Nonlinear Vibration
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Lecture - 17
Super and Sub Harmonic Resonance Conditions.

Welcome to today class of Non-Linear Vibration. So, today class we are going to discuss more regarding the bifurcation stability and bifurcation of equilibrium points. So, already we have studied regarding the single degree of freedom system free and forced vibration; we have used different methods to solve this equation of motion.

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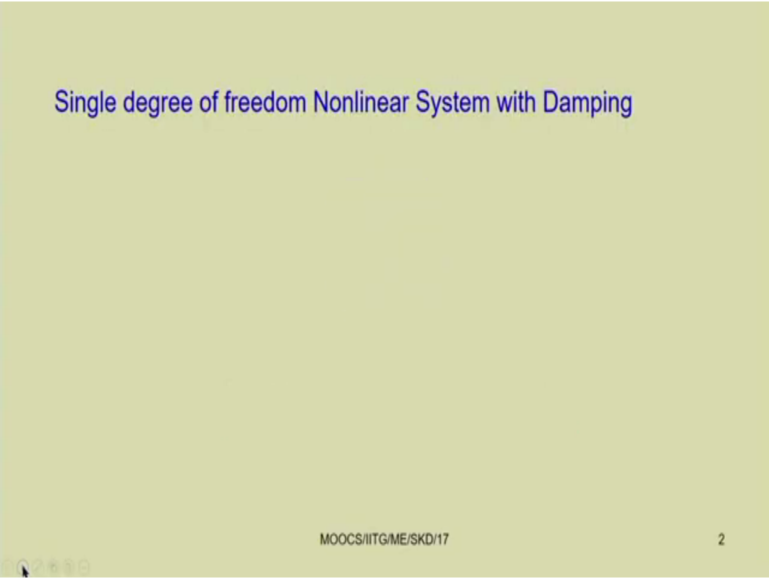

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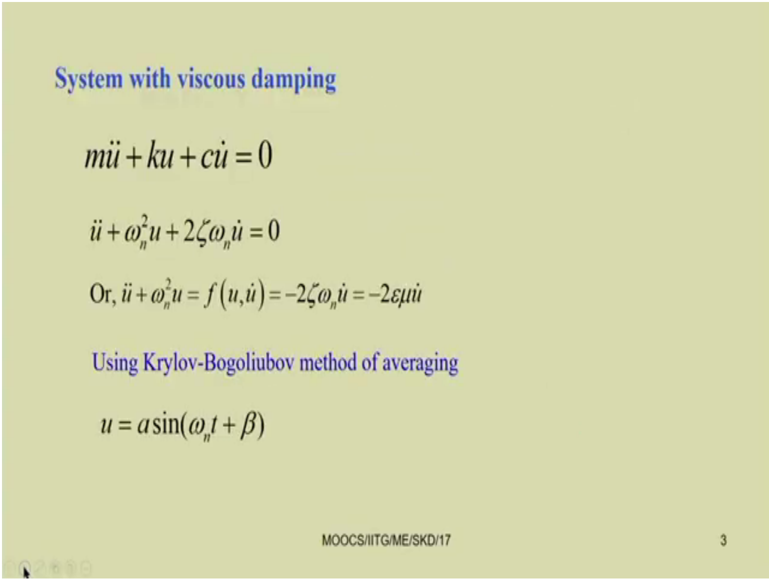

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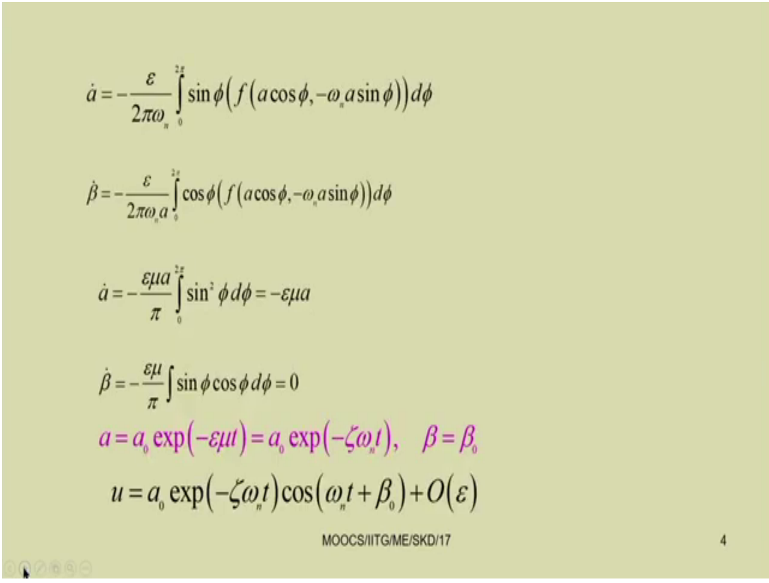

And we have taken different type of damping's. For example, we have taken the viscous damping; then we have taken the coulomb damping, quadratic damping.

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$$u = \exp(-\zeta\omega_n t)\left[u_0 \cos\omega_d t + \left(\left(\dot{u}_0 + \zeta\omega_n u_0\right)/\omega_d\right)\sin\omega_d t\right]$$

$$\omega_d = \omega_n\sqrt{1-\zeta^2}$$

For over damped

$$u = \left(\dot{u}_0 + \left(\zeta + \sqrt{\zeta^2-1}\right)\omega_n u_0\right)/\left(2\omega_n\sqrt{\zeta^2-1}\right)\exp\left(-\zeta + \sqrt{\zeta^2-1}\right)\omega_n t +$$
$$\left(-\dot{u}_0 + \left(-\zeta + \sqrt{\zeta^2-1}\right)\omega_n u_0\right)/\left(2\omega_n\sqrt{\zeta^2-1}\right)\exp\left(-\zeta - \sqrt{\zeta^2-1}\right)\omega_n t$$

For critically damped case

$$u = \left(u_0 + \left(\dot{u}_0 + \omega_n u_0\right)t\right)\exp(-\omega_n t)$$

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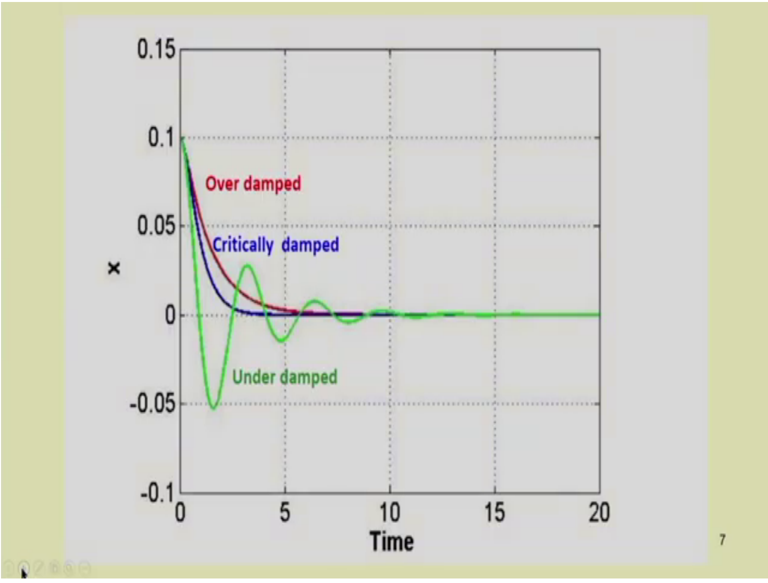

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Single degree of freedom system with quadratic damping.

$$\ddot{u}+\omega_n^2 u = f(u,\dot{u}) = -\varepsilon \dot{u}\,|\,\dot{u}\,|$$

$$u = a\sin(\omega_n t+\beta)$$

$$\dot{a} = -\frac{\varepsilon}{2\pi\omega_n}\int_0^{2\pi}\sin\phi\big(f(a\cos\phi,-\omega_n a\sin\phi)\big)d\phi = -\frac{\varepsilon a^2\omega_n}{2\pi}\int_0^{2\pi}\sin^2\phi|\sin\phi|d\phi$$

$$= -\frac{\varepsilon a\omega_n}{2\pi}\left[\int_0^{2\pi}\sin^3\phi\, d\phi - \int_\pi^{2\pi}\sin^3\phi\, d\phi\right] = -\frac{4}{3\pi}\varepsilon a^2\omega_n$$

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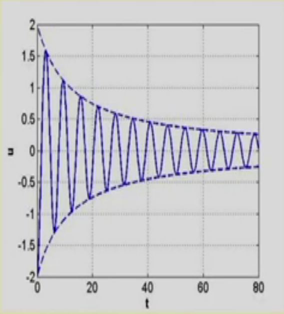
$$\dot{\beta} = -\frac{\varepsilon}{2\pi\omega_n a}\int_0^{2\pi}\cos\phi\left(f\left(a\cos\phi, -\omega_n a\sin\phi\right)\right)d\phi = -\frac{\varepsilon\omega_n a}{2\pi}\int_0^{2\pi}\sin\phi\cos\phi\left|\sin\phi\right|d\phi = 0$$

$$u = \frac{a_0}{1+\dfrac{4\varepsilon\omega_n a_0}{3\pi}t}\cos\left(\omega_n t+\beta_0\right)+O\left(\varepsilon\right)$$



It may be noted that unlike the linear system the response does not decreases exponentially but decreases algebraically.

