Nonlinear Vibration
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Lecture - 18
Bifurcation Analysis of Fixed-Point Response

Welcome to today class of Non-linear Vibration. Last class we have discussed regarding the stability and bifurcation of the non-linear systems two different type of bifurcations we have studied or we have discussed. So, one is static bifurcation and other one is dynamic bifurcation. So, today class we will deliberate more on the static and dynamic bifurcations along with stability analysis.

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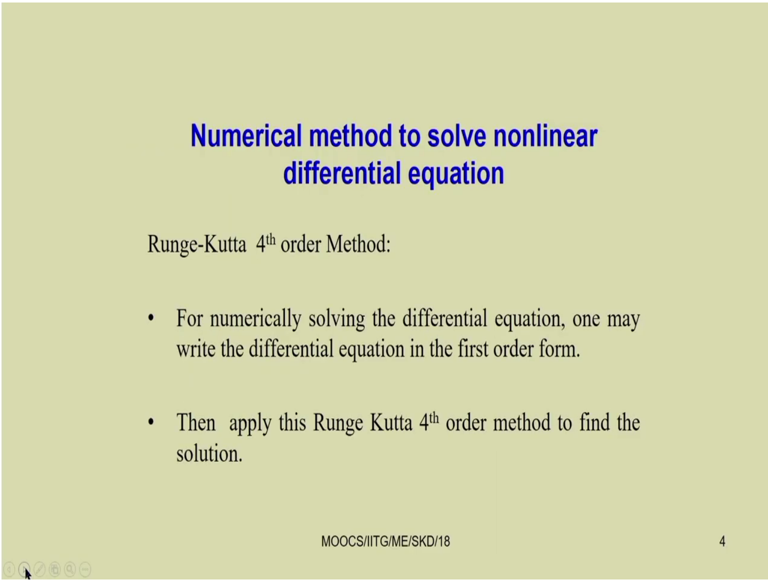

And we have taken the example of these duffing equations and also to study whether a system is stable or not, we should also know the numerical methods to solve the differential equations. Already we have discussed this thing many times in this course. So, for example, so you have to use this Runge-Kutta 4th order method.

Either you can use the codes available in different software packages, like in MATLAB, you can use this ode four five if you have stiff equation then you can ode 2 3 or other software's also you can use for different types of non-linear equation solvers.

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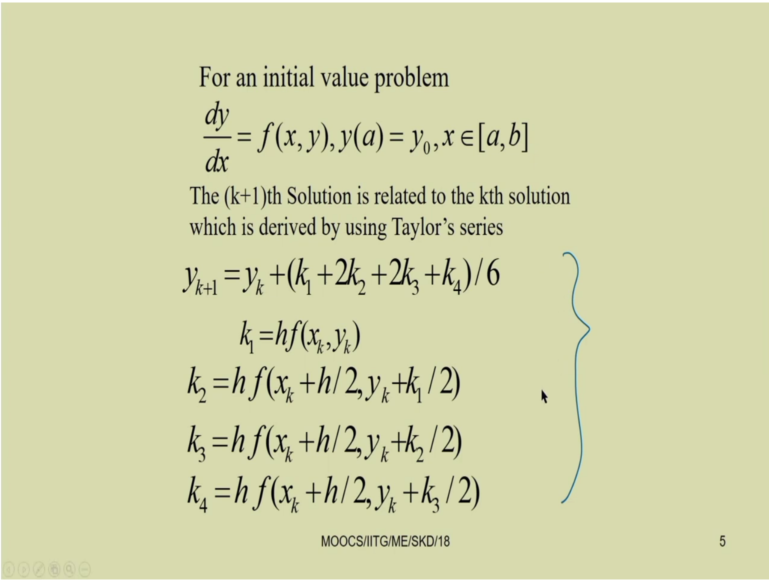

In case of you are using this Runge-Kutta method you can write your own code also. So, where you can use in this 4th order Runge-Kutta method. So, you can write these solution in this form y k plus 1 equal to y k plus k 1 plus 2 k 2 plus 2 k 3 plus k 4 by 6, where k 1 equal to h F x k y k k 2 equal to h f x k plus h by 2 y k plus k 1 by 2 k 3 equal to h f x k plus h by 2 y k plus k by 2 and k 4 equal to h f x k plus h by 2 y k plus k 3 by 2, where h is the time interval.

So, if you take this time interval very small your accuracy of the solution will be more or it will be more accurate solution you can get. So, after choosing these k and the initial condition. For example, so you can start with a initial condition taking these initial condition y k you can find the next y k plus 1 what is the response after the time interval h. So, in that way you can write down this code of your own and you can solve this equation.

