Nonlinear Vibration

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Lecture - 16
Nonlinear system with hard excitations

So, welcome to today class of Non-linear Vibration. Today class we are going to discuss regarding different type of damping in non-linear systems.

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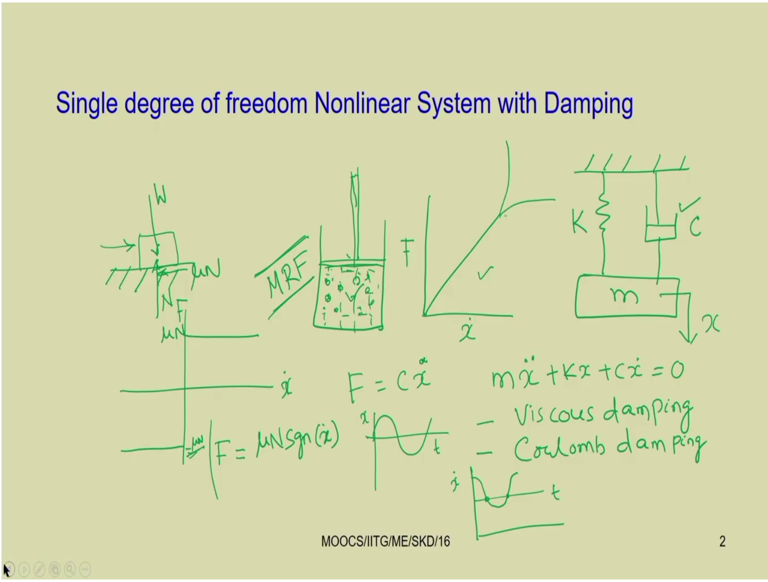

So, we will take the single degree of freedom, non-linear systems or single degree of freedom system with different type of damping. And we are going to use different perturbation methods what we have studied before for finding the response in case of the systems. So, we are familiar with the system with a damper. For example, this spring mass damper systems.

In case of a linear spring mass damper system in case of a linear spring mass damper system, so we know by using different method, so the equation of motion can be written so this is C this is K. The equation of motion can be written, so if this is x, then it can be written in terms of mx double dot plus Kx plus Cx dot equal to 0 for the pre vibration and for force vibration. So it will be some force forcing term can be given.

So, today class we are going to see the free vibration response in case of single degree of freedom system with different type of damping. So these type of damping where we are use a damper, so we can use a damper, so the damper is nothing but so a cylinder piston arrangement we can give. So, it is a cylinder piston arrangements.

So inside these things we have fluid. So, in this we are applying for example; so, this is the cylinder piston arrangement and in the cylinder piston arrangement so we are applying this force.

So, here the force is proportional to, so force is proportional to x dot. So the force can be written as F equal to C x dot. So, in case of, so there are several other type of damping also present in the system. So, one just now I told you that is the viscous damping. We may have these dry friction or Coulomb type of damping.

So, coulomb damping; in case of Coulomb damping, so we know so, if two surfaces are there and they are moving on each other, so for example, this is a surface and on the surface, so let this is the body moving on this thing, then we can have or the these two body. So due to this inter molecular friction so they will experience this Coulomb friction.

And, in case of Coulomb friction the force we know it will be retain in this form that is, so let the weight is W. So let the weight is W. So, here the reaction force equal to N. So, the friction force if the, if we are applying a force in this direction, so a friction will generate and that friction force will act in this direction. So, here in this direction and magnitude equal to mu N.

So we know, this friction force F in this case can be written as mu N signum x dot. So it depends on the velocity. So, if we plot the velocity x dot versus this force F, friction force F, so it will have a value of mu N. So this is equal to mu N and for less than x dot less than 0, so it will be equal to minus mu N. So, it will have two value.

So, plus mu N for x dot greater than 0 and minus mu N for x dot less than 0. So, when the velocity changing from minus to plus, so it will subjected to a jump. It will jump from minus mu N to plus mu N or plus mu N to minus mu N. In a particular vibrating system, so let if it is vibrating in a sinusoidal way if the displacement is taking place in sine curve let this is x versus t, so we know the velocity will takes place. So it will be, velocity can be cos term. So if x is sin then the velocity will be cos.

So, here so we can see x dot versus t this is x dot versus t. So, in a particular cycle, so, you can observe that the velocity changing from positive to negative here and also from negative to positive here. So, in a particular cycle the velocity changes from plus to minus or changes its sign twice.

So, during this period so or during this time the response or the force will be changing from minus mu N to plus mu N. So, due to this change, so this the system is not linear, the system is a non-linear system.

So, in case of the viscous damping also, the force what we have taken x versus x dot, so it may be linear like this what we have shown or it may be non-linear. So, we can have a non-linear force it can be extended this way or this way. So, the force can be non-linear also and it can be passive or active.

For example; so, in this case in this particular case, it is passive; that means, this force whenever we are applying this force would exert a force on this thing but we can make it active. For example, by applying some iron particles. Let us put some iron particle in this and apply magnetic field.

So, when we are applying a magnetic field this iron particle will align themselves. The iron particle will align themselves and the viscosity the property the viscosity will be changed, the viscosity of the system will change.

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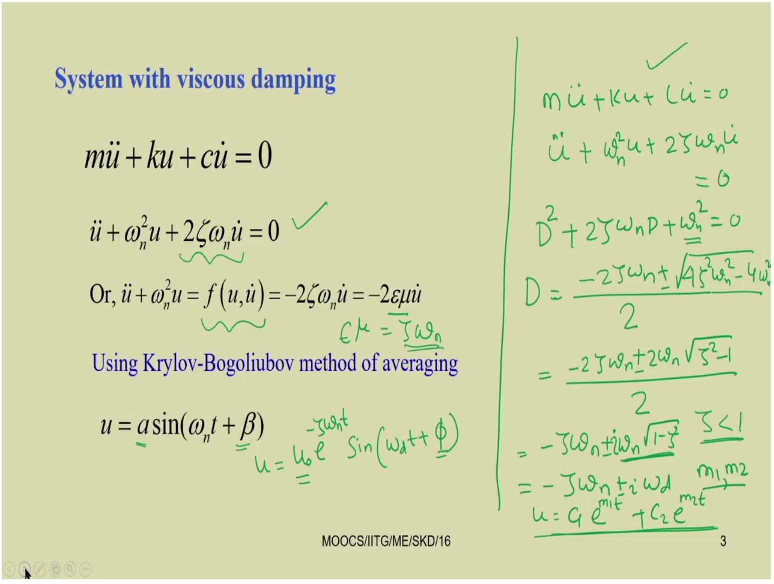

In that case so whenever we want we can change the damping property of the system by applying external magnetic field. As we are able to change the damping by applying this magnetic field then we can make the system or we can make the damper a active one. Here particularly, if you see this MRF, Magneto Rheological Fluid, so by using this magneto rheological fluid type of things we can make this damper active. Otherwise by simple fluid it can be a passive damper.

Now you know about the passive damper and the active damper also. So, in case of active damper, so we can change the property damping property by applying some external stimulation. For example, so we have applied in this case the magnetic field and by applying this magnetic field, so we have changed the damping property.

Let us now see the analysis of a system. So, let us start with a simple linear system with this damping. So in this case, already you are familiar with the solution. So you know, so if your system is m u double dot plus k u plus c u dot equal to 0. Now, by dividing this m you can write this equation equal to u double dot plus k by m k by m is nothing but omega n square u plus c by m that is equal to 2 zeta omega n u dot. So, it is equal to 0.