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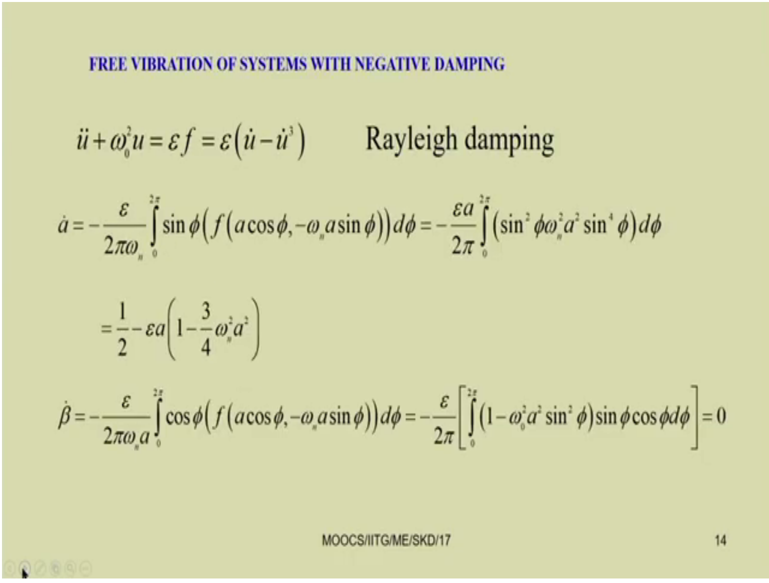

And we have found the response for all these methods; also we have taken this Rayleigh damping, then Van Der Pol oscillator.

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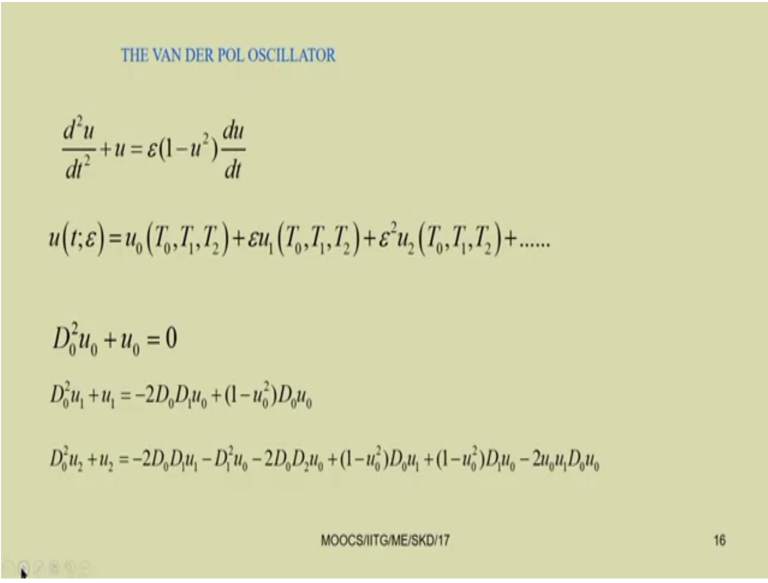

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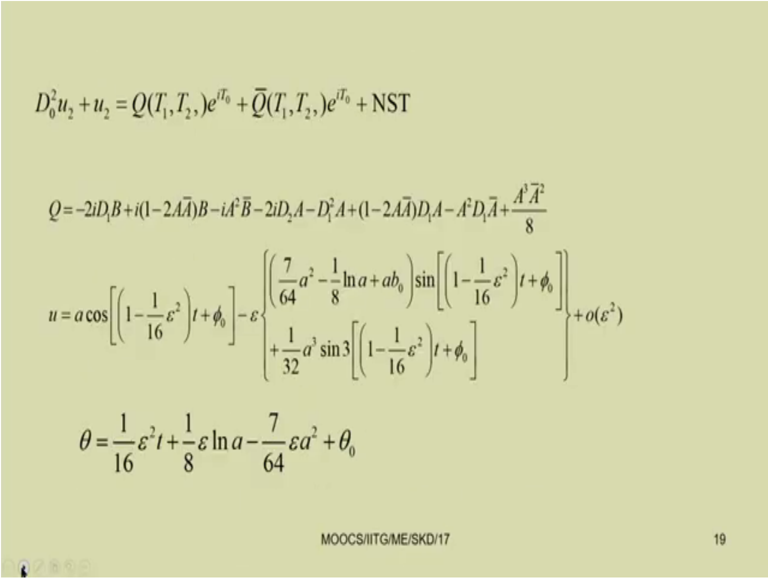

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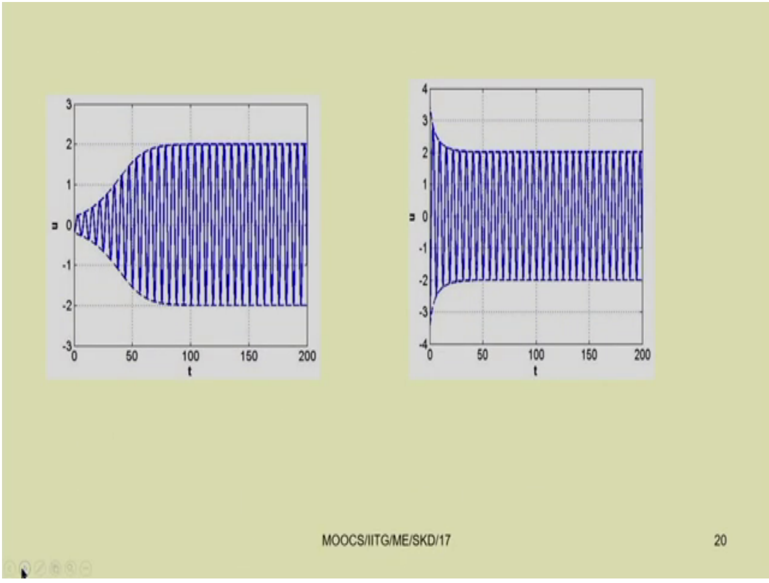

And we have studied the free vibration response of the system in case of systems with different type of damping's. Further we have taken these single degree of freedom systems with Duffing oscillator. So, here we have studied the systems with both weak and strong forcing term.

So, in many cases, the system may not be only single degree of freedom system. So, it may be a multi degree of freedom system or continuous systems. So, in case of multi degrees of freedom system, so one can reduce this multi degrees of freedom system to that of a single degree of freedom system by using modal analysis method.

So, briefly let me tell how we can reduce; for example, let us take a 2 degrees of freedom system and see how we can reduce that thing to that of a single degree of freedom system. So, let us take the coupled equation of a coupled 2 degrees of freedom equation.

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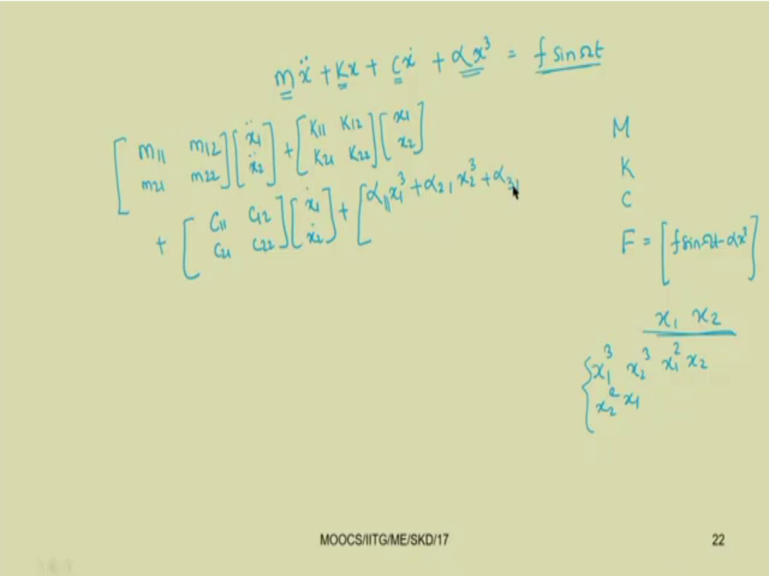

So, in case of single degree of freedom system, we are writing these equation in this form; that is M x double dot plus K x plus C x dot. So, let me take some non-linearity also plus alpha x cube, that is equal to f sin omega t. In case of multi degrees of freedom system, so M will be replaced by a mass matrix, M will be matrix; similarly this K will be replaced also by a matrix K matrix, C will be replaced by a damping matrix.

And these alpha x cube term; so this non-linear term and the forcing term, we may take to right hand side and write them as a single vector. So, that is, so we can take a vector for example, let with f a vector or let us take a vector F, which contains both. So, it will contain, let it is f sin omega t minus alpha x cube term it will contain. So, this is a vector, so in that way we can take.

So, for example, let us take a mass matrix like this. So, for example, this is M 11, M 12, M 21, M 22. So, this is equal to x 1 double, dot x 2 double dot plus similarly I can write this is K 11, K 12, K 21, K 22. So, this is x 1, x 2. Similarly, for damping I can write this equal to C 11, C 12, C 21, C 22 into x 1 dot x 2 dot.

Now, for this dump, this non-linearity let me write this in this way. So, as it is we are taking x equal to x 1 and x 2. So, our non-linearity can be in this form; that means it can be of x 1 cube, it may be of x 2 cube, it may be of x 1 square x 2 or x 2 square x 1. So, these are the four options for this non-linearity.

So, you have these four options. So, using these four options, I can write for example, I can write this is for example, it is alpha 1 x 1 cube, alpha 1 x 1 cube plus alpha 2. So, let us put the subscript 11 also. So, in this case, so it will be alpha 21, alpha 21 x 2 cube or alpha 21 x 2 cube plus alpha 31, ok.