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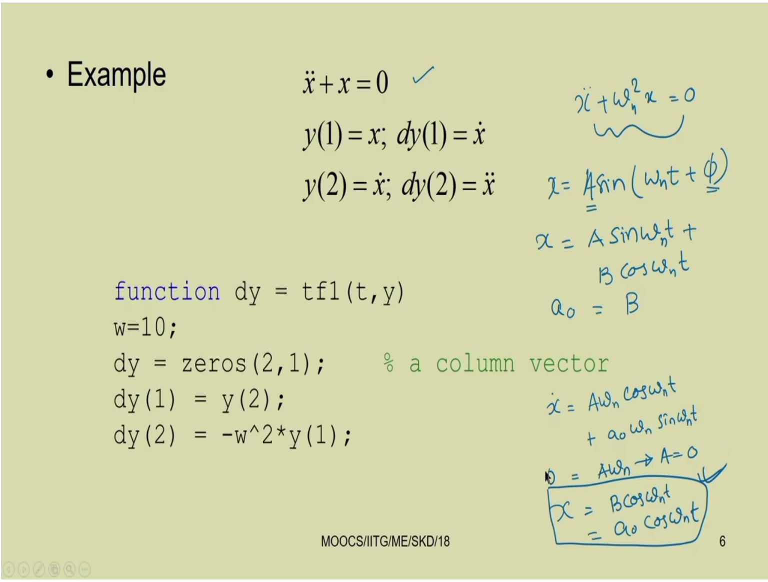

Previously also we have taken these perturbation methods. So, by using the perturbation methods you can get the reduced equations. Those reduced equations also you can use this Runge-Kutta method because these those reduced equations are first order equation.

So, for steady state as they will not be function of time directly you can get a set of algebraic or transcendental equations which you can solve. Otherwise you can use this Runge-Kutta method to solve this first order differential equations to find the solution.

If you have a secondary differential equation for example, x double dot plus x equal to 0. So, in that case you know the closed form solution of this thing which can be written as A sin t here omega n square equal to 1. So, omega n square, so this equation can be written in this form x double dot plus omega n square x equal to 0.

So, this is the equation of a spring mass system where this omega n equal to root over omega n square equal to k by m or omega n equal to root over k by m, but its solution already you know it is equal to x equal to A sin omega n t omega n t plus phi. So, A sin omega n t plus phi, here A and phi depends on the initial condition. So, you can take the initial displacement and initial velocity as initial conditions and you can find x.

So, here to write down the code in MATLAB. So, you have to initially you just assume let your y 1 equal to x and d y one equal to x dot. Similarly y 2 will be equal to x dot. So, d y 2 equal to x double dot, but your x double dot is nothing but in this case equal to minus x. So, you can write your equation in this form that is d y 1 in the function file you can write. So, here the code is written below.

So, you can see the code. So, in the function file it is written for example, so you have taken two equations. So, initially you just initialize with 0. So, d y equal to zeros 2 1. So, this is two row and one column. So, your d y 1 equal to y 2 and d y 2 equal to minus omega square y 1. So, this is the function file.

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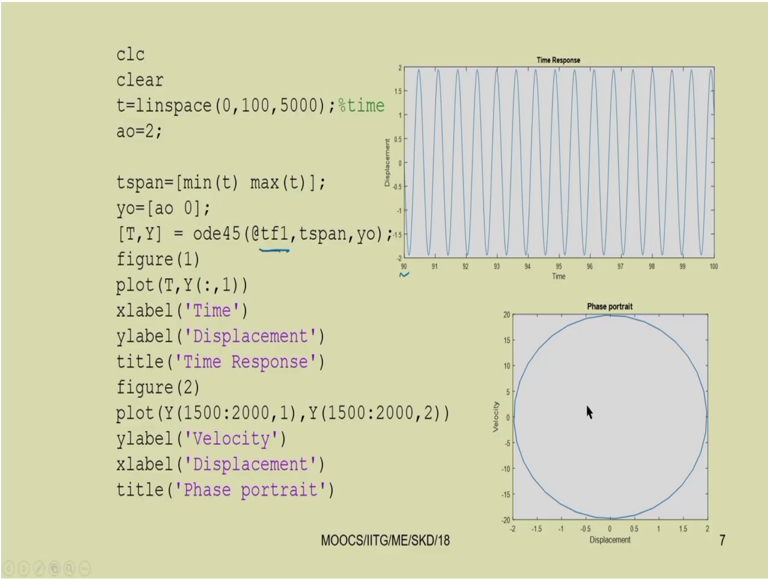

And in the main file you can call the function to find. So, this is the main file. So, here you have to decide or you just declare the time space t equal to line space 0 to 100. So, if you have to do it up to 100 with 5000 divisions it has been taken initial a 0 equal to taken 2. So, tspan minimum time and maximum time you can specify. So, y 0 initial conditions for example, y 0 equal to a0 0; that means, initial displacement equal to a0 and initial velocity equal to 0.