So, the solution of this equation already we have found by using or we know by writing down the auxiliary equation. So, in this case the auxiliary equation is nothing but this is D square plus 2 zeta omega n D plus omega n square equal to 0. And also we know, so here the root D will be equal to minus b, that is equal to minus 2 zeta omega n plus minus root over b square minus 4 ac by 2 a, b square equal to 4 zeta square omega n square minus.

So beta square minus here we have 4 b square minus 4 ac 4 into omega n square, c equal to omega n square by 2 a by 2 a. We have this value. So, already we know, so if you take this 4 outside 4 omega n square outside so this becomes minus 2 zeta omega n. So, plus minus 2 omega n root over, so this becomes zeta square minus 1 divided by 2.

So, depending on the value of zeta for example, it if it is under damped. So, zeta will be less than 1, if it is over damped, so zeta will be greater than 1 or for critical damped zeta will be equal to 1. So, let us take the case of when it is under damped. So in case of under damped, so this zeta is less than 1. So inside term is negative. We can take this minus outside, so this root over minus 1 so; that means, this is becomes i.

So, this thing we can write equal to minus zeta omega n plus minus omega n into root over 1 minus zeta square and here we can have this i. Here the two roots. So, we have obtained two roots. So, the roots are minus zeta omega n plus minus i omega n into root over 1 minus zeta square, but this term generally we tell this as omega n into root over 1 minus zeta square we tell as omega d damped natural frequency.

So, this will be equal to minus zeta omega n plus minus i omega d. So, the solution u will be equal to C 1 e to the power, so e to the power minus zeta, this roots we can write, so this is equal to D 1 e to the power D.

So, this is D 1 and D 2. I can write or I can write the roots as m 1 and m 2. If the roots are written as m 1 and m 2, then it will be equal to e to the power m 1 t plus C 2 e to the power m 2 t. And already you know by simplifying this thing so the solution can be written in a very simplified way.

So, the solution can be written. So, we will see equal to u will be equal to we can write u 0 e to the power minus zeta omega n t sin omega d t plus phi. Where this u 0 and phi are constant which can be obtained from initial condition.

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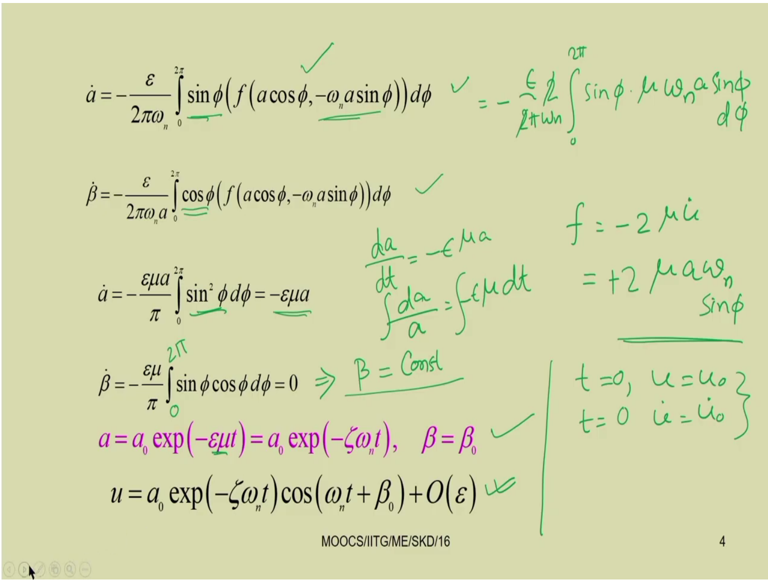

This derivation is familiar to all of us. So, which we derived in the vibration course, linear vibration course. But, let us now apply this Krylov-Bogoliubov method of averaging or any other perturbation method to find the solution of this equation. Now, this m u double dot dividing by m already we have written this equation in this form.

And we can write by using this book keeping parameter also, so we can write to apply this method of averaging. So, left side we can keep this u double dot plus omega n square u and other terms we can shift to right hand side and write that thing as f u u dot. So here f u u dot, so only this damping term will go to right hand side, so this will becomes minus 2 zeta omega n u dot. So, we can write that thing using the book keeping parameter in this way. So this will be equal to minus 2 epsilon mu u dot.

So, minus 2 epsilon mu u dot. So here epsilon mu is nothing but zeta omega n. So, it is equal to zeta omega n, epsilon mu we can write it equal to zeta omega n. So, now by applying this solid, we know this Krylov-Bogoliubov method. And, by applying this Krylov-Bogoliubov method, so we can write the solution u equal to a sin omega n t plus beta.

So, here the difference from the linear equation and this non-linear solution method is. So, this a and beta are slowly varying function of this displacement and velocity; that is u u dot. So here, a and beta are not constant, but a and beta are function of time, so they also vary with time. So, a and beta are slowly varying function of u and u dot or slowly varying function of time.

So, by taking this u equal to a sin omega n t plus beta, so already we have derived this in this Krylov-Bogoliubov method and this a and beta can be written equal to a dot equal to minus epsilon 2 pi omega n integration 0 to 2 pi. So, we have to multiply sin phi for a dot we have to multiply sin phi and for beta dot we have to multiply cos phi with the f, and here u will be replaced by a cos phi and u dot will be replaced by minus omega n a sin phi.

So, by performing this integration so we can find a dot and beta dot. This is the standard procedure for using this Krylov-Bogoliubov method. So, now this f is nothing but so this is equal to minus epsilon by 2 minus epsilon by 2 pi omega n integration 0 to 2 pi. So, we have to multiply the sin phi into 2 zeta. So, our f so we can check what is this f. f equal to you can see this is equal to minus 2 minus 2 epsilon mu u dot minus 2 epsilon mu u dot.

So, you just see this f equal to minus 2 mu epsilon mu u dot. So, here then I should substitute. So, this minus minus plus then, so this 2 I can take outside also, then so this is equal to or epsilon also can be taken outside then this epsilon 2 we have taken. So this is mu, mu also is a constant that thing also can be taken outside, but for the time being you put it here. And u dot, so for u dot so we have to substitute.

So, this again this plus into multiplication of minus, so this becomes minus; so minus omega n a sin phi, so sin phi d phi. So, this integration becomes so this is d phi. So, this integration becomes sin phi multiplied by sin phi. So, sin phi multiplied by sin phi so this becomes sin square, so this becomes sin square phi and we got this minus.

So, minus epsilon mu a by pi, so 2 2 cancel already you have taken this epsilon. So, this becomes minus epsilon mu a by pi integration 0 to 2 pi sin square phi d phi.

So, this integration becomes minus epsilon mu a. Similarly, for beta dot so for this, so this thing can be written as minus 2, so you just see f equal to epsilon f, we are writing epsilon f. So, f will be equal to this epsilon will go mu for u dot it is equal to minus minus plus a omega n sin phi a omega n into sin phi.

So, this is the thing you have to introduce in this beta dot. So, the integration will be minus epsilon mu by pi integration sin phi into cos phi d phi. So, integration of sin phi into cos phi so we multiplied 2 and divided by 2. So this integration is for 0 to 2 pi. So this is equal to sin 2 phi sin 2 phi by so 2 you have multiplied and divided, so 2 sin phi into cos phi equal to sin 2 pi, so this integration becomes 0.