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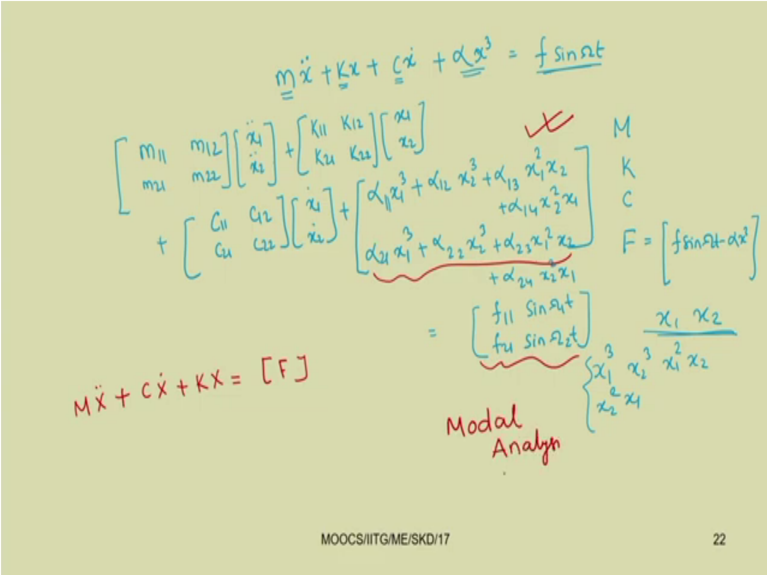

So, let me put this way. So, alpha 12, so this will be alpha 13. So, next time we will put alpha 13, alpha 13 x 1 square x 2 plus alpha 14 x 2 square x 1. So, this way you can write. Similarly, for the bottom part it will be or for the other one it will be x 21, x 1 cube, x; not x this is alpha 21, alpha 21 x 1 cube alpha 22 x 2 cube alpha 23 x 1 square x 2 alpha 24 x 2 square, x 2 square x 1.

So, this way we can write all the non-linear terms, then plus this forcing. So, forcing can be written for these two cases. So, it will be f 11 sin omega t; similarly the next one I can write f 21 sin, so f 21 sin omega t. So, or you can write x, you can use omega 1 or omega 2 here also. So, now, you can see; so we can uncouple these equations. So, if I will take. So, let me take this part, this part and this. So, I can take these to the right hand side and can write this equation in this form that is M X double dot plus K X plus C X dot equal to.

So, M X double dot plus C X dot plus K X equal to; I will take all the other things to right hand side and let me write this equal to F. So, F contains these non-linear terms and this forcing term. So, it contains the non-linear term and the forcing term of the system. So, as you are familiar with the modal analysis; so in modal analysis how we proceed? So, we can do this modal analysis.

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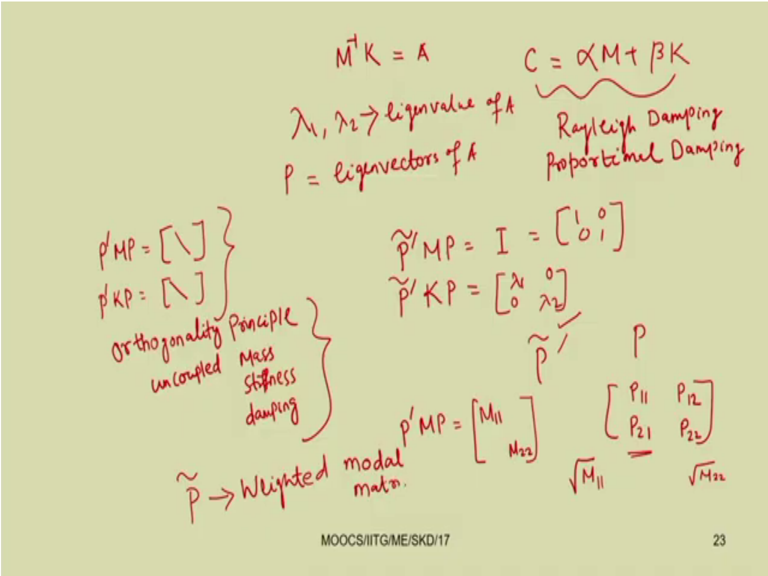

So, generally, in case of modal analysis, we take this M inverse K; first step is take M inverse K equal to A. So, after find taking this M inverse K equal to A; then we can write, also we can take this damping as the Rayleigh damping. So, Rayleigh damping can be, let me take this is equal to lambda 1 or alpha we can take this is equal to alpha M plus beta K. So, this way we can take.

So, these are known as the proportional damping. So, this is known as Rayleigh or professional damping. So, Rayleigh damping or proportional damping. So, in case of this proportional damping, the damping term is considered to be proportional to this mass matrix and this stiffness matrix. So, after taking this M inverse K equal to A; so then we can find the eigenvalue and eigenvector of A.

So, let lambda is the eigenvalue. So, if lambda is the eigen. So, this is 2 is to 2; so we can have lambda 1 and lambda 2 has the eigenvalue of A. So, corresponding to this eigenvalue; so we can have the eigenvectors also. So, let P is the eigenvectors of A. So, after finding these eigenvalue and eigen vector; so we know the property of these eigenvalue and eigenvector.

So, we know that P transpose M P is a diagonal matrix. Similarly P transpose K P is also a diagonal matrix, otherwise you can take. So, using this orthogonality principle; so, using this orthogonality principle, so we can reduce or we can uncoupled; so, we can uncoupled, we can uncouple these mass, stiffness, and damping matrix. So, we can uncoupled it.

So, we can take P transpose M P, P transpose K P and we can uncouple; because this will become diagonal.

So, otherwise one can take this weighted model matrix also; so P weighted model matrix, P weighted model matrix transpose M P. So, this is nothing but the I matrix and then; so P transpose K P, so that is I matrix and you can verify that P transpose this weighted model matrix K P is nothing but. So, it will contain. So, if we are taking 2 by 2, so it will contain lambda 1 and lambda 2.

So, P transpose K P will contain these, this one. So, this is equal to also your 1 0 0 1. So, how to find these P weighted model matrix? So, P weighted model matrix can be obtained; so by dividing this P, that is the modal matrix divided by the square root of the, square root so it should be divided each column of these things.

So, now for example, so this is P 11, P 12, P 21 and P 22. So, in each column, so if we are dividing square root of this M; that means so, let us first find these P transpose MP; P transpose MP already we know that this is equal to a diagonal matrix. So, in this diagonal matrix, this will be let it is M 11 and this is M 22.

So, the first column should be divided by root over M 11 and the second column should be divided by root over. So, the first column should be divided by root over M 11 and the second column should be divided by root over M 22. So, this way by dividing these M 11 and M 22; so here root over M 22. We will get the weighted modal matrix P.

So, this is known as P weighted modal matrix, weighted modal matrix. So, either by using this modal matrix method or by using this weighted modal matrix method, so, we can find or we can uncoupled the equation and by on. So, now, let us see. So, now, our equation becomes.

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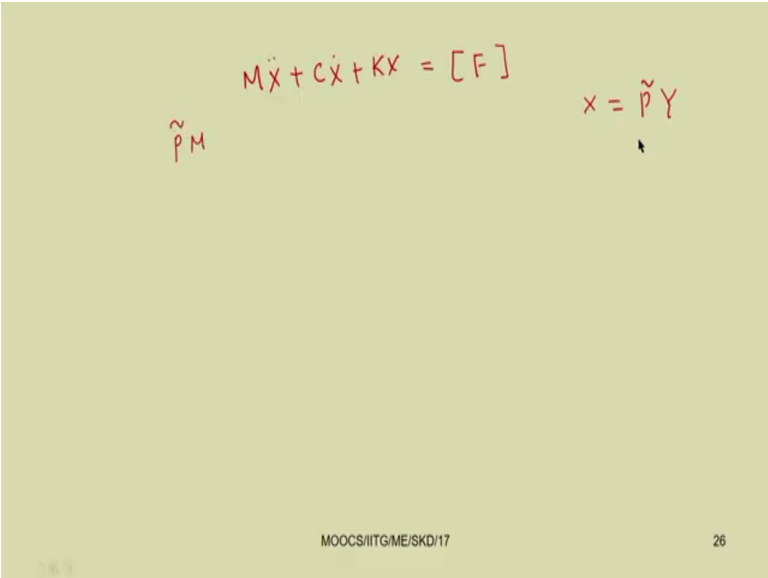

So, for example, let me use this weighted modal matrix. So, our equation, original equation is M X double dot plus C X dot plus K X.

So, we I have taken all the terms, non-linear terms to right hand side. So, this is equal to, let it is equal to F. So, here what we are doing. So, now, we are substituting let us substitute X equal to P weighted modal matrix into Y. So, X equal to P weighted modal matrix into Y. So, in this case I can write this P weighted modal matrix, M. So, for X, I can write it is equal to.

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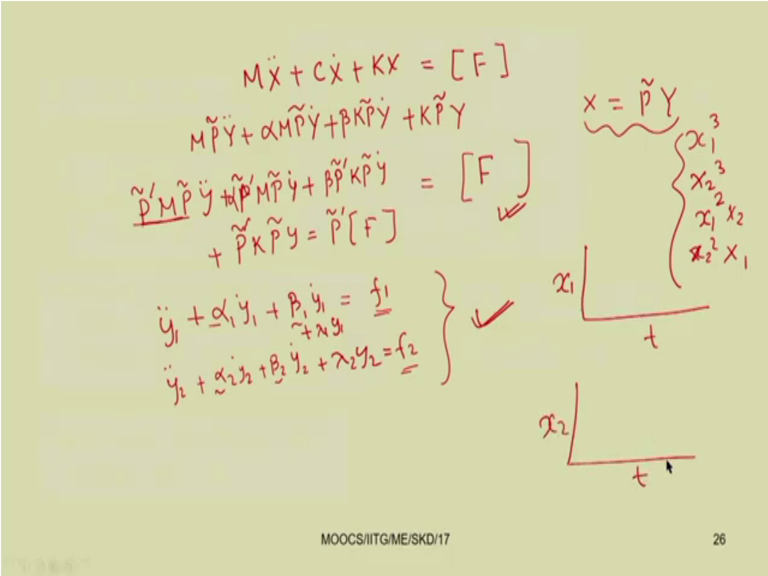

So, let me first write so, MP weighted modal matrix Y double dot plus this for C, I can write, this is equal to alpha MX dot. So, alpha M X dot can be written alpha MPY dot plus beta KPY dot. So, plus KP, this is weighted modal matrix, this is weighted modal matrix, weighted model matrix Y.

