equal to 0 so, a minus lambda b, c d minus lambda equal to 0.

So, if you take the determinant of these things so, a minus lambda into d minus lambda minus b c equal to minus b c equal to 0. So, this thing can be written in this form that is lambda square minus trace lambda plus determinant equal to 0.

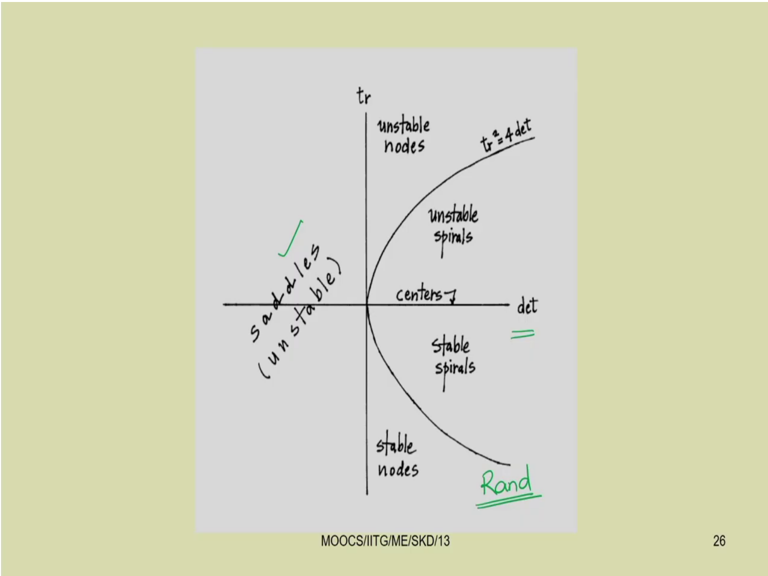
So, where so, for these by solving thisthing you can write this lambda equal to trace by 2 plus minus root over trace by 2 square minus determinant. So, here trace equal to so, this trace equal to this plus this your matrix equal to a b c d. So, trace becomes a plus d and determinants becomes ad minus bc. So, this way so, trace become a plus d and determinant becomes a d minus ad minus b c.

So, depending on so, actually depending on this factor. So, this root over inside this term so, if it is real then you can have depending on the nature of the response of this thing. So, you can have different type of lambda. So, lambda may be real, it may be complex. So, in case of the real, so, the real part may be positive, the real part may be negative. So, all of all or we know.

So, if the rest if the eigenvalues lies in the left hand side of the s plane. So, this is s plane. So, this is the real part, this is the imaginary part. So, we know that so, if these eigenvalue lies in the left hand side of the s plane; that means, it becomes negative then this will be e to the power minus this lambda is negative. So, the response will so, the response will decay, but if this lambda is positive the response will exponentially grow.

So, if the response will exponentially grow so, then the system becomes unstable. So, we must to have a stable system so, we must have this real part of the lambda must be must lie in the left hand side of the s plane. So, if it is lying in the right hand side, any of the eigenvalue is lying in the right hand side of the s plane then the systems becomes unstable.

(Refer Slide Time: 48:05)



So, this way you can see. For example: so, if one plot this determinant versus trace, so, this part is taken from the book by R Rand non-linear vibration book non-linear oscillation book by R Rand. So, if you plot these determinant versus trace so, you can see we can have different type of response.

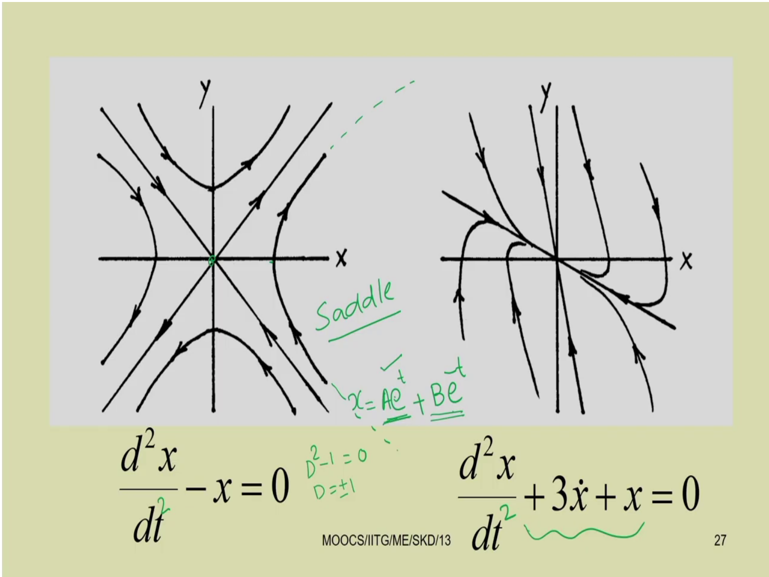
In the previous case we have plotted these real part and imaginary part, but here we are plotting we are plotting this determinant versus the trace. So, when you are plotting this determinant versus trace. So, you can see if in the left hand if it is in the left hand side if the point in the left hand side, the you will get a response so, which is unstable.