So, this beta dot equal to 0. So as beta dot equal to 0, so beta it must be equal to d beta by d t equal to 0 so beta must be a constant. So beta must be a constant. So, this constant can be taken as beta 0. So, here we got a dot equal to minus, so this a dot means so d a by d t. So a dot equal to d a by d t we got equal to minus epsilon mu a. So, d a divided by or d a by epsilon mu a equal to minus d t.

So, now integrating both the side, so this is so or it is d a by so we can put this d a by a equal to d a by only simply a you divide so d a by a will be equal to minus mu epsilon mu d t. So, this is l n a integration both the side. So, it becomes l n a and here it becomes so l n a equal to it l n a equal to minus epsilon mu t.

So, a will be equal to a equal to e to the power minus epsilon mu t or now substituting this mu equal to zeta in terms epsilon mu in terms of zeta omega n. So, we can write this equation a equal to a 0 e to the power minus zeta omega n t and beta equal to beta 0. So, now we got this expression for a and so u can be written in terms of a cos so already we have written u in terms of a.

So here we are assuming u to be a sin omega n t plus beta, so already we have seen a and beta are not constant, but they are slowly varying function of time. So substituting this value of a and beta so we can write u equal to a 0 e to the power minus zeta omega n t cos omega n t plus beta 0.

So now, by applying this initial condition for example, let us take at t equal to, so if we put at a t equal to 0, that is initial condition, if u equal to u 0 and a t equal to 0 u dot equal to u 0 dot, so we can substitute it in this equation and its derivative and we can find.

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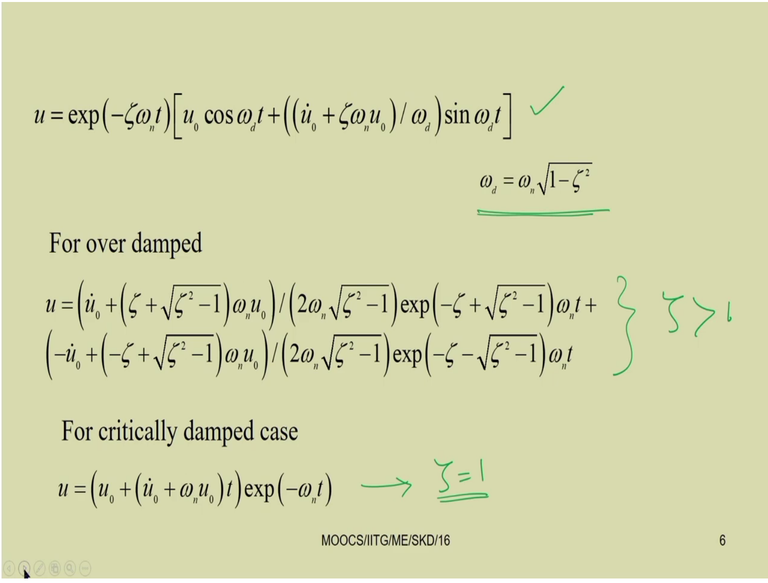

So, you can find that thus u will be equal to minus zeta omega n t u 0 cos omega d t plus u 0 dot plus zeta omega n u 0 by omega d sin omega d t, so, where omega d t equal to omega d equal to omega n into root over 1 minus zeta square. So, from the auxiliary equation also we got this similar expression. So, either by using this Krylov-Bogoliubov method or by using this classical method so you are getting the similar expression.

So, now for similarly for over damped system also we can find this expression. So, for over damped case, so in case of over damped we can find this expression and for critically damped. So for over damped case this zeta greater than zeta greater than 1 and for critically damped case zeta equal to 1. So, in both the cases similarly we can get this expression.

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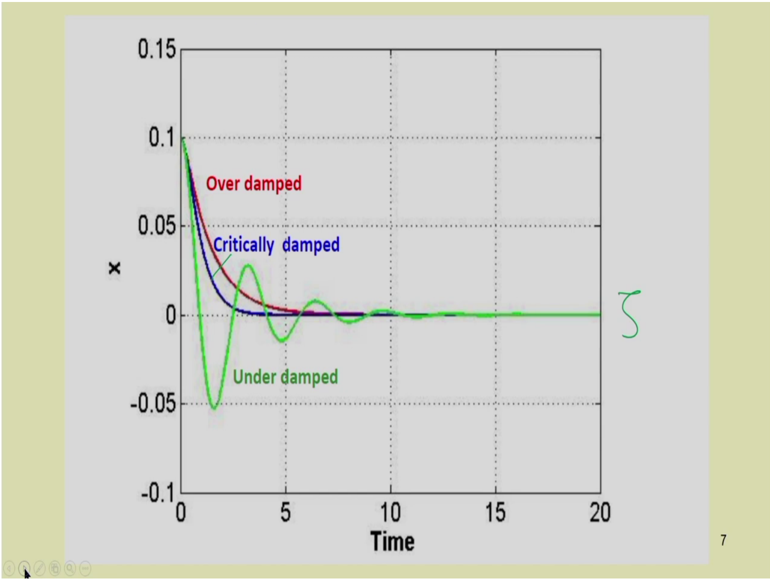

And, by plotting this all this three, so you can see this is the over damped system, this response is for a over damped system. The middle 1 is for critically damped and the oscillatory motion is for under damped case. So, in case of under damped system the response gets or reduces to it is this steady state value and come to the steady state value as t tends to infinite.

So, depending on the value of zeta that is the damping parameter, so the time can be determined. So, how much time it will take to reach to the steady state oscillation. So, by increasing the value of zeta so it will take less time to reach this steady state, but with lower value of damping so the system will oscillate for more time and it will take more time to come to the steady state solution.

So, already we are familiar with this viscously damped linear system, but here we have applied this Krylov-Bogoliubov method to find the response pre vibration response of a single degree of freedom system. So, you can add this cubic non-linearity into the system and follow the similar method to find the solution for the pre vibration response. So, in that case this u can be written so there will be minor modification in this expression.

So, this a dot can be only this f term can be changed. So, if we are having for example, let us take the dropping equation with this non-linearity.

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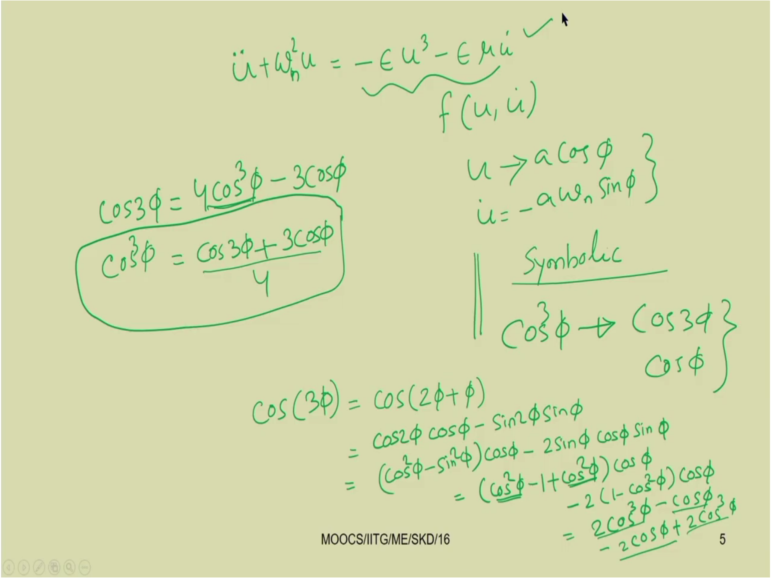

So, in that case the equation can be written u double dot plus omega n square u or omega 0 square u. So, it will be equal to minus epsilon u cube, then minus epsilon mu u dot, so this is the expression. So, this term will be equal to f u u dot. So, already you know so for u you have to substitute a cos phi and for u dot. So, it is equal to minus a omega n sin phi.