So, similarly in this F, so we have to substitute this F; in a F of also we have the term with X cube. So, or we have this X square. So, we have all the three terms are there; one is X 1 cube, second one is X 2, small x 1. So, x 1 cube x 2. So, we have four terms are there.

So, x 1 cube x 2 cube, then x 1 square x 2 and x 2 square. So, these are small x 1 square x 2 and x 2 square x 1. So, we have these four terms inside this thing. So, there also we have to substitute this value and we can get this force, the forcing term which is a function of.

So, this is a non-linear term, what it will contain only single-single terms; so two terms will come into picture here. So, the non-linear terms we have kept it in this and other terms are in the left hand side. So, our equation is now, now let us P multiplied by P weighted modal matrix transpose.

So, P weighted modal matrix transpose, MP weighted modal matrix Y double dot plus P weighted modal matrix transpose M. So, alpha into MP weighted modal matrix Y dot. So, plus beta P weighted modal matrix, KP weighted modal matrix Y dot plus P weighted modal matrix KP, this is the transpose weighted modal matrix Y equal to P weighted modal matrix into this F.

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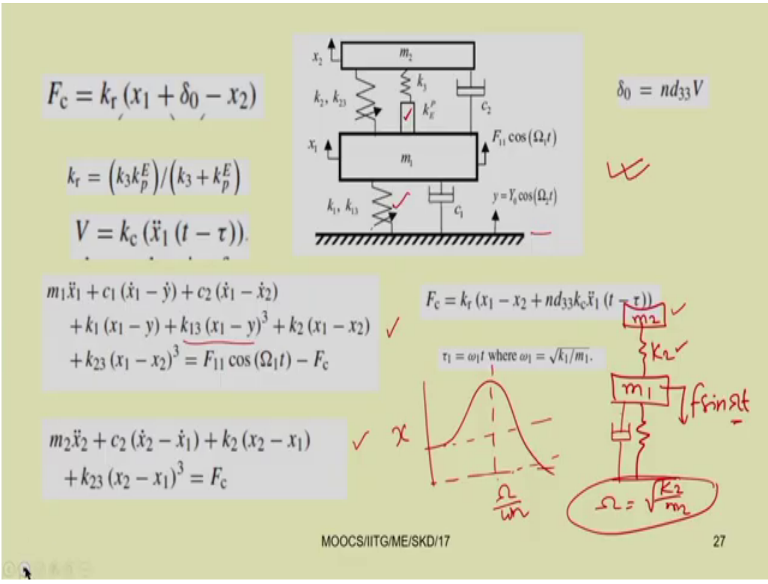

So, this way we can, now this left hand side you can see, left hand side can be written. So, by uncoupled form, so you will have one equation this way. So, this part is I. So, this is y double dot. So, y 1 double dot. So, you can have this equation y 1 double dot plus. So, you can use some term for example, alpha 1 y dot y 1 alpha 1 y 1 dot plus beta 1 y 1 dot. So, this will be equal to some terms, so which is a function of all these things.

So, it will be there. So, it can be converted into y form. So, it will be equal to F1; similarly we can have the second equation, so which is uncoupled also in this. So, this is y 2 double dot plus alpha 2 y 2 dot plus beta 2 y 2 dot, ok. So, here also we can have another term, that is lambda 1 plus lambda 1 y 1.

So, here we will have lambda 2 y 2. So, this is equal to f 2. So, this way you will get two equations, but note that these right hand side f 1, f 2 contain this couple term of x 1 x 2; but in the left hand side, we have the terms uncoupled terms. So, it is reduced to similar to that of the first. So, similar to first single degree of freedom systems and you can use different methods to solve this equation.

So, particularly you may be using, you may use the OD 45 or the fourth and fifth order Runge Kutta method to find the solution of this thing. First you have to find these y 1, y 2 and then you can replace or use this equation again to convert back to in terms of x. So, you can plot this x versus t or x 1. So, you can plot x 1 versus t and x 2 versus t to find the time response of this thing.

So, this way you can reduce your multi degrees of freedom system to that of a single degree of freedom system and you can solve this thing. For example, so we have a equation we have a system; so where this is the case of a vibration observer, where we have two mass. So, in case of already you are familiar with the tune vibration observer, where the primary system is, the vibration of the primary system is completely absorbed by using a secondary tune mass damper system.

So, for example, so if you have a system like this; so this is the primary system and you have a secondary system. So, already you know that this is m 1 this is m 2; already it is subjected to a force f sin omega t. It is known to you that, at omega equal to omega n; so you have maximum response amplitude.

So, if you have a damper; so otherwise it will go to in finite. So, in without a damping, so the value is in finite; but if we have a damper, then this is reduced to, it has a finite value. So, this is your x 1 or x versus t; so x verses omega by omega n. But, if you put a secondary spring mass system; for example, this is k 2 m 2, k 2 m 2.

So, you have to choose this parameter k 2 m 2 in such a way that, so this omega will be equal to root over k 2 by m 2. So, this is the principle of tune vibration observer. But in many times, this omega by changing this omega; so you have to change these spring and mass again and again to avoid that thing.

So, you can put a piezoelectric plates here. So, by putting a piezoelectric plates by applying the sectional electric field. So, you can change the stiffness of the system and you can control the or observe the vibration of the system actively. So, by putting the damper actually, so as you know; so the damping for example is not applicable for a value, higher value of omega by omega n.

So, you can find by using this damping; so the system will have different type of response or the system response can be analyzed in a different way. Also one may see that there may be some time lag between when we are applying the force and when the response is taking place.

So, to accommodate all the things, so you can have a complicated system; when you can take the time delay in the system, also further you can complicate the system by using non-linear spring. For example, here you have taken the non-linear spring in the primary case; you can take the non-linear spring in the secondary case also.

So, if we are taking the non-linear spring the primary case; so as you can take also the base to be vibrating. So, let the base is vibrating with y equal to y 0 cos omega t.; and so, the vibration of the primary mass equal to x 1; so there will be a relative displacement of the spring. So, the spring, so this will be equal to x 1 minus, x 1 minus y; similarly for the damper it will be x 1 dot minus y.

Similarly, for the case of the secondary spring; so the displacement will be x 2 minus x 1. And for the damper also it will be x 2 dot minus x 1 dot. So, this way you can write down this equation of motion, so which will be in this form of coupled equations. So, you can see this, here you have x 1 double dot, so another x 1.

So, though the mass matrix is uncoupled; but the stiffness and damping matrix are coupled along with, you have the non-linear terms also. So, in this case by using this modal analysis method, first you can uncouple this equation; particularly only up to these linear spring or linear damper, all the non-linear terms you may shift to the right hand side and you can do your analysis.

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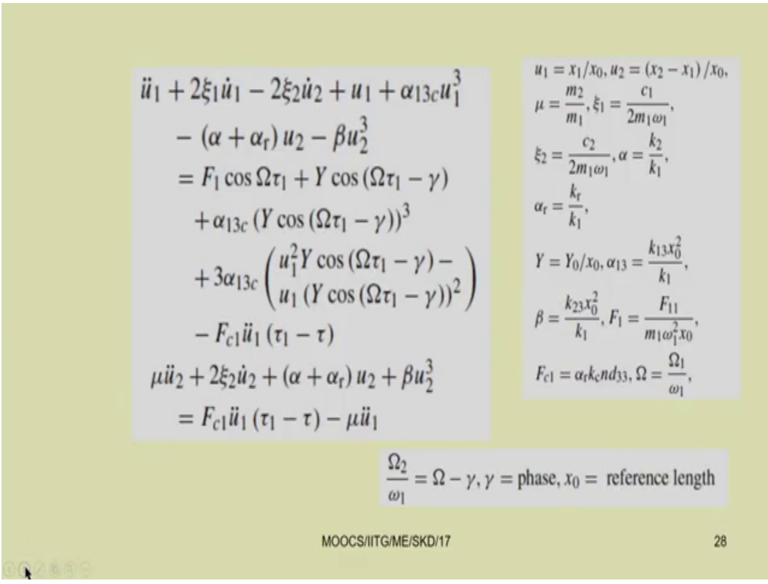

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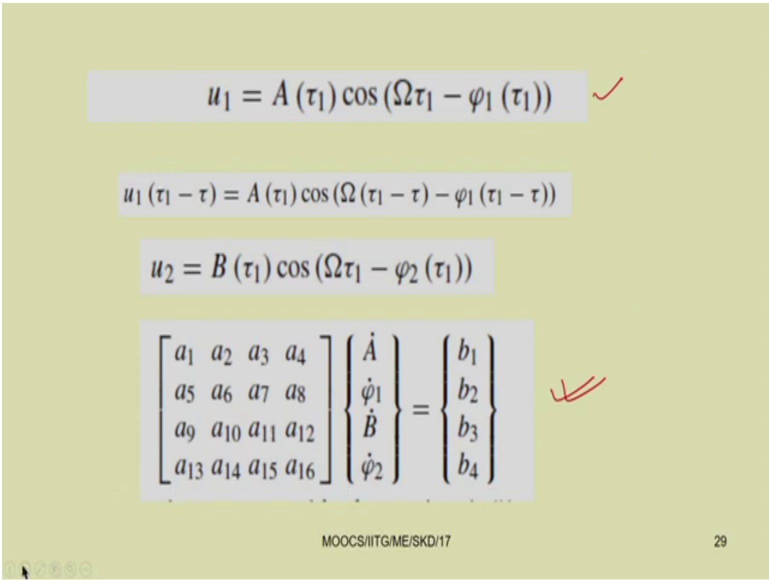

Otherwise, you can apply this method of multiple scales what do you have studied before and you can or you can, you may use this harmonic balance method. So, here the harmonic balance method is used in this case. So, u 1 equal to A t 1 cos omega t 1 minus psi 1 tau 1, so to where tau is the.