So, in that case so, if it is in the left hand side actually so, in that case so, it becomes positive one of the eigenvalue real part will becomes positive. So, that is why you will get the saddle points and so, here also different types of things are there with examples I will show. So, one is your centre. So, then you can get stable spiral, stable node, unstable spiral, unstable node.

So, depending on these tr square equal to 4 determinant. So, tr square equal to 4 determinant from these things. So, this becomes tr square minus 4 determinant. So, this part becomes so, you can take these half outside. So, this becomes tr square minus 4 determinant.

So, depending on tr square, if tr square greater than 4 determinant then this part becomes positive and you can have real roots here. So, if this tr square less than 4 times determinant, then this becomes. So, the root over will be. So, this is a negative number so, the root over will be imaginary. So, in case of imaginary so, you can have the complex root. So, depending on the where it lies, if it lies in the left hand side of the s plane then it becomes stable otherwise it is unstable.

(Refer Slide Time: 50:37)



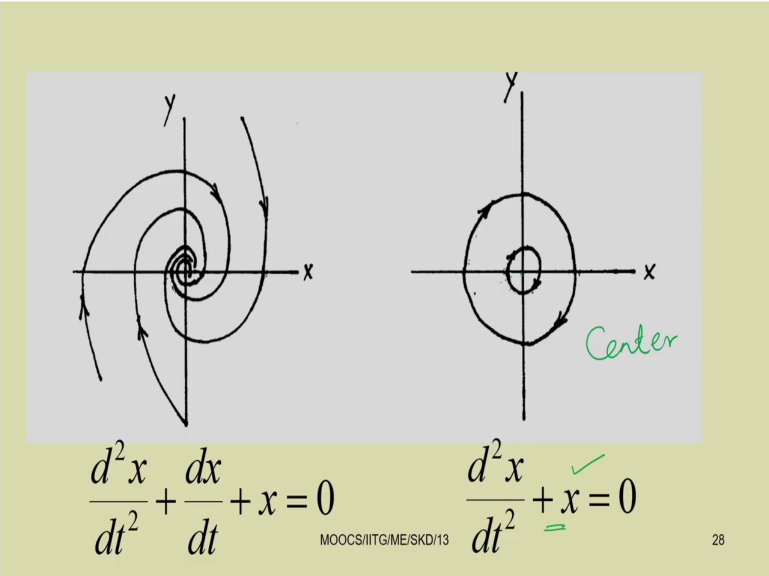
So, let us take this example. So, far for example, let us take this equation d square x by dt square minus x equal to 0. So, the auxiliary equation becomes d square minus 1 equal to 0. Or, so, if you see or d equal to d square equal to 1 so, d equal to d square equal to 1; so, d equal to plus minus 1.

So, if d equal to plus minus 1, so, the response becomes x equal to; so, you can see the x equal to A e to the power so A e to the power, that is one solution is 1. So, this is t plus A e to the power t plus B e to the power minus t. So, due to the presence of this E to the power plus t; that means, the response amplitude will grow with time. So, this part will reduce it, but this part will increase it.

So, if you plot these x versus y that is x versus x dot in this case, so, you can find the solution or the response to be like this. So, it will exponentially grow. So, one part is so, due to the presence of this thing so, it is coming to this one the 0 line but, due to the presence of this part exponentially it will grow and it will go to infinite. So, this way it will start from infinite so, and this way it will go to infinite. So, you can find so, this point is the saddle node point.

So, you can have a saddle node response. So, this type of response is known as saddle or this type of point what you will get, so is the saddle point. So, now let us take equation this thing. So, this is similar to that of a pre vibration of a single degree of freedom system. So, in this case so, you will get the spiral. So, it is spiralling into the system. So, you have a stable point, stable spiral point.