So by substituting this so for u cube, so it will be equal to a cube cos cube pi, so then for a dot so we have to multiply sin and for beta dot we have to multiply cos and integrate it and by so we can do this integration symbolically using the symbolic software. We can use this symbolic software like this maple, mathematica or in MATLAB itself also we can do that thing or manually also we can do as this is this involve only cubic order cos cube theta.

So first this cos cube theta you can write in terms of cos 3 theta and 3 ok cos 3 theta in terms of cos 3 theta can be written and cos 3 theta and cos theta or this cos cube phi first you must convert it write in terms of cos 3 phi and cos phi. So this thing already you know. So, just you expand this cos 3 phi. So, if you have forgotten, so this way you can do also.

So, cos 3 phi so you can write this equal to, if you have forgotten this formula from the first principle also you can do plus cos 2 phi plus phi. So cos a plus b equal to cos square 2 phi minus sin square phi. Cos a plus b equal to cos a into so cos a plus b, so cos a plus b equal to cos a into cos b minus sin a into sin a into sin b sin phi. So, now, so this is equal to so you can write this cos 2 phi equal to cos square phi minus sin square phi into cos phi minus for sin 2 phi 2 sin phi into cos phi into sin phi.

So this gives rise to, so here again what you can do? So, this is equal to cos square phi; for sin square phi you can write equal to 1 minus cos square phi. So these becomes 1 minus minus plus 1 plus cos square phi into cos phi minus so this is sin phi into sin phi sin square phi, so 2 into sin square phi equal to 1 minus cos square phi 1 minus cos square phi into cos phi.

So, this way you can derive and you just see all the terms have been written in terms of cos. So, this is cos square phi plus cos square phi, so this is 2 cos square phi into cos phi, so this becomes, so 2 cos cube phi. So, this becomes for the first term it becomes 2 cos cube phi cos square 2 cos square. So then this becomes minus cos phi.

So, for the first term this becomes 2 cos cube phi minus cos phi and these next term becomes, so minus 2 cos phi minus 2 cos phi, then minus 2 cos square phi into cos phi minus minus plus plus 2 cos cube phi. So, this we can write this cos 3 phi equal to so you just see this cos 3 phi can be written. So, you have seen this cos 3 phi equal to so from this thing we can we have found.

So, this is 2 cos cube phi plus 2 cos cube phi, so this becomes 4 cos cube phi then minus so this is cos phi this is minus 2 cos phi then this becomes minus 3 cos phi. So, for the u cube term what we have found, so we can replace this u cube that is cos cube. So, it will be u cube cos cube phi. So in case of cos cube phi so we can write this cos cube phi in this case. So, cos cube phi will be equal to cos 3 phi plus 3 cos phi divided by 4.

So, this divided by so this divided by 4. So, we can use this expression while doing this integration manually, otherwise if you want to do this integration using the symbolic software also, so you can do it. So, this way so instead of a linear system so if you have a dropping equation with this damping non-linearity, you can use this KB method Krylov-Bogoliubov method to find the solution of the system ok.

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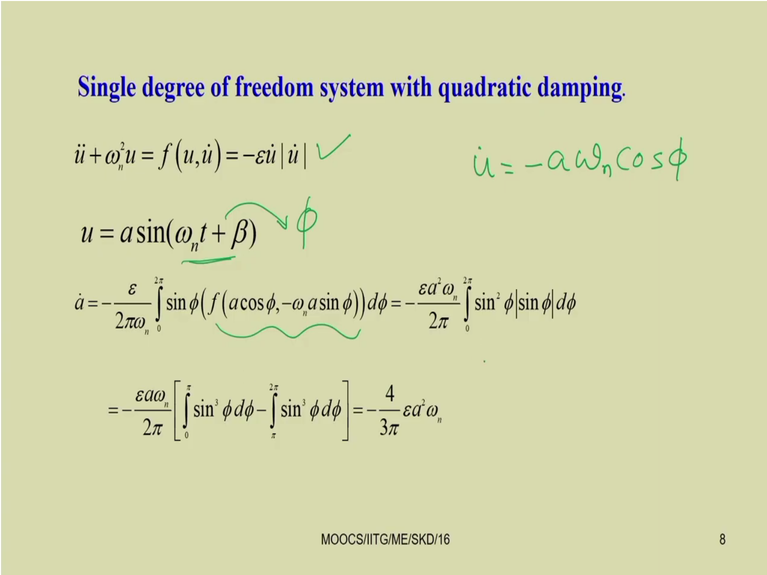

So, let us see some other type of damping also. So, single degree of freedom system with quadratic damping. So, we can take the quadratic damping in this form that is; u double dot plus omega n square u plus epsilon u dot multiplied by u dot. So, in this case so we can write this is the quadratic damping. So now, let us apply this similarly KB method or method of averaging.

So, here also we can write this u equal to solution u equal to a sin omega n t plus beta; where a and beta are slowly varying function of time. So, here what we can do, so we can write this a dot equal to so already we have written f equal to epsilon u dot u dot and already you know. So, this u dot can be written as minus a omega n a omega n. So this is sin we have taken, so it will be u dot will be equal to cos phi.

So, this omega n t plus beta we are taking omega n t plus beta as phi. So, this a dot equal to minus epsilon by 2 pi omega n 0 to 2 pi sin phi into this f. So for f already I told you, so u dot can be replaced by the term, so this becomes so we can have this thing a dot equal to sin phi into u dot into u dot and similarly. So, here we can take actually the solution to be u equal to a sin omega n t plus beta.

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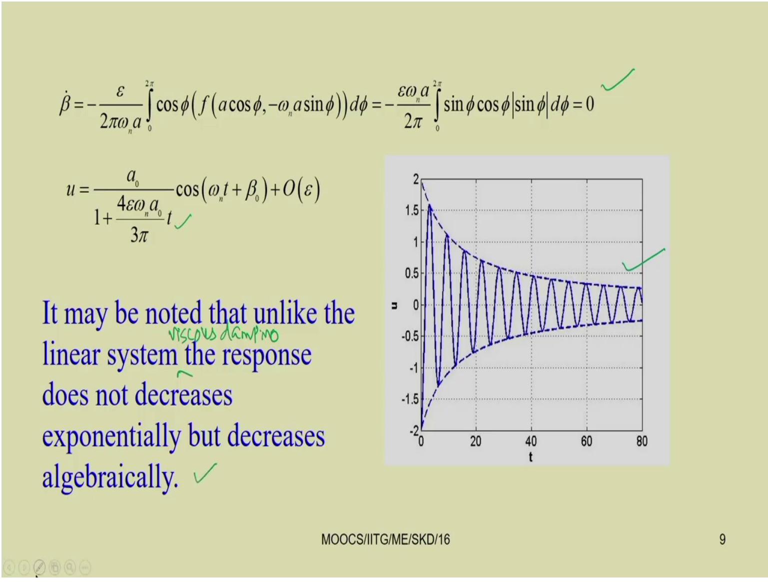

Then beta dot equal to cos phi into this term we can put. So, then this become we can this integration become 0. So, we can find this u equal to a 0 by u 1 plus 4 epsilon omega n a 0 by 3 pi t. So, here you just see this t term. So, we have a t term in the bottom. So into cos omega n t plus beta 0.