So, here the tau you can take as the time, non dimensional time also, tau 1 tau 2 are non-dimensional time this or you can take tau as the delay also. So, by using these ways, so you can reduce this original equation to this form. So, that is a 1 a 2 a 3 a 4. So, you have taken the state vector as A 1 dot A 1, A 1 psi 1 b 1 psi 2, where A and psi are the amplitude and phase of the primary system. And B and psi are the amplitude and phase of the secondary system.

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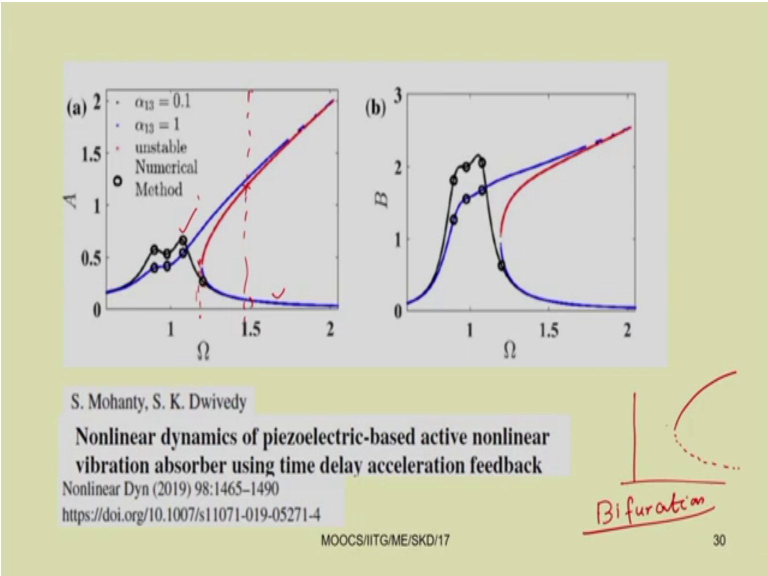

So, that way you can by using this harmonic balance method, so you can solve this equation to find the response. So, this is the response of the systems. So, you can get. So, already we know how to study the stability; today we are going to study more on the stability and bifurcation. So, the purpose of showing you this thing is that, so you can observe that there are multiple equilibrium points, multiple equilibrium solutions for a given frequency.

So, for example, for frequency omega equal to 1.5, so you have; so this is one response, so you have another value. So, if you can draw a line at 1.5. So, you can see. So, you have this is one response, this is also the other response. So, this way you can see that, non-linear system contain multiple solutions and out of all those solutions, some solutions may be stable, some may be unstable.

And you can see for example, let me draw only this part of this curve; you can observe that, just you note the colour of these things. So, this is, this as one colour and this is the other colour. So, it is shown that, this red colour is unstable and this colour is stable. So, you just see, previous to this point, previous to this point the system has only.

So, these curve belongs to the linear system and so, this is for the linear system and this curve for the non-linear system it is plotted. So, for the linear system; so you can observe that for certain value of system parameter, so you can obtain two equal peaks approximately equal peaks in the response. And for the non-linear curve, so you just observe for the non-linear curve.

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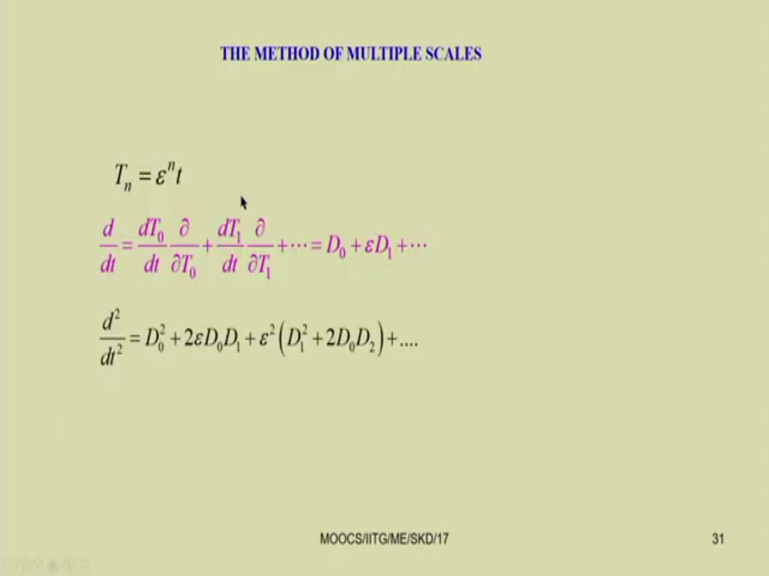

So, up to certain value of omega; so you have only single solution. What after that thing? But after these things; so you have multiple solutions. So, the system has multiple solution. So, out of the three solution two solutions are stable and one solution is unstable.

So, previously also I have shown you the basin of attraction to study the response or to know this multiple response based on which initial conditions; so we are getting the stable and unstable response of the system. Now, before this point, so you have only one solution and after this point; so you have three solutions. So, that is a qualitative change, qualitative or quantitative change in the number of solutions.

So, whenever there is qualitative or quantitative change in the system response occur; so those points are known as bifurcation points. So, today class we are interested to study the Bifurcation points. So, already we know what we mean by stable point, unstable point; if the response keep with slight perturbation in the response, if the response come back to the original position, then it is stable response.

But if with slide perturbation, if the system response grows; then this becomes unstable response. So, sometimes also we use this word asymptotically stable. So, if with time t tends to infinity, the system response is stable or it come back to the; if we displace from the original position, it come back to the original position, then it is asymptotically stable.

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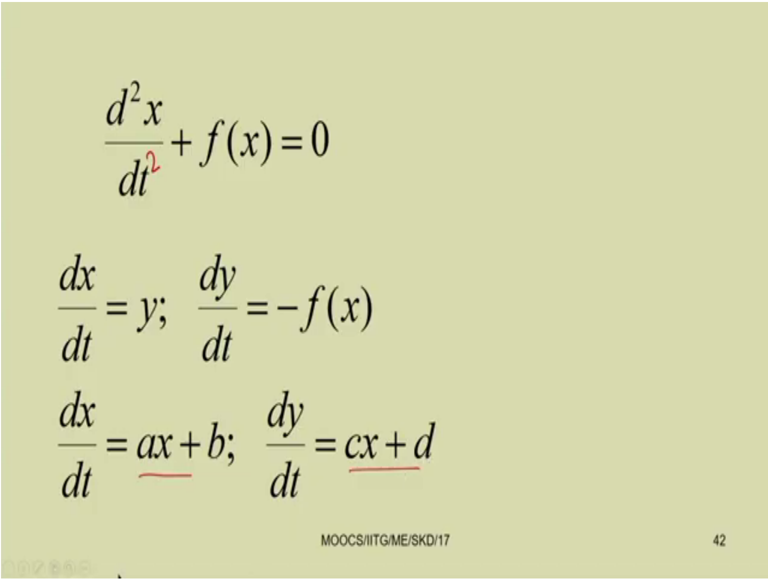

So, all these things we have already seen and today class also, we will see what we mean by this bifurcation, already we have studied this thing. So, for a given system d square f by dt, d square f by dt square plus f x equal to 0, second order equation. So, if you are, we can write them in to first order equation; that is dx by dt equal to y and dy by dt equal to minus f x. And by writing this dx by dt in this form ax plus b and dy by dt in the form of cx plus d.

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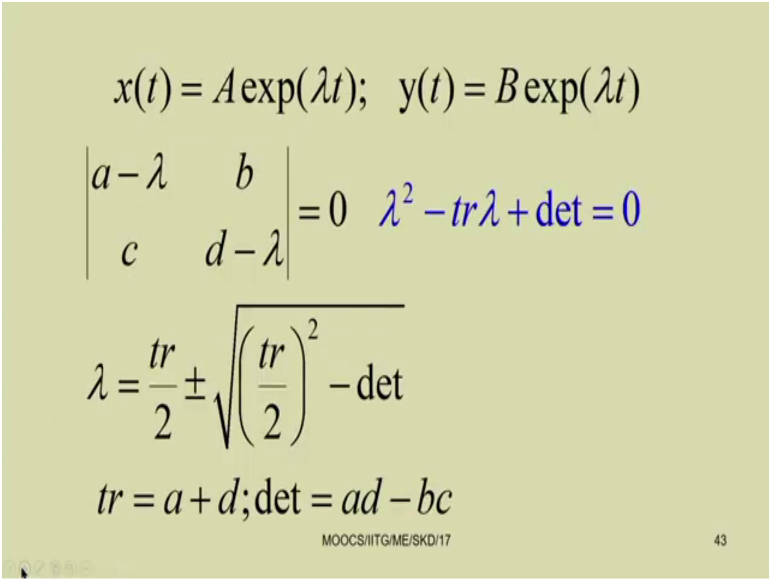

So, already we know, we can find this at Jacobian matrix and finding the eigenvalue of the Jacobian matrix. So, we know, we can write the in this form of; so these eigenvalues lambda square minus stress lambda plus determinant equal to 0. So, from that thing, so we know our lambda that is the eigenvalue and we know. So, if the eigenvalue is, if the real part of the eigenvalue is negative; then the system is stable, otherwise the system is unstable.