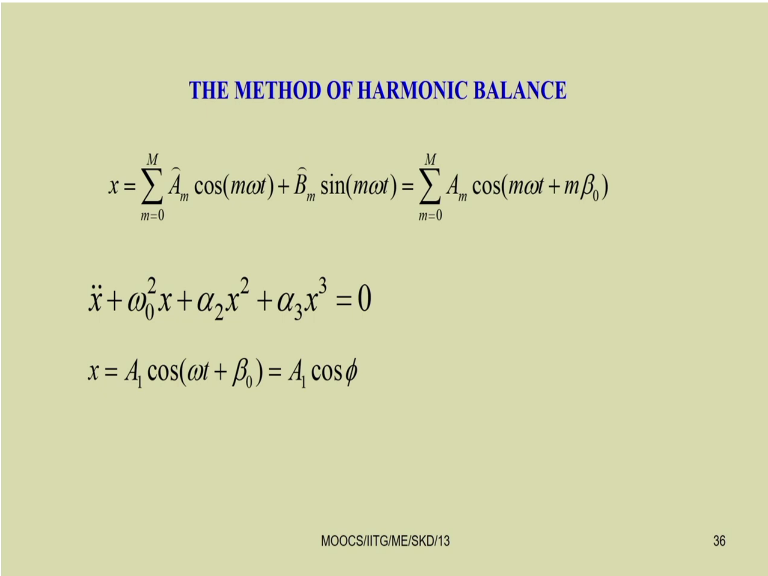
(Refer Slide Time: 52:39)



So, in this case also so, you just see so, you have a stable point. So, here d square x by dt square plus damping plus x, so, it is spiralling here. But, in this previous case so, all the points are coming to the same point. And, if you take this one this is similar to that of a undamped system. So, here so, always, so, this omega equal to 1 so, here omega square equal to 1. So, here so, you will have a response like this. So, this is the center. So, this is center.

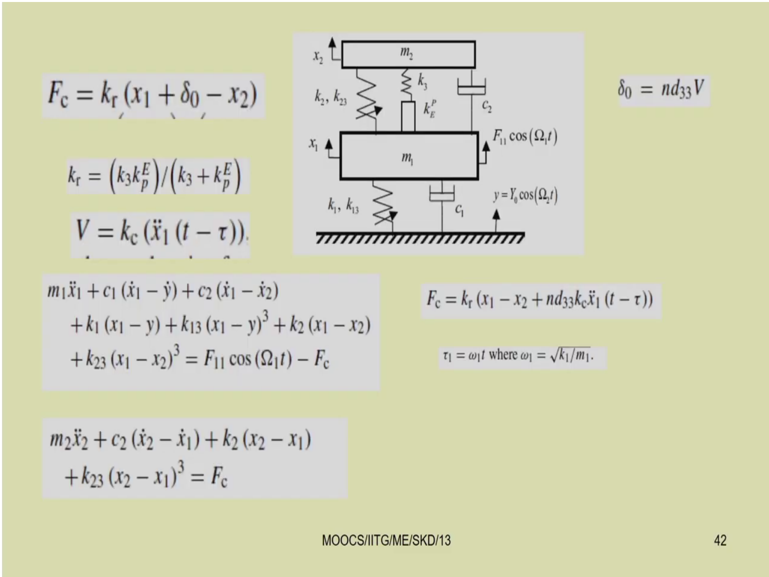
So, depending on the initial conditions so, you can get different types of circles. In this way so, you can analyze a given equation and you can find sometimes also depending on the coefficient. So, you may get the response to be chaotic. So, we will study the force vibration case.