So, you can take different things also. For example, if the system is subjected to some impulse type of forcing then the displacement you can take 0, but the velocity will be equal to f delta t by m, you can find the initial velocity and you can put it. So, depending on the application, so you have to choose these initial condition here by taking these initial condition y0 equal to a0 0 you can take. So, you can write this ode T Y equal to ode 4 5 at the rate tf1.

So, you just see this tf1 is the function filename. So, tf1 tspan y0 you can write. So, there are some other options also available. So, we can check this is the simplest one you can put, you can write in for example, in figure 1 let you plot the time response in figure 2 let you plot the phase portrait in figure 1. So, plot T Y colon 1, so that means, in Y.

So, you are storing the first column you are storing this Y and in second column you are storing this Y dot. That means, you are storing this x and x dot that is displacement and velocity in these two column. The first column of Y you will store the displacement the second column you storing the velocity. So, when we are plotting this phase portrait that time you can plot Y1 comma Y2. So, you can see here then this ylavel. So, you can put plot T comma Y colon 1.

So, that means, give you the first column. So, all the first column that is displacement. So, this plot you can see displacement versus time. So, already you are familiar with the solution y equal to Y 0 sin omega t, here by putting these t equal to 0 displacement equal to 2. So, it will start from.

So, you have to put that equation. So, a0 equal to A sin omega t. So, two way you can write this you just see the closed form solution either you can write it in this way or you can write x equal to A sin omega n t.

So, if you want to compare your result with respect to the analytical solution this way you can do also. So, B cos omega n t, so at t equal to 0. So, we have taken x equal to a 0, so a 0 equal to, so a 0 equal to B now you can put x dot. So, x dot equal to A omega n A omega n cos omega n t plus for B it can be a 0 a 0. So, for cos differentiation equal to a 0 omega n sin omega n t. So, at t equal to 0. So, as it is equal to 0 you are taking. So, 0 will be equal to A omega n.

So, A omega n equal to 0. So, as omega n is not 0 so A equal to 0. So, the solution becomes x equal to B cos omega n t B cos omega n t where B equal to a 0. So, this becomes a 0 cos omega n t the solution becomes a 0 cos omega n t. So, you can compare this thing with the solution what you have obtained by using this Runge-Kutta method.

Here you just see it has been plotted from 90 that is why you have not seen at t equal to 0. So, what is the values by changing this axis property. So, it has been said from 90 to 100 to the response to be periodic. So, here also the phase portrait is not plotted for the whole range 0 to 100. So, it has been plotted from only 90 to 100.

So, you can choose. So, for example, here you just see the plot command. So, plot Y 1500 to 2000. So, you have taken only the data available from 1500 to 2000 column 1 comma Y 1500 to 2000 column 2 that is this is displacement versus this velocity we are plotting. So, you can label these xlevel as ylevel as velocity xlevel as displacement then title phage portrait. So, this is the phase portrait we have seen.

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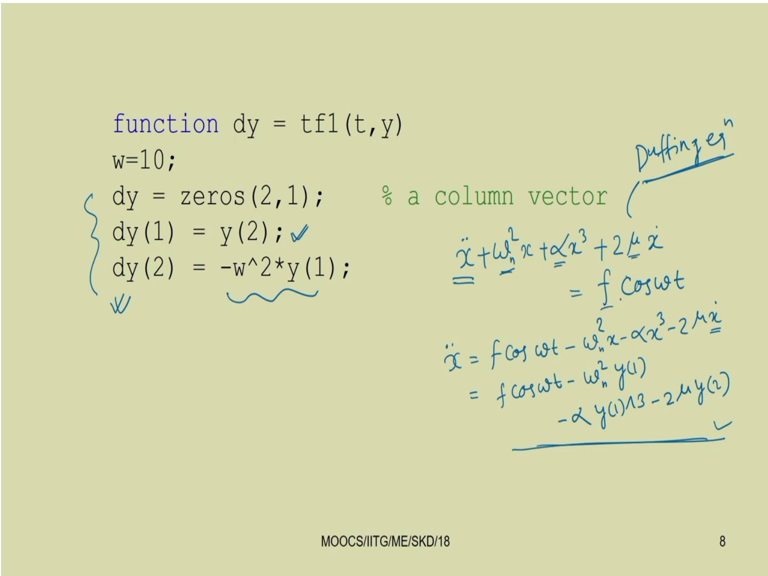

So, now, as an assignment, so you can check by using this way. So, you can check the duffing equation also. So, in this duffing equation if you want to find the response. So, in that case only these function has to change. Here the first one will remain same also your duffing equation let you right the duffing equation in this form that is x double dot plus omega n square x plus alpha x cube plus 2 zeta or 2 mu x dot you can put 2 mu x dot will be equal to f sin omega t or f cos omega t let us put, you can put f sin or f cos omega t.