So, by taking different initial condition, for example, taking this initial condition a 0 equal to 2 and different value of omega n, so, if you plot then you can get a response like this. So, it may be noted that or like in case of linear system the response does not decreases.

Exponentially in the previous case we have seen in case of viscous damping on like in case of the linear system with viscous damping with viscous damping. So, in case of viscous damping so you have seen the response decreases exponentially, but in this case the response is not decreasing exponentially, what it decreases algebraically.

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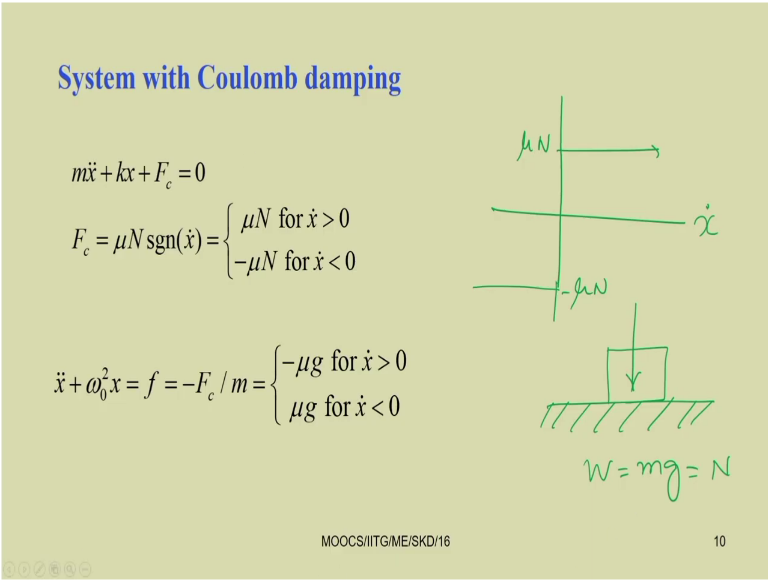

So, similarly we can take another case also system with coulomb damping. So, in case of coulomb damping already I told you so here the forcing so we can it can be written minus mu N. So, this is mu plus mu N and minus mu N, so this is minus mu N this is x dot. So, this for example, let us have a system like this mx double dot plus k x plus. So, you have a system.

So, in which so this is the mass so total force equal to mg, total weight is acting. So that weight W equal to mg.

So, that is equal to the normal reaction force N. So, it is equal to F c equal to mu N signum x dot, so in that case it is mu N for x dot greater than 0 minus mu N for x dot less than 0. So the equation reduces to by dividing this m equation reduces to x double dot plus omega 0 square x equal to F, F is nothing but minus F c by m.

So, which will be so by substituting N equal to mg, m m will cancel and this will be equal to minus mu g for x dot greater than 0 and mu g for x dot less than equal to 0.

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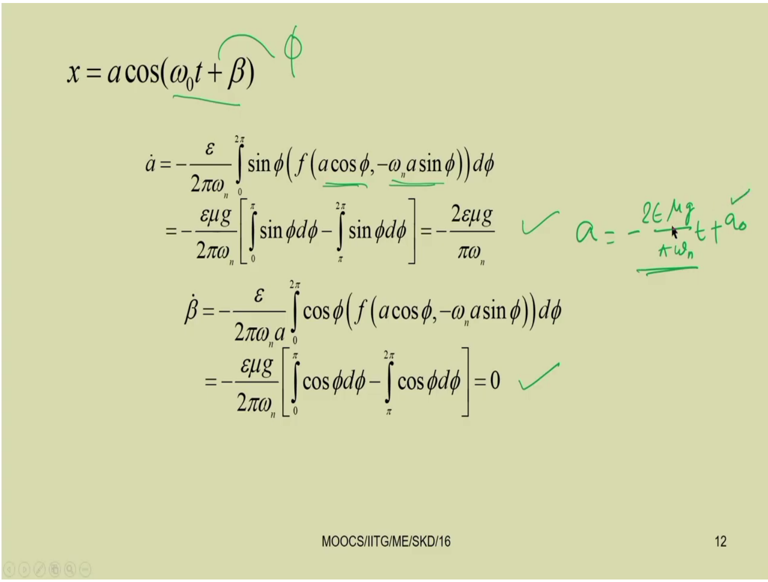

So, now by applying, so, writing this x equal to a cos omega 0 t omega 0 t plus beta which is equal to phi which is equal to phi. So we can write this equation equal to minus epsilon 2 pi omega n 0 to 2 pi sin phi into f. So for u we are substituting a cos phi and for u dot we are substituting minus mu n a sin phi.

So, d phi, so this integration is giving rise to minus 2 minus 2 epsilon mu g by pi omega n. So, beta dot equal to this integration becomes 0, so a dot we got equal to minus 2 epsilon mu g by pi omega n. So, this a, so here right hand side it not a function of a. So this is a constant then. So, beta dot equal to 0, so this is also constant.

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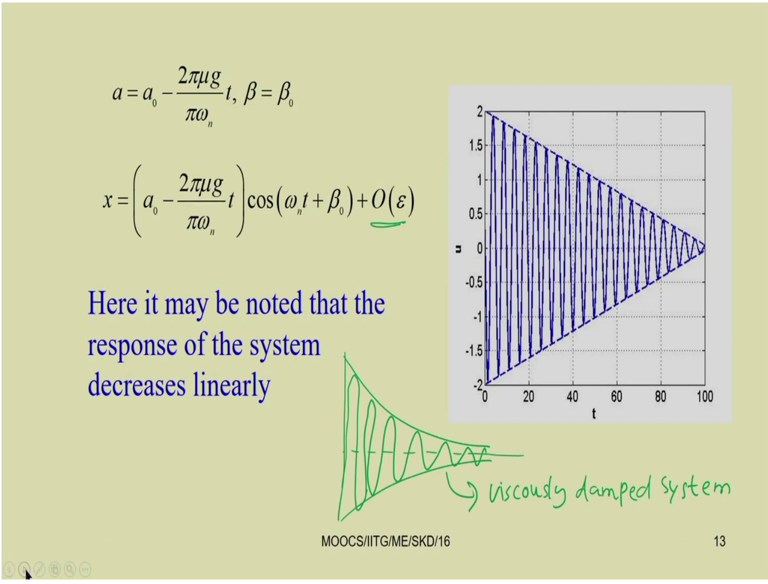

So, equal to so we can write this a equal to a 0 minus by a equal to a 0 minus 2 pi mu g by pi omega n t and beta equal to beta 0. So, you can see this thing by integrating this a dot equal to this, a will be equal to minus 2 epsilon mu g by pi omega n t plus a 0. So, this part is constant. So this is constant, so integration of this thing is into t. So, this is plus a 0. Now, by substituting this two. So we can write this x equal to a 0 minus 2 pi mu g by pi omega n t cos omega n t plus beta 0 plus order of epsilon.

Here, you can see here it may be noted that the response of the system decreases linearly. So, this is here decreasing in a linear fashion. So previously we have seen in case of the viscous damping so it decreases exponentially. So, in case of viscous damping the response decreases exponentially and in case of a decreases exponentially and in case of in case of negative damping the second case we have taken u u dot into multiplied by mod u dot.