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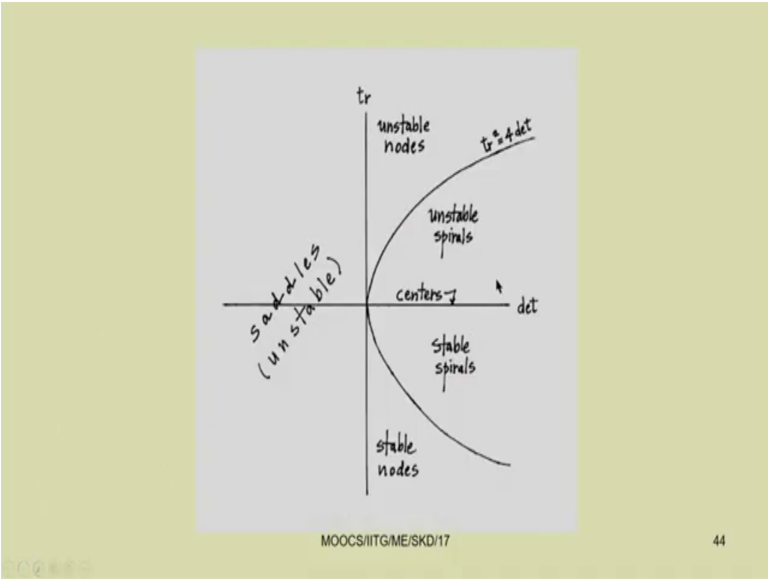

And by plotting between a relation between determinant and stress also we know, we will obtain the node, saddle unstable nodes; then these unstable spirals, center, stable and unstable nodes also.

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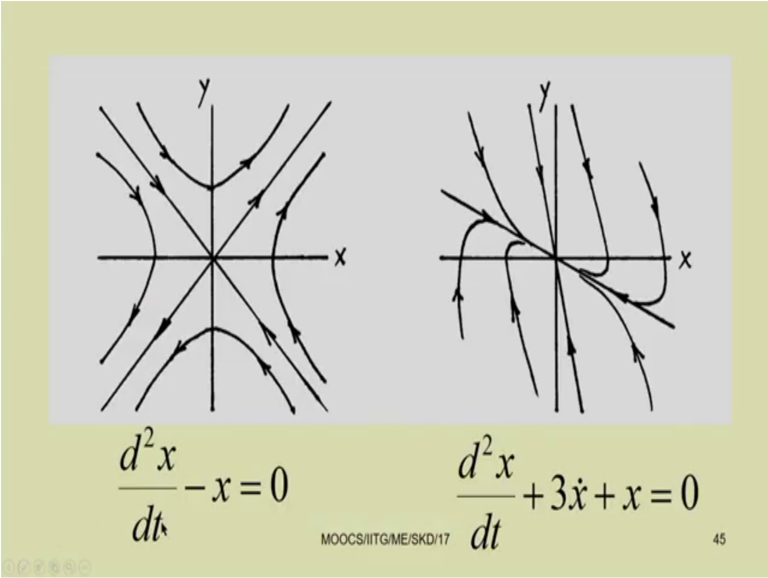

So, with different examples also we have seen. So, d square x by dt square minus x equal to 0. So, in this case, we can get a point; similarly in this case also d square x by dt square plus 3 x dot plus x. So, spiral we are getting.

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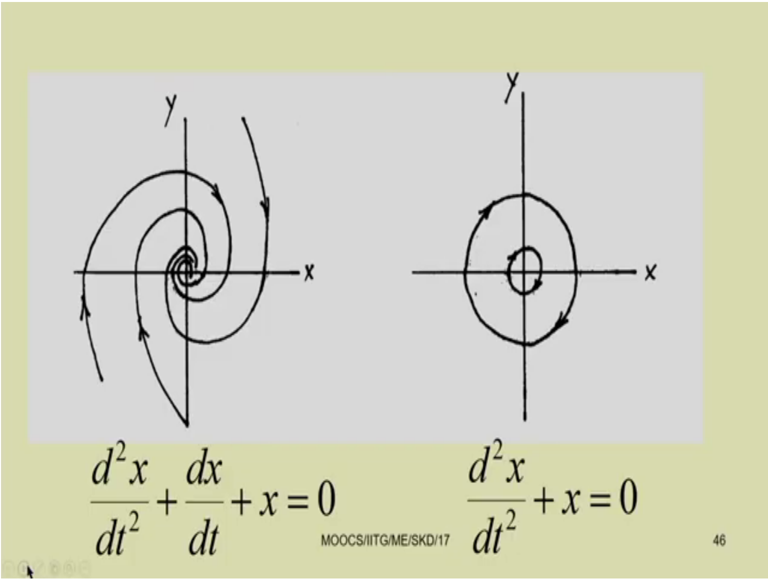

And here also we are getting the spiral and here we are getting. So, d square x by dt square plus x equal to 0; so here is the center we are getting. So, this way we got different type of responses.

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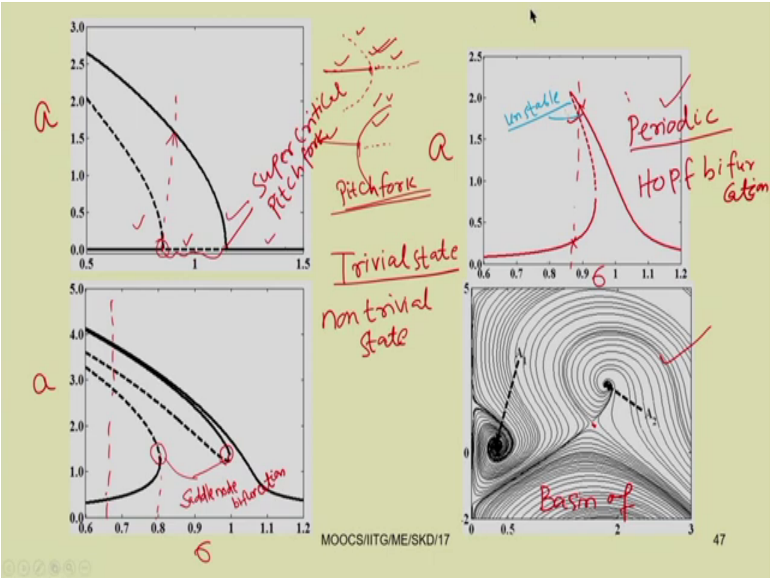

Also some of the typical response part we may see also in these non-linear systems. So, for example, let me plot this omega or a by omega by omega n natural frequencies; sometimes you may plot these things with respect to the detuning parameter sigma also. So, a versus sigma, amplitude versus sigma if you plot, so in these cases; so you can see, sometimes the response will looks like this, sometimes you can have multiple response.

Here you just see, here you can observe. So, up to this point, this is 0 line. So, 0 line is also known as trivial state. And so, you should be familiar with the trivial state and non trivial state, so non-trivial state. So, x, so the response; when the response is 0, so that is trivial state.

So, if you have nonzero response, so this is non-trivial state. So, up to this point, up to for example, sigma equal to this; so the system has a trivial state. Similarly after these things also the system has a trivial state. So, between this and this point, so you have seen.

So, by performing this eigenvalue analysis; so we have seen the system response is unstable. Up to this point the system has a stable and another unstable response and also one more stable response. So, what is happening here? So, if you take a point here. So, it is going from stable to unstable.

So, the system by slightly increasing the sigma value, the system will jump from this response to this response. So, it will go from this stable to this stable point. So, the system will jump from this to this; similarly at this point, the system will have only the stable trivial state response.

So, here you can observe, the stability of the system is changing. So, the stability, the trivial state stability is changing from stable to unstable. And so, before this bifurcation, so these, this is known as a bifurcation point. So, if there is any qualitative or quantitative change in the response occur for a particular critical point; that critical point is known as the critical bifurcation point and so, these are known as the bifurcation points.

There are several different types of bifurcation point; for example, in this type of bifurcation point, so we can see. So, we have plotted only the upper half of the curve. So, one can plot the lower half of the curve also. So, the response, so initially the trivial state was stable.

Now, the trivial state become unstable, leading to another stable non trivial state. So, from this trivial, so from the trivial stable state; it is going to non trivial stable state and it looks like a pitch fork. So, this type of bifurcation is known as pitch fork bifurcation, pitch fork bifurcation.

So, we will see how the eigenvalue changes in case of the pitch fork bifurcation. So, pitch fork bifurcation can be. So, here it is going from a stable trivial state to stable non trivial state; in this case, the pitch fork bifurcation is a continuous bifurcation point. So, it is a continuous bifurcation point.

But in this case you just see, here we have a stable branch and here also we have unstable branch and then we have a unstable trivial state. So, here we have a unstable trivial state; that means this stable trivial state and this unstable non trivial state meets at this point, but it leads to an unstable trivial state.