So, in this case, so your first equation will remain same, second equation is nothing but. So, this part will be replaced by x double dot. So, x double dot in this case equal to x double dot equal to f cos omega t minus omega n square x minus alpha x cube minus 2 mu x dot here. So, you have to substitute this is equal to f cos omega t minus omega n square.

So, for x, so x equal to your y 1, so 1 minus alpha x cube. So, this becomes y 1 whole cube you just see y 1 whole cube you can put, whole cube. So, minus 2 mu for x dot. So, it can be so it is equal to y 2. So, this way you can write your function 2. So, this function 2 can be written dy 2 can be replaced by this one and you can use this same ode 4 5 function to find the different solution for a duffing equation.

So, different solution for duffing equation. So, you can verify for different value of different coefficients the response may be periodic, sometimes it will be periodic.

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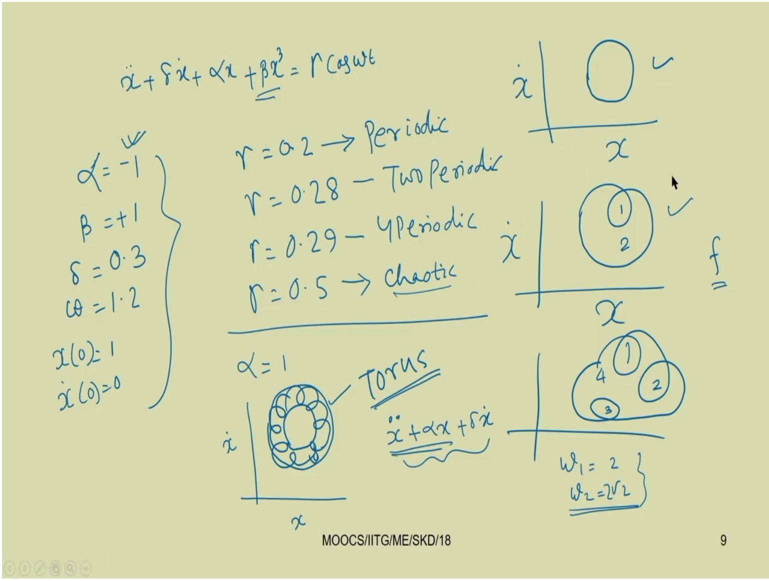

So, in case of the periodic if you plot the displacement versus x verses x dot. So, in case of periodic, so you can see the phase portrait will be. So, the phase portrait will be a closed solution. So, in case of for some other value of the system parameter. So, you can see the phase portrait maybe may contain two loops.

So, for some other value if you go on changing your system parameter for example, if you go on changing this value of forcing f. So, amplitude of forcing parameter if you go on changing. So, you can see it will go on. So, this is periodic, this is two periodic then you can have 4 periodic. So, in case of 4 periodic, so there will be 4 loops. So, here you just see this is loop 1, this is loop 2.

Similarly, you can have 4 loops. So, for example, this is one, this is the second one, another one and this one. So, you can have 4 loops. So, this is 1 loop, this is the 2nd loop, this is the

3rd loop and the whole thing is itself is a loop, so that is 4 loops. So, by changing the system parameter if you write the equation particularly. So, if you write this equation in this form.

For example let me write and you can verify this thing. So, the equation if it is written x double dot plus delta x dot plus alpha x, let me write this way plus beta x cube equal to gamma cos omega t. So, if you take the value for example, take the value of alpha equal to minus 1, then beta equal to plus 1, beta equal to plus 1, then delta equal to damping delta equal to let you take equal to 0.3.

So, damping equal to 0.3, then omega let us take omega equal to 1.2 and initial condition, let me take x 0 equal to 1 and x dot 0; x dot 0 equal to 0, displacement equal to 1 and velocity equal to 0. So, if you take these parameters. So, for you can see for gamma equal to 0; gamma equal to 0 you can see the response or gamma equal to 0.2, gamma equal to 0 is the free vibration response.

So, gamma equal to 0.2, if you put then you can check that the response is periodic, the response is periodic like these things, so you can get a response. So, if you take this gamma equal to 0.28 for example, 0.28. So, you can find this is two periodic. Similarly if you take gamma equal to for example, you just take gamma equal to 0.29; gamma equal to 0.29, so it is 4 periodic.

And if you take higher value of gamma for example, you just take gamma equal to 0.5. So, you verify this thing gamma equal to 0.5, so the response will be chaotic.

Chaotic means, it contains many harmonic terms more than 16 times or more than around 30 harmonic terms it will contain, so the response is chaotic. So, the chaotic response are deterministic response, but it will contain many harmonics and you cannot distinguish from the time response and from the phase portrait.