So, there also we have seen it decreases algebraically and here it is decreasing linearly. So this is in case of the viscously damped system. So, this is the case of viscously damped system. This way given a system so you can use this averaging method to find the response of the system.

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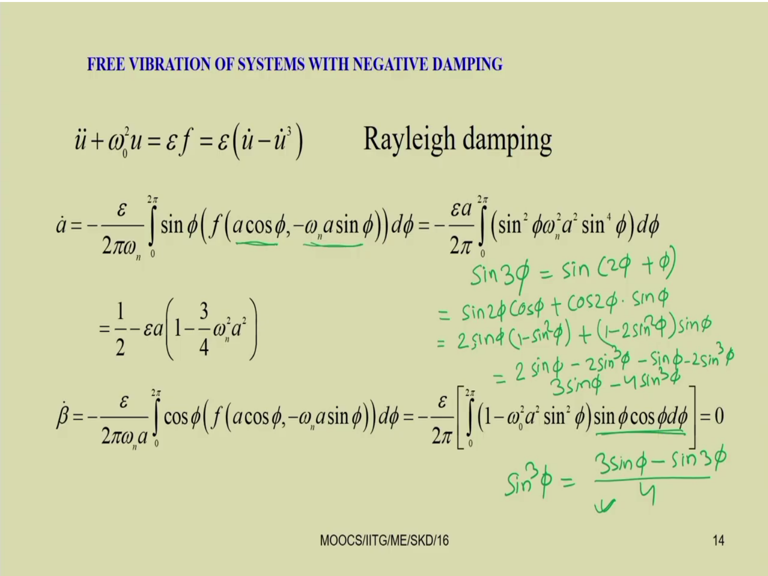

So, let us see one more type of system here we will take the Rayleigh damping. So, in case of Rayleigh damping the equation can be written in this form that is u double dot plus omega 0 square u equal to epsilon f equal to epsilon into u dot minus u dot cube. Here, you can see already you are familiar with this Rayleigh damping and you know that it will leads to a limit cycle. Let us find this response. So, in this case a dot will be equal to minus epsilon 2 pi omega n 0 to 2 pi sin phi f a cos phi minus omega n a sin phi.

So for u you are substituting a cos phi and for u dot you are substituting minus omega n a sin phi. We have this u dot and u dot cube. So, for u dot so the equation will be epsilon into u dot, that is minus omega n a sin phi then minus u dot cube minus u dot cube. So, then it will give rise to minus a cube omega n cube sin cube phi.

Like we have derived this expression for cos cube phi. Here also you can derive this expression for sin cube phi you can expand the term sin 3 phi and you can find that thing. For example, sin 3 phi you can write. So sin 3 phi you can write equal to sin 2 phi plus phi. So you can express all this things in terms of sin so that it can be written as sin a into cos b cos phi plus cos a into sin phi.

So, this sin 2 phi is nothing but 2 sin phi into cos phi. So cos phi into cos phi this becomes cos square phi and this cos square phi you can write in terms of 1 minus sin square phi. Similarly, for this cos 2 phi it is cos square phi minus sin square phi. This cos square phi can be written as 1 minus sin square phi. So again sin square phi so 1 minus 2 sin square phi 1 minus 2 sin square phi into sin phi.

You can write this equal to this becomes 2 sin phi 2 sin phi minus 2 sin cube phi, here you got this minus sin phi minus 2 sin cube phi. This becomes 3 sin phi 3 sin phi minus 4 sin cube phi. So from this thing you can find the expression for sin cube phi. So, sin cube phi becomes sin cube phi will be equal to 3 sin phi, 3 sin phi minus sin 3 phi divided by 4.

For u dot cube, so you will have sin cube phi a cube omega n cube sin cube phi and for sin cube phi you can substitute this thing. So this will be easier for your integration if you are doing it manually. But if you are using some symbolic software like MATLAB, maple, mathematica, then you need not have to expand it or if you want to expand also you expand and then substitute to find the expression.

Now this a dot you can find in this form. So, a dot equal to minus a dot equal to half minus epsilon a into 1 minus 3 by 4 omega n square a square. You can write this a dot equal to d a by d t, so d a by d t equal to this. So d a by this term will be equal to d t. So, now by integrating this you can get this expression for a. similarly, beta dot equal to, so now beta dot as it contain the sin phi into cos phi. So, this integration becomes 0. So beta dot equal to 0, so you got beta equal to constant.

So, beta equal to constant, but a equal to so you got a some expression for a, so from there you can get this a square also.

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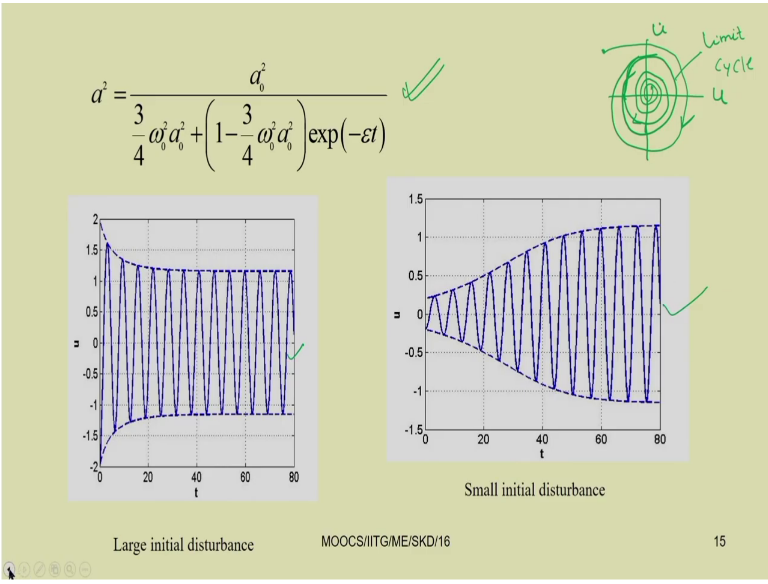

So, a square equal to so the expression becomes a square equal to a 0 square by 3 by 4 omega 0 square a 0 square plus 1 minus 3 by 4 omega 0 square a 0 square e to the power minus epsilon t. This way you can find the expression for a versus t that is the time response and you can plot. So, this is written in terms of x I think, so this is u. So, if you plot this u versus t, so this is the required curve.

So you just see, if you are having large initial condition. So for example, you have started with 2. So if you have started with 2, so you just see it decreases 2 or minus 2 so it decreases and reach to that mean so you can if you plot this u versus u dot. So, this is u versus u dot, it will be limit cycle. So if you start from any other point outside this, so it will spiral down and it will come to this position.

So, it will spiral and come to this orbit this limit cycle. And if you are taking starting from some for example, you have started from this initial condition, so it will spiral off and it will go to this limit cycle. So, it will here, it will increase. So, in the second case if you are starting with small initial condition then it will goes on increasing, the response will goes on increasing and reach this limit cycle limit cycle.

Or if you take a value start with an higher initial condition, so it will decrease and come to the response amplitude decrease and come to this response this limit cycle. So that thing you have clearly observed it here. So, with higher value, so, it is decreasing and for lower value so it is increasing and you are getting this periodic response and finally, it goes to the limit cycle. So, this way you can solve Rayleigh damping. So here also you may note so if you have let us add the term of let us add the term alpha u cube or epsilon alpha u cube.