(Refer Slide Time: 53:41)



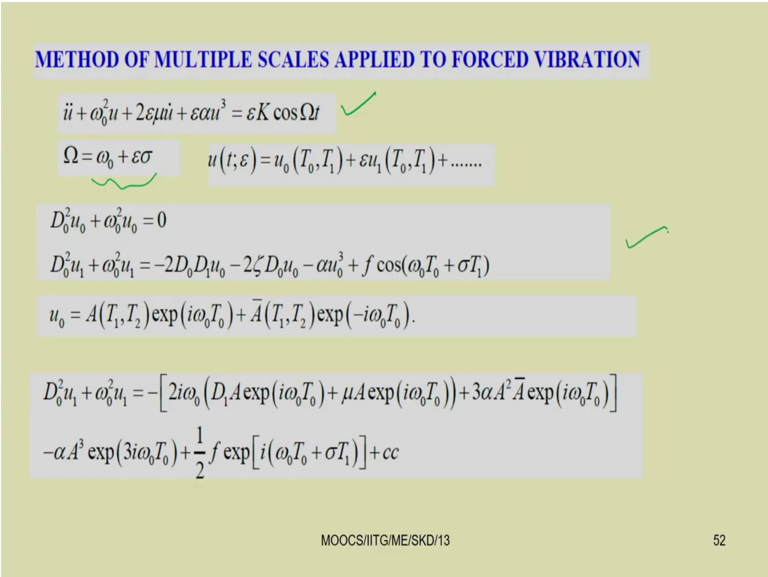
So, for example, already we have seen the force vibration case by using this method of multiple scales.

(Refer Slide Time: 53:49)



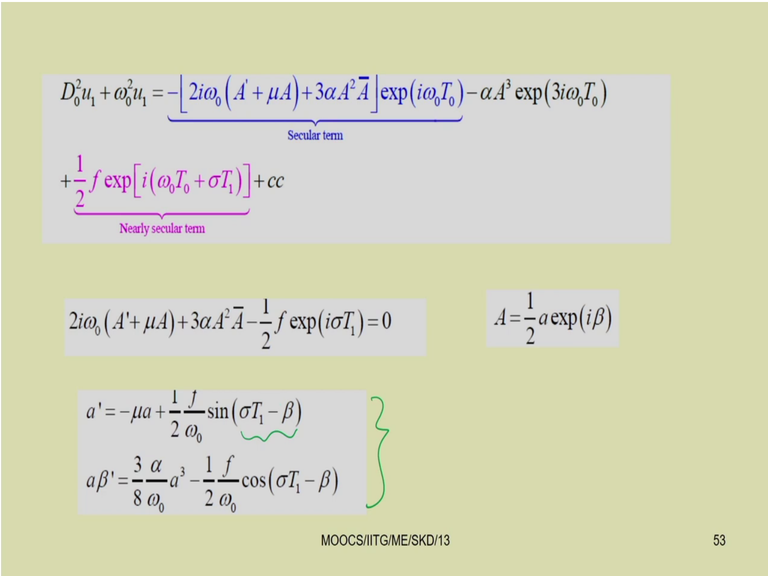
We have taken several examples while we are solving these equations.

(Refer Slide Time: 53:59)



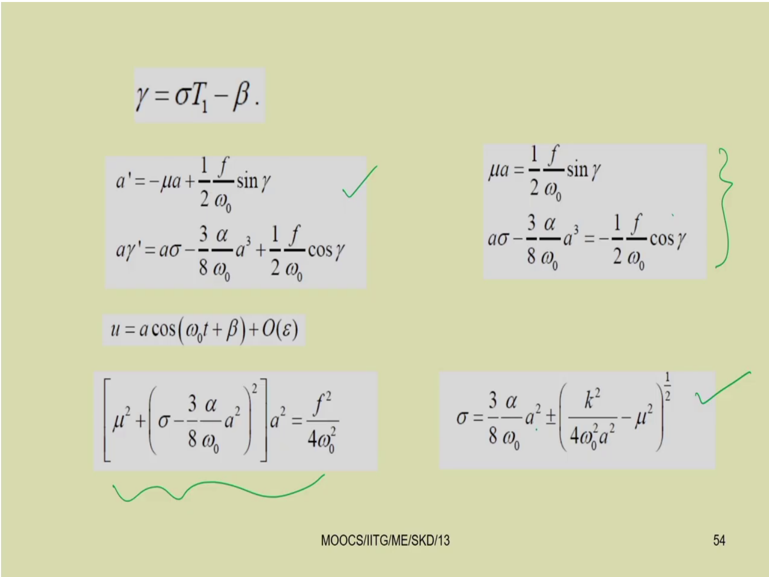
So, for example, we have taken the equation this is the equation we have taken u double dot plus omega 0 square u plus 2 epsilon mu u dot plus epsilon alpha u cube equal to epsilon K cos omega t. So, this is the weak null weak forcing type of thing. So, already you know for the primary resonance conditions we can take omega equal to omega 0 plus epsilon sigma.