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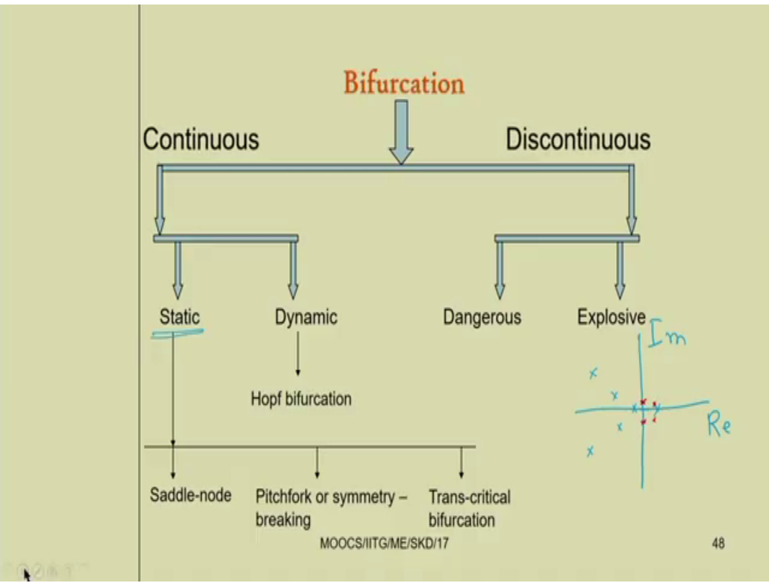

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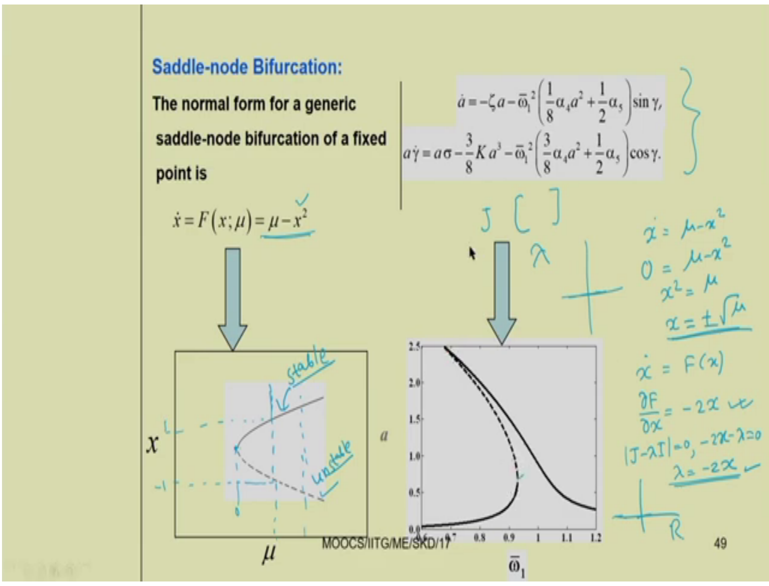

So, this type of bifurcation, so here also you have a pitch fork type bifurcation. So, we just see, a pitch fork towards the left, the curve what I have drawn; so it pitch fork towards left. What? So, after this thing it is unstable. So, unstable means after these things so it will have a tendency to jump up or because there is no other stable solution available near to this vicinity, so it will jump up.

So, this type of pitch fork bifurcation are known as unstable, this is subcritical pitch fork bifurcation. So, this pitch fork bifurcation, so which is continuous one; so this is super critical pitch fork bifurcation. And this pitch fork bifurcation which is discontinuous, after these it discontinue; so the response will discontinue due to the presence of unstable state. So, it will be subcritical pitch fork bifurcation; you will now know two different types of pitch fork bifurcation.

So, one is supercritical pitchfork or continuous pitch fork bifurcation, and the other one is sub supercritical and subcritical pitch fork bifurcation. So, this is subcritical pitchfork or discontinuous bifurcation. Similarly, here also you can observe that thing. So, you have a pitch fork bifurcation here.

So, if you decrease this value of sigma, it is going from stable trivial state to stable non trivial state. So, this is a super critical. So, this point is super critical pitchfork bifurcation point. So, this is a super critical pitch fork bifurcation point. So, now, you just see here. So, you can have multiple. So, 1, 2, 3, 4, 5; so five states are there. So, multiple solutions are there.

So, more than two stable states may be there also. So, if there are two stable states, then is known as bi stable. So, if there are three stable state, you will have tri stable. So, that way, so you can have many stable state. At this point you can observe here or at this point you can observe; so there is a at this bend, at this bend point you can see, it is changing from unstable to stable.

So, here also it is changing from unstable to stable. So, these types of bifurcations are known as saddle node bifurcation. So, this is and this. So, these two points are known as saddle node. So, these are saddle node bifurcation point. You just see after these things, there is no solution exists.

So, in this branch there is no solution, though there are some other solutions; but in this branch there is no, after this thing in this branch there is no solution. So, these are known as saddle node bifurcation, ok. So, this is these correspond to, so the sigma is known as saddle node bifurcation points.

So, that sigma is known as saddle node bifurcation point. You can observe another type of bifurcation here; so in this branch, part of these branch. So, there is no bending here, but in this case what is happening; so suddenly the stable state becomes unstable. And if you perform this analysis you can see, it gives rise to; though this fixed point response is unstable, but it gives rise to periodic response.

Near this thing if you perform this analysis; so you can see, it will give rise to another periodic response here. So, this is non-trivial fixed point response is unstable; but there occur a periodic response which may be stable or unstable. So, these types of bifurcations are known as Hopf bifurcation, Hopf bifurcation.

You have seen pitchfork bifurcation, saddle node bifurcation, and Hopf bifurcation of the system; already I told you, you can plot the basin of attraction. So, this is known as basin of attraction, basin of attraction. So, when you have multiple solutions available; so by taking different initial conditions, by taking different initial conditions.

So, you can find the, so for what initial condition; so it is going to which attractor, which stable point it is going those things you can study. So, here you can see clearly that, this point is a saddle node bifurcation point, saddle node point. So, these are the stable nodes so, stable points.

So, corresponding to, so if you take a point for example, here. So, this is one stable, so this is one stable; but this one is the unstable branch. So, this one is the unstable branch. So, this is the unstable. So, this point is unstable and other two are stable points, ok. This way already by performing or finding the Jacobian matrix. So you can study, so whether the response is stable or not.

But now, you must know how to find the bifurcation points or what is the characteristic of this bifurcation point. This bifurcation, so it may be continuous bifurcation; so it may be discontinuous bifurcation. So, in this continuous case, it may be static bifurcation or it may be dynamic bifurcation.

So, in case of static bifurcation, so we have the saddle node bifurcation, pitch fork or symmetry breaking bifurcation or we may have the trans-critical bifurcation. And always these discontinuous bifurcation are explosive or dangerous bifurcation, which one has to avoid.

So, in case of continuous bifurcation; so we have seen the static and dynamic bifurcation. So, you know the system to be stable or unstable based on the real part of the eigenvalue, by checking the real part of the eigenvalue; so we generally tell that the system is stable or unstable.

If you plot the real part and imaginary part of all the eigenvalues of a system, just you plot the real part and the imaginary part. So, let you have a multi degree of freedom system. So, in case of multi degrees of freedom system, you will have multiple eigenvalues.

If the eigenvalue, so now, by changing the system parameter. So, for example, you have changed one of the system parameter and observed how the eigenvalues are changing. So, in that case if it is passing through; for example, now you have, you can have a pair of complex conjugate number eigenvalue, these pair of complex conjugate late now it is crossing.

So, then it becomes a real value to equal roots. Now, it is crossing this thing, now it is crossing the imaginary part of the, imaginary axis with value 0. So, it crosses by changing the parameter, so you have seen; if it crosses the imaginary axis with 0 value, that means it will have a real part only, then this type of bifurcation is known as static bifurcation.

So, in case of static bifurcation, the eigenvalue will have zero imaginary part and zero real part. So, it will have zero. So, our critical point it will have a zero value; zero real part and zero imaginary part also, so for static bifurcation. For a dynamic bifurcation, so it will have; so it will cross the imaginary axis with nonzero value.

So, it may cross the imaginary axis with nonzero value. So, if it is crossing this imaginary axis with a nonzero value; that means the real part is 0. So, eigenvalue has imaginary, a pair of imaginary roots.

So, due to this presence of this pair of imaginary roots; so it will have a frequency, which will give or leads to a periodic response. If the imaginary part are non-zero; so when at this critical point, then it leads to Hopf bifurcation point; but if the real parts are nonzero value; if the imaginary parts are 0, then it will, you will have these static bifurcation.

So, in case of static bifurcation, so you have three different types of bifurcation; one is the saddle node bifurcation, second one is the pitch fork or symmetry braking, and third one is the trans-critical bifurcation.

So, let us see the saddle node bifurcation, its generic form and how we can find it. So, in case of the saddle node bifurcation, the normal form or the generic form of saddle node bifurcation can be given by this equation; that is x dot equal to mu minus x square.

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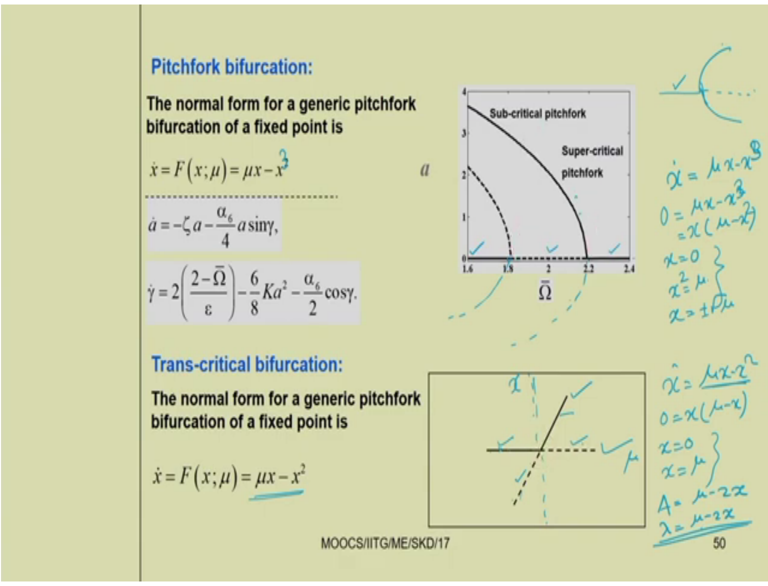

So, let us write down this equation, x dot equal to mu minus x square. We will take this equation and then as x dot equal to mu minus x square. So, in this case the fixed point will be;

so by putting this x dot equal to 0, so we will get the fixed point 0 equal to mu minus x square or x square equal to mu or x equal to plus minus root over mu.