So in that case, so you have to take this f term to right hand side that epsilon alpha cube term to the right hand side and replace this u by a cos phi and perform this integration and find the response. So, you can solve a non-linear equation or non-linear equation with cubic non-linearity and Rayleigh damping and find the respective solution by using this averaging method or averaging by method by KB method; Krylov-Bogoliubov method.

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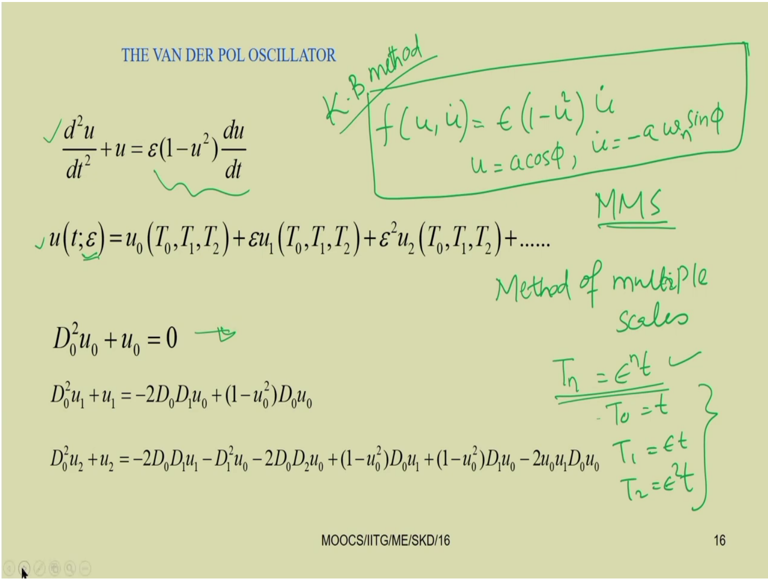

So, let us take another type of example also. So, in this case we are taking this Van Der Pol oscillator. So you already know that we can reduce this Rayleigh oscillator to that of the Van Der Pol oscillator also by changing the variable. Here the equation Van Der Pol equation can be written in this form that is d square u by d t square plus u equal to epsilon into 1 minus u square into d u by d t.

So, you can conveniently use this method of averaging that is KB method to find the solution by substituting u equal to a cos phi in this equation. So, in that case so this will be your f u in this case your f u u dot. So if you are taking KB method I am just explaining, but here we are we have taken another method that is method of multiple scale and we have solved, but as an assignments, so you can take KB method also and solve this thing.

So, function f u u dot equal to epsilon 1 minus u square u dot; here you have to substitute u by a cos phi phi equal to omega n t plus beta and u dot equal to minus a omega n sin phi and perform this integration to find a dot and beta dot and find the response. So, if you want to use this KB method, so in KB method Krylov-Bogoliubov method.

So, let us use another methods. So here we are using method of multiple scale method of multiple scales. So, already you are familiar in this method that we are taking different time scales that is T n equal to epsilon n t, and then we are expanding this derivatives and taking this T n equal to epsilon n t, so that T 0 equal to t, T 1 equal to epsilon t and T 2 equal to epsilon square t.

So, these are different time scales we have taken. So, this is t we have taken different time scales and we can perform this analysis. Similarly, here u is assumed to be u as a function of t and epsilon parameter book keeping parameter. So we are writing u equal to u 0 using different time scale that is T 0 T 1 T 2, so that way we have written. u equal to u 0 T 0 T 1 T 2 plus epsilon u 1 T 0 T 1 T 2 plus epsilon square u 2 T 0 T 1 T 2.

We can take only two terms or we can take 3 terms also or more number of terms also we can take. So, taking these three terms, so, we can write substituting this equation in this first equation. We can have and separating different order of epsilon, we can write down this equation D 0 square u 0 plus u 0 equal to 0 D 0 square u 1 plus u 1 equal to minus 2 D 0 D 1 u 0 plus 1 minus u 0 square D 0 u 0.

Similarly, D 0 square u 2 plus u 2 equal to minus 2 D 0 D 1 u 1 minus D 1 square u 0 minus 2 D 0 D 2 u 0 plus 1 minus u 0 square D 0 D 1 plus 1 minus u 0 square D 1 u 0 minus 2 u 0 u 1 into D 0 u 0. So, this way we can write of different order of epsilon.

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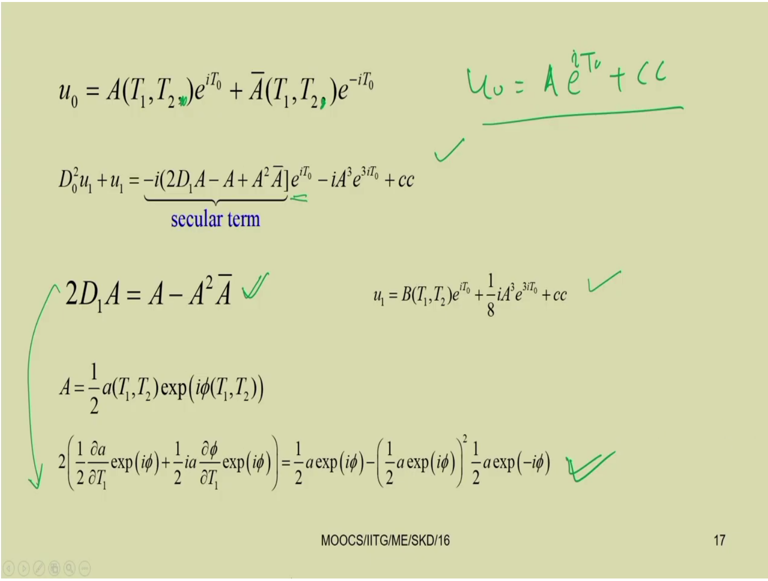

So, now the solution of this D 0 u 0 plus u 0 is known to you that is u 0 equal to, so we can write this u 0 equal to A T 1 T 2 as a cannot be a function of T 0, so it can be a function of T 1 and T 2. So it can be written a T 1 T 2 e to the power i T 0 plus a bar T 1 T 2 e to the power minus i T 0, sometimes we use to write u 0 equal to a e to the power i T 0 plus cc complex conjugate.

So, instead of writing complex conjugate you can write put the terms also, then substituting this u 0 in the D 0 square u 1 plus u 1 equation, so we get this one. So, here you can see the coefficient of e to the power i T 0 will leads to the secular term.

So, we must eliminate the secular term as we know the response will contain or response is bounded now by to kill the secular term. So what we can do? This term can be eliminated if and only if this term equal to 0.

So this term will be equal to 0 when this 2 D 1 A will be equal to A minus A square A bar. So, the solution so now eliminating that term, so we can write the particular integral for u 1. So, u 1 equal to beta T 1 T 2 e to the power i T 0 plus 1 by 8 i A cube e to the power 3 i T 0 plus complex conjugate.

So, now by substituting this u 0 and u 1 in the other equations, we can write or if you want to have the first order solution. So then what we can do? So, here itself you can substitute A equal to half a e to the power i beta or i phi. So by substituting A equal to half a e to the power i phi. So this is equal to so this term can be written equal to 2 D 1 A.