So, we can easily check that the response to be chaotic. This chaotic response, in case of chaotic response it depends on the initial condition by changing these initial condition so you can check that. So, you can get different type of attractors, in one case for example, by taking

1 0 you will get one type of attractor. By changing that initial condition you can verify that you are getting different other attractor the attractor will not be same.

So, you can verify these thing another conditions also you can take and check. So, if alpha equal to 1.

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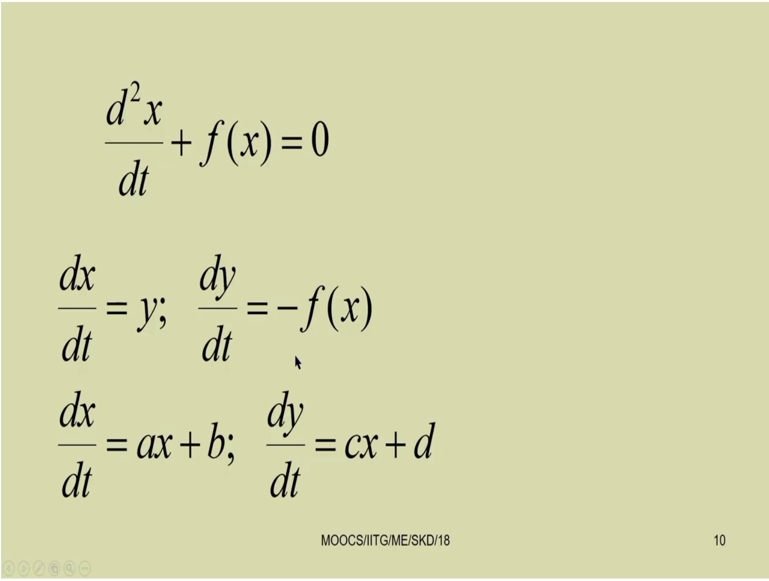

In this case in this particular case you just see alpha you have taken to be minus 1. So, in this case you have taken alpha equal to minus 1, particularly if you see this equation x double dot minus alpha x. So, you just see the system is unstable, so in case of a linear system. So, x double dot minus alpha x alpha is positive or you just take this plus and now alpha is negative.

So, if alpha is negative plus delta x dot in linear case actually the system to be in the linear case the system is unstable, but by adding these by adding these cubic non-linearity and applying some force. So, you can see it yields two different type of other different type of responses. We now understand that in addition to fixed point response.

So, there are other type of responses also available in our system. So, the they are periodic response. These periodic may be single frequency it may be multi frequency. So, in case of single frequency this is single period you got. So, in case of 2 frequency, so you can get this type of 2 lobes in case of 4 frequency 4 lobes. So, in all these cases if you check carefully. So, the frequency ratio are in the form of integer numbers.

So, this let it is omega 2 and omega 1. So, omega 2 by omega 1 must be in the form of integer, but if they are irrational number, so let us check take some irrational number. For example, 1 frequency omega 1 equal to 2 and other frequency omega 2 equal to root 2 root 2 times 2, 2 root 2. So, in that case you can verify; you can verify that the phase portrait will no longer be similar to these, what it will be similar to a torus. So, it will be similar to a torus. So, what is a torus? So, you can check in case of a torus.

So, the you have two circles which are connected by so connected by this. So, this way they will be connected, so this is torus. So, we will study more regarding this torus when we will discuss regarding the quasi periodic response. So, now let us again come back.

So, these are the different type of response you can get. So, given a differential equations. So, now, you can use this ode 4, 5 from the MATLAB to get different type of response. So, these response you can try and you can find.

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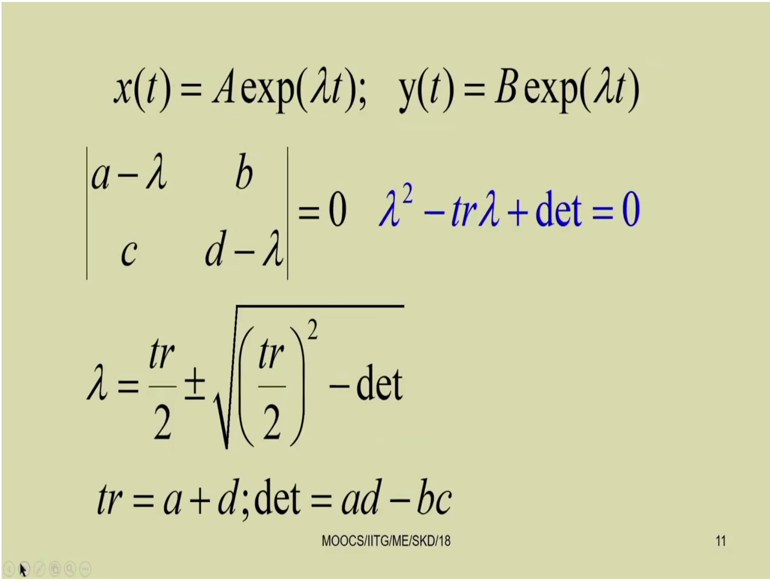

So, let us come back to. So, again so let us briefly again check that, if we have a equation. So, we can study the stability of that equation by studying or by finding the trace and determinant also. So, this part already we have discussed.