So, this is the equilibrium point. So, the equilibrium point is x equal to plus minus root over mu. So, if we plot mu versus x; so we can get this curve, we will have a real value actually. So, if mu is greater than 0, so before. So, mu less than 0, so mu; so this is, this correspond to mu 0. For mu less than 0, so there is no solution available; but mu greater than 0 greater than equal to 0, so we will have two branch of the solution.

So, in this case you just see. So, we have x equal to plus minus root mu. So, by taking a particular value; for example, let us take mu equal to 1. So, our equation becomes x dot equal to 1 minus x square or we know. So, we can write this x dot equal to F x; the Jacobian matrix to find the Jacobian matrix, this is a single degree of, single degree of freedom equation, so the or first order equation.

So, in this case, the Jacobian matrix you can find by del F by del x. So, del F by del x. So, in this case, this is equal to mu minus 2 x square. So, this becomes minus 2 x. So, del F by del x becomes minus 2 x. If you put this Jacobian J minus lambda I form equal to 0; so J minus lambda I equal to putting J minus lambda I equal to 0, so you got minus 2x.

So, minus 2x minus lambda equal to 0 or lambda equal to minus 2x. So, lambda equal to minus 2x. Now, so you have two value of mu corresponding to one value of mu. So, you have two value of, so you have two value of x; for example, if you take mu equal to 1, so x equal to, x equal to plus minus 1.

So, this is x equal to plus 1. So, this is x equal to minus 1. For x equal to plus 1, so if I will take x equal to plus 1, so this lambda becomes minus 2. As lambda becomes minus 2, this is negative, so this branch is stable branch. So, this is, so you can take one or two more points and you can verify that this branch is stable; because the eigenvalue, the real part.

So, the eigenvalue is only real. So, minus 2 and the eigenvalue is negative. So, lambda equal to minus 2. If you take some other point; for example, if you take mu equal to 4, then it will be plus minus 2, taking this plus 2. So, another point if you take mu equal to 4; so then this is plus 2. So, if it is plus 2, then lambda equal to minus 4.

So, lambda equal to minus 4, the real part is negative. So, this branch is stable branch. Similarly, now come back to these branches. So, in this branch, x equal to minus 1; so lambda equal to minus 2 into minus 1, so this was 2; so, this is a negative, this is positive. So, as the lambda is positive; the real part of the eigenvalue is positive, so this branch is unstable. So, this is a unstable branch.

So, what you have seen? So at this critical point. So, this is the critical point at which; so both stable and unstable branches, so it merge here at this point. So, at x mu equal to 0, so mu equal to 0; so before that, thing there is no solution and after the solution.

After mu equal to 0, so, you are two branch of solution. So, there is a qualitative change in the number of solutions. So, that is why mu equal to 0 correspond to the bifurcation point. So, here the bifurcation is. So, you just see as you are moving this value of mu, so, it is crossing the 0 line, so imaginary part. So, it is crossing the real part. So, if you plot the real and imaginary, so always you have.

So, you can find that, it crosses the real axis or it crosses this s plain through the 0 value, that is imaginary part is also 0. So, that is why this is static bifurcation and these type of static bifurcations are known as saddle node bifurcation.

So, this is for a first order equation. So, if you have second order equations, so you can write using two first order equation. So, this is the case of a typical case, similar to that of a Duffling equation or some other type of equations also. So, in this case you can see or you can observe, so you have two saddle node bifurcation points.

Here also similarly you can find this Jacobian matrix. So, after finding this Jacobian matrix, you just find lambda. So, after finding lambda, so corresponding to these points; you can check that, you can plot the real part and the imaginary part of the eigenvalue and you can verify that, it is passing through at this critical point at this point and this point, it crosses the imaginary axis with 0 value.

So, these are and it looks like saddle point. So, this is known as saddle node bifurcation point. So, this is these two are the saddle node bifurcation point. Similarly, let us see one more case, pitch fork bifurcation. So, already I have shown this pitch fork bifurcation. Here the mu, so let me take this pitch fork bifurcation.

So, x dot equal to; so the first order equation if you understand, then you can understand the other equation mu x minus x square, for fixed point this x dot equal to 0; so, 0 equal to mu x minus x square. So, if you take x common. So, this becomes x into mu minus x, x equal to 0 is a solution and x equal to mu is also a solution; this should be mu x minus x cube, mu x minus x cube.

So, in this case x cube, so this becomes x square. So, mu, so x square equal to mu. So, you have two solutions; so one is x equal to 0, so that is the real state and the other one is mu equal to x square or x equal to plus minus root mu as you have seen in the previous case. Corresponding to this plus minus mu, so you will have two branch; so this is one more branch, this is also one more branch, in addition to that you have the trivial state.

So, this is the trivial state and these two are the non-trivial state. Similarly here, so you have these two are the non-trivial state and corresponding to this is the trivial state. If now for example, you reduce this value; by reducing these from the stable non trivial state, so this type of bifurcation is known as supercritical pitchfork bifurcation.

And here, so from a stable it is going to unstable, so that is why or from unstable it is going to another unstable and stable. So, this is known as subcritical pitch fork bifurcation.

So, this is subcritical and this is super critical pitch fork bifurcation. So, this is the generic form of this thing. So, you can prove these things, so by plotting this one. So, by plotting, so actually by plotting these things, so you can check; so, one is your 0 line. So, 0 line is stable.

So, up to these it is stable and you have all you can verify that, these two branches are stable and this becomes unstable after this point.

So, this is a pitch fork bifurcation point. And this type of bifurcation point, where mu x minus x square you are taking. So, if x dot equal to mu x minus x square by putting 0 equal to; if you take x common, so this is the things, x equal to 0 is a solution and x equal to mu is also a solution. So, these are the straight lines. So, x equal to 0 this is the trivial state and x equal to mu. So, this line is x equal to mu, so this is another line.

So, now, to study the stability, now you find the Jacobian matrix. So, your F x equal to mu x minus x square. So, differentiating this thing, so your A will be equal to, A equal to mu minus 2 x. So, A minus lambda i equal to 0 if you do. So, lambda will be equal to mu minus 2x. So, lambda equal to mu minus 2x.

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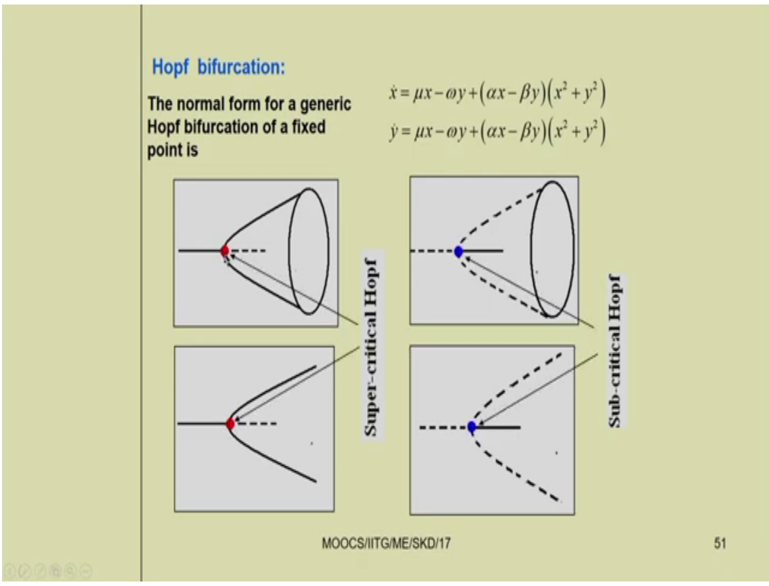

So, now let us take a point here. So, if you take a point there. So, you can see. Let us take a point and check. So, this is on the 0 line, x equal to for example, let us take minus 5. So, if we will take. So, for example, let us take mu value equal to 0. So, this lambda equal to mu minus 2x.

So, this line is mu and this is your x line. So, you can call this x and this is mu, x axis and mu. Just you verify; so what is the value? So, here you can verify for x equal to 0. So, you have taken x equal to 0. So, x equal to 0, so in this case, lambda will be equal to; when you are taking x equal to 0, so lambda equal to mu.

So, if mu is negative, then automatically lambda is negative and hence the system is stable. So, up to this then this is stable. So, after that, so if mu is positive, this part is positive; that means the eigenvalue is positive, so the response is unstable. So, you have a stable response here, so you have an unstable response here.

Now, take the these branches. So, if you take these branches. So, in this case, this is x equal to mu. So, we have taken x equal to mu. So, lambda will be equal to mu minus 2 mu, this becomes minus mu for. So, mu less than 0; so this becomes minus, minus plus, so this becomes positive. So, that is why this real part is positive, this becomes unstable.

And for positive part, this is minus mu; so this becomes negative. So, this eigenvalue becomes negative, real part of the eigenvalue becomes negative, so that is why this is stable. So, here what you have observed. So, a stable branch, stable non trivial stable trivial and unstable non trivial branch change their sign here and these becomes these.

Non trivial branches become stable, so from unstable to stable, and the trivial branch become from stable to unstable, they are changing their stability. So, this type of bifurcation is known as trans-critical bifurcation. Today class we have studied the static bifurcation points.

So, these four different type or three different types of static bifurcation points; that is saddle node bifurcation point, pitch fork bifurcation point, and trans-critical bifurcation points. And tomorrow class, we are going to see or study about the Hopf bifurcation point.

So, here we can see at the bifurcation point, stable fixed point branch becomes unstable fixed point branch; but it will give rise to a periodic response. So, tomorrow class, we will study about the Hopf bifurcation and also we will study other different types of responses for the Non-linear systems.

Thank you.