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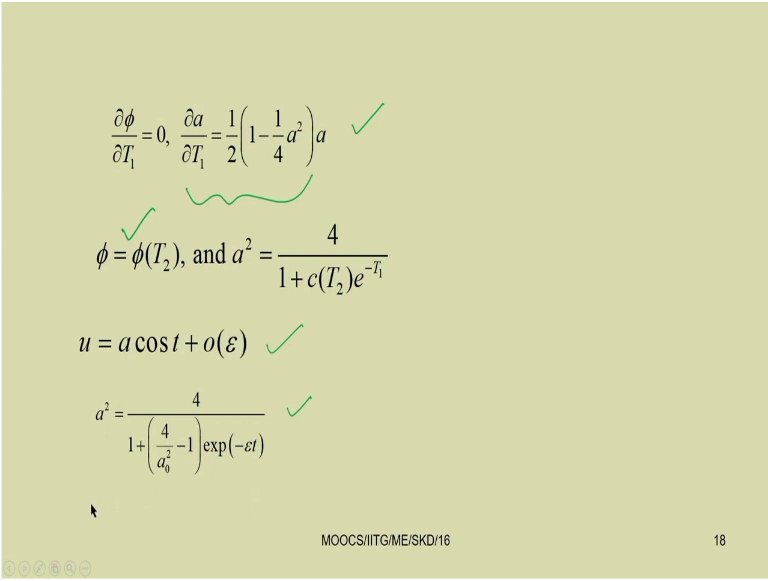

So by substituting so we got this equation. So now, by separating the real and imaginary part real and imaginary part we can obtain this thing. So, that is d phi by d T 1 equal to 0 and d a by d T 1 equal to half 1 minus 1 by 4 a square into a. So from this thing, so as this is constant, so phi equal to phi will not be a function of T 1, so it will be a function of T 2.

Similarly, a square from the second equations we can take this d a by this whole term will be equal to d T 1 and by integrating that thing. So we can get this a square equal to 4 by 1 plus c T 2 e to the power minus T 1. Now, we can substitute u equal to a cos t, where this a can be written by using this expression. So, from this thing, one can plot the response with respect to time. So here you just see a is not a constant, but a is a function of time.

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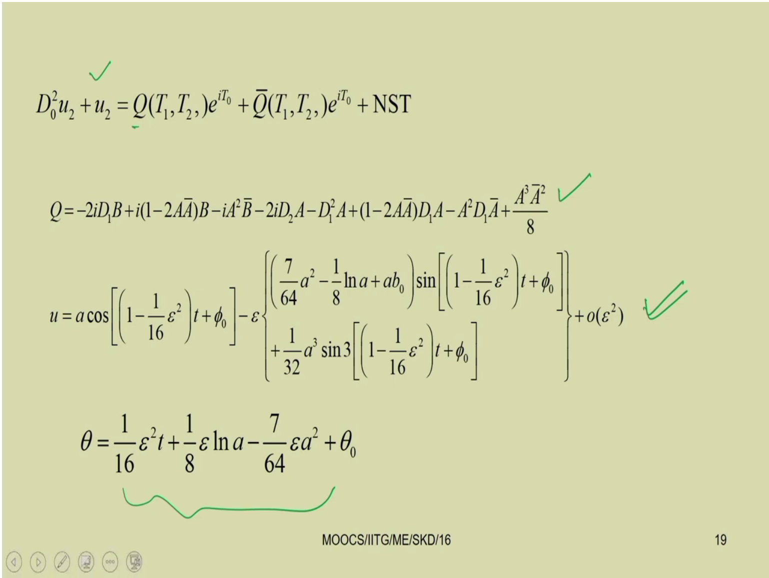

But if we want to go for this higher order solution, can take substitute this expression for u 0 and u 1 in the third equation; that is D 0 square u 2 plus u 2 equal to the whole term, where this Q can be written using this. Now, by substituting again a equal to half A e to the power i beta. So, we can get this expression for u in this form where theta equal to, so where you can see this theta equal to 1 by 16 epsilon square t plus 1 by 8 a cos.

This whole term can be written 1 by 16 epsilon square t plus 1 by 8 epsilon l n a minus 7 by 64 epsilon a square plus theta 0.

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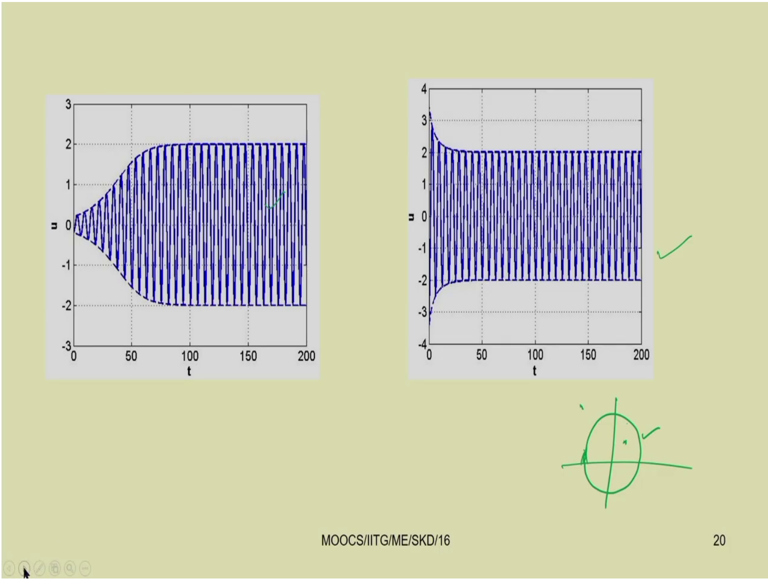

So, in this way you can write and then you can plot this expression so you have 60. You can find the response in this case. So, here also you just see if you are starting from a small value if the initial condition is small then it is growing and it is reaching to a the limit cycle. Similarly, if you are taking this initial condition from minus 3 that is a higher value. So it is come coming back to the limit cycle. So the limit cycle is at 2. So you just see at 2 the limit cycle is at 2.

So, if you plot this u versus u dot so you will have a limit cycle and if you are taking some initial condition outside this thing, so it will come to this limit cycle, it will decrease and come to the limit cycle and if you are taking a point inside this thing it will grow and it will come to this initial point.

So, you can actually physically understand this by looking this equation of motion. So, what is happening? We are taking a small term or that is a term inside this and if you are taking a term outside this, how the damping is affected and how it is reducing the response to the limit cycle you can understand physically.

So, in addition to these things in this case also. So, you can take the equation of that of a dropping Van Der Pol type of equation, so in which so you can add the term D square u by d t square plus u with a cubic order non-linearity or both cubic and quadratic and non-linearity. So, your equation that time will becomes D square u by d t square plus u equal to epsilon into 1 minus u square d u by d t then minus epsilon alpha u cube minus epsilon beta u square and you can proceeds.

So, either you can use this KB method or method of multiple scale to find the response of the system. This way today class we have discussed different type of damping present in the physical systems. For example; we have started with a viscous damping, then we have taken a negative damping, then coulomb damping also we have discussed then both Rayleigh damping and Van Der Pol oscillator also we have discussed.

Though we have taken the system simpler system, but the systems can also be extended to include this cubic non-linearity or quadratic non-linearity and those equations can be solved conveniently by using either this Krylov-Bogoliubov method or by using method of multiple scales, other methods other averaging methods or harmonic balance method or the other methods what we have studied also can be used to solve this equations.

And we have seen in this cases that how the damping, the effect of damping on the response these damping cases can be conveniently used for many different applications, many different physical applications. And some of the applications we are going to study in the coming classes also.

Thank you.