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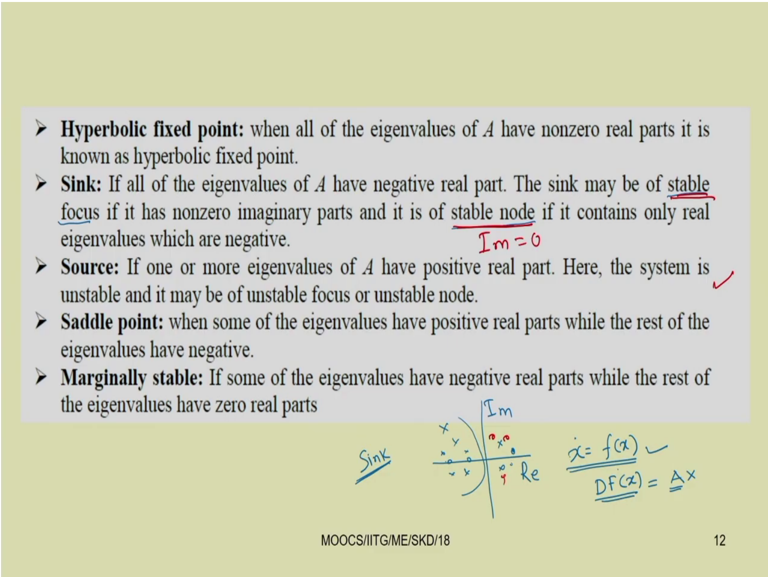

So, now let us see few definitions. So, one is this hyperbolic fixed point. So, we know now. So, given an equation for example, let we have a equation x dot equal to f x. So, this is the given equation and we want to study the stability of this equation. So, in that case, so we know the equilibrium point or the fixed point.

So, when the equilibrium point is said to be. So, there are several definitions are there. So, one is hyperbolic fixed point when all the eigenvalues of A. So, we can find these DF x. So, you can find these DF x these DF x can be written as AX. So, this a matrix is the Jacobian matrix.

So, when all the eigenvalue of the Jacobian matrix have nonzero real parts, it is called or it is known as hyperbolic fixed point. So, already we have discussed regarding these eigenvalue and we have seen. So, if all the eigenvalues. So, if you plot this is the real part of the eigenvalues, this is the imaginary part. So, you take all the eigenvalues for different system parameter, if the eigenvalues are in the left hand side of the x plane.

So, if they are on the left hand side of the x plane then the system is stable, here the hyperbolic fixed points. So, when all the eigenvalues of A have nonzero real parts. So, it may be in the right hand side or in the left hand side of the x plane, but if it is nonzero real part then they are known as hyperbolic fixed point.

So, sink if all the eigenvalues of A have negative real parts if all the eigenvalues of A. So, they are negative real parts, so then it is known as sink. So, that means so if they have negative real part, that means, the system is stable. So, if the system is stable. So, it is known as sink the sink may be stable focus if it has nonzero imaginary parts and if it is node if it contains only real eigenvalues which are negative. So, out of all these eigenvalue if all the eigenvalues are only. So, it has 0 imaginary parts then we can get ignored.

So, if they are complex conjugate then we will get oscillatory motion. So, that time it will be. So, it may have nonzero imaginary part. So, in that case we will get focused. So, it is known as stable focus and in case of only node so, we will get stable node if it contains. So, if it contain only the real part if the imaginary part is 0.

So, we can have two options either imaginary part is 0. So, if imaginary part is 0 then we will have the stable node, if imaginary part is 0 then we can have stable node. So, this is imaginary part 0.

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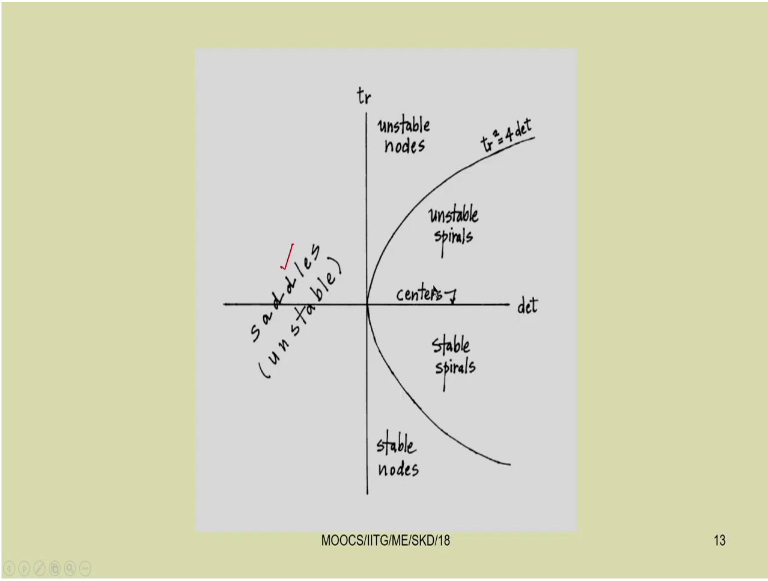