(Refer Slide Time: 54:25)



And, if we follow the simple procedure of method of multiple scales and eliminate this eqaul terms so, we can get a set of equations. So, where a dash equal to minus mu a plus half f by omega 0 sin sigma T 1 minus beta and a beta dash equal to 3 by 8 alpha by omega 0 a cube minus half f by omega 0 cos sigma T 1 minus beta where the sigma is known as the detuning parameter. So, in this case you just see this is autonomous equation because it is written in terms of the time. So, to make it autonomous you can take the sigma T 1 minus beta equal to gamma.

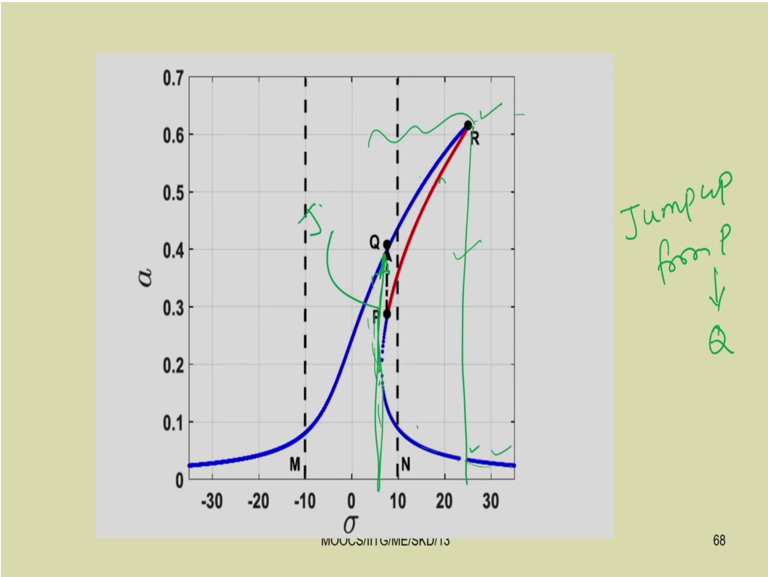
(Refer Slide Time: 55:03)



And, this a dash and gamma dash equation can be written in its autonomous form. And, again for steady state this a dash and gamma dash will be equal to 0 and you can have a set of equations. So, this mu a equal to half f by omega 0 sin gamma and a sigma minus 3 by 8 alpha by omega 0 a cube equal to minus half f by omega 0 cos gamma.

So, by squaring and adding so, you can get a equation this. So, which is sixth ordered in case of the amplitude and second order or quadratic in terms of sigma. So, writing a quadratic equation in terms of sigma, one can solve to get this equation sigma equal to 3 by 8 alpha by omega 0 a square plus minus k square by 4 omega 0 square a square minus mu square root over.

(Refer Slide Time: 55:57)



So, one can plot the frequency response plot. So, a typical frequency response plot looks like this. So, here so, you can observe that so, already just now I told you regarding the stability part. So, you can see that these brands. So, in case of these type of systems, so, you have multi brand system. So, up to these you just see the response we have only single response, but here to here.

So, we have multiple response are there. So, 1 2 3 so, we have three solutions are there. So, before this thing we have only one solution and after this thing also we have one solution. So, we have multiple solutions between these two this position, these versions. So, this is known as the frequency response plot.

So, if we go on increasing the frequency that is sigma is detuning parameter. So, by increasing sigma so, we are increasing the frequency we can see at this point R so, it will jump down to this response. So, because after this thing as it is not continuous. So, the system will jump down from R to this point.

Similarly, by increasing so, due to this jump down the system may break or fail, similarly by increasing the response by increasing or decreasing the sigma so, it will follow this path and at this position it will have a tendency to jump up. So, it will jump up here. So, you can have a jump up phenomena.

So, you can have a jump up phenomena. So, from point P so, it will jump from P to Q and then further decreasing so, it will follow this path Q then M and it will come back this. So, due to this jump up or jump down phenomena the system may have catastrophic failure.

So, next class we will discuss more on these jump up and jump down phenomena and we will study the if the forcing is strong; so, in that case also we will see simultaneous, if there is some simultaneous resonance conditions or sub harmonic or super harmonic resonance conditions, so.

Thank you very much.