So, if imaginary part is there then we can have if imaginary part is there, then we can have stable focus. So, here so it will spiral actually, it will spiral down to get the response. So, already we have plotted those things and we will see it again. So, then source if one or more of the eigenvalues of A have positive real parts. So, then some of the roots or some of the eigenvalues will be in the right hand side of the x plane.

So, in that case it is known as source, so we will have source. So, if one or more of the eigenvalue of a have positive real parts, here the system is unstable and maybe unstable focus or unstable node. So, if it the complex number that is nonzero imaginary part that case we will get focused unstable focus and if it is 0 imaginary part, but it is in the right hand side then it will be unstable node, saddle point.

So, when some of the eigenvalues have positive real part and some other have negative real part. So, that means, so it is in both the side of the x plane. So, then we can tell it is a saddle node saddle point, this is saddle point. So, in case of saddle point equilibrium point can be told to be saddle point, if some of the eigenvalues have positive real parts while the rest of the eigenvalues have negative real parts.

Marginally stable, if some of the eigenvalues have negative real parts, while the rest of the negative eigenvalues have zero real part, it is marginally stable. So, if most of the eigenvalues have negative real part, but some of the eigenvalues have zero real part. So, none of the eigenvalue have real part in the right hand side of the x plane.

So, if some of them are on the right hand side of the x plane then it will be source or the system will be unstable with these things. So, we know when the system will be stable. That means, when the system response will be bounded and unstable when the system response will be unbounded. So, from the trace and determinant also we can get the same thing. So, we know the saddle point, so the saddle point will be here.

And then so if you plot the determinant versus trace. So, we can see this trace square equal to 4 into determinant, so this curve we can plot. So, here it will be centers on stable spiral stable spiral. So, if it is in this region it will be stable spiral. So, in this region it will be stable nodes above this line.

So, above this line it will be on this line it is center and it is unstable spiral and unstable nodes, nodes is imaginary part equal to 0. So, if the eigenvalue has 0 imaginary part actually you will get that thing.

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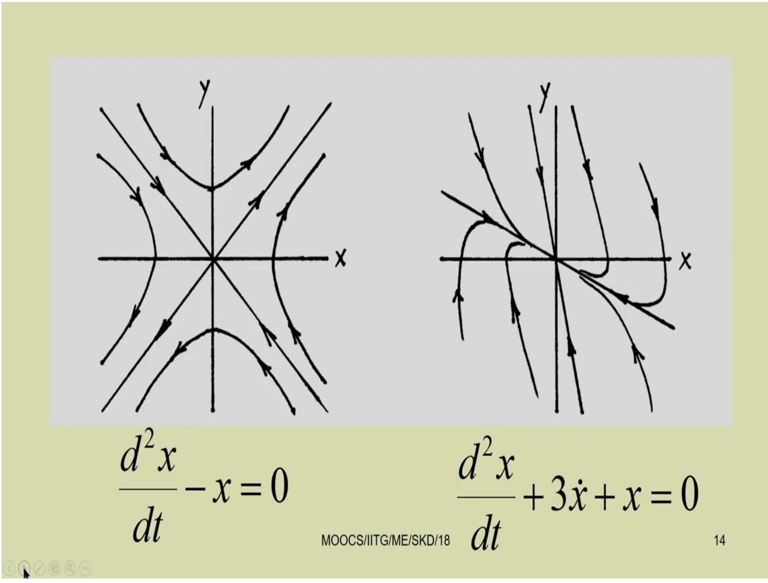

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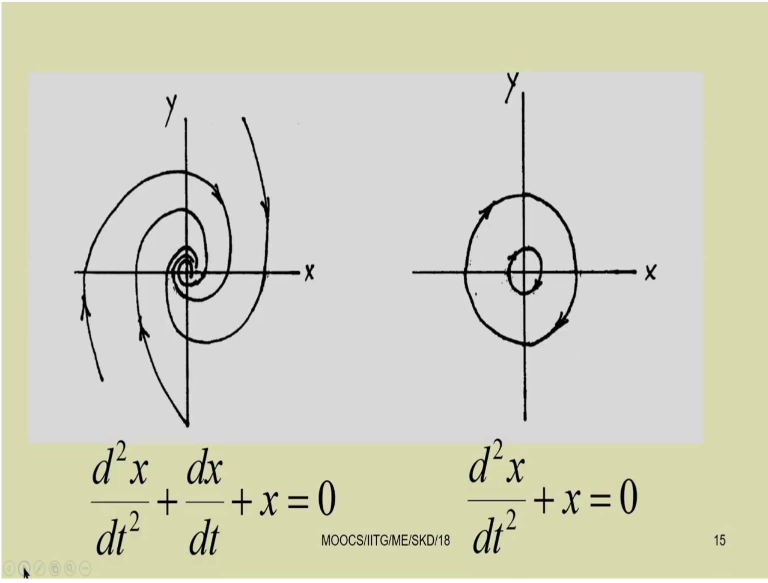

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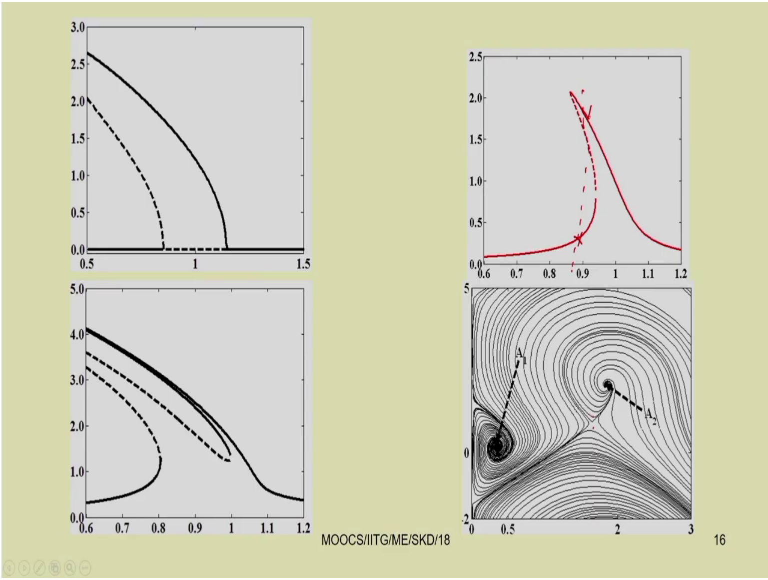

Already we have plotted this thing and we have seen and these are different examples also we have taken previously and we know regarding the multiple branch in case, multiple equilibrium solutions or multiple equilibrium points in case of the non-linear system. And for these multiple equilibrium points, we must use this basin of attraction method to find for which equilibrium for which initial points. So, one obtain which equilibrium points.

So, for different for example, you can see for all these initial points it is coming to this thing and for all these initial point, so it is coming to A 2, so A 1 A 2. So, you can have bi stable region here, so if you draw a line here. So, you can get the bi stable region and these in this bi stable region. So, you can find this is one solution bi stable solution. So, bi stable, but three solutions you have. So, out of this thing one is unstable. So, this point is the unstable saddle point.

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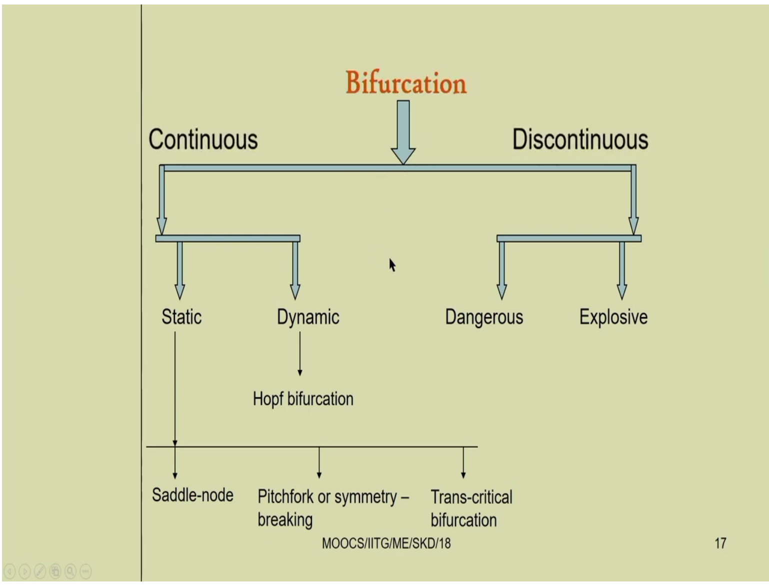

So, you can have a saddle point here. Now, let us revise this thing this bifurcation. So, you have continuous bifurcation discontinuous. So, you have static bifurcation dynamic bifurcation. So, in case of static, so you have saddle node bifurcation, pitch fork or symmetry breaking bifurcation and trans critical bifurcation and in case of dynamic. So, you have this hopf bifurcation ok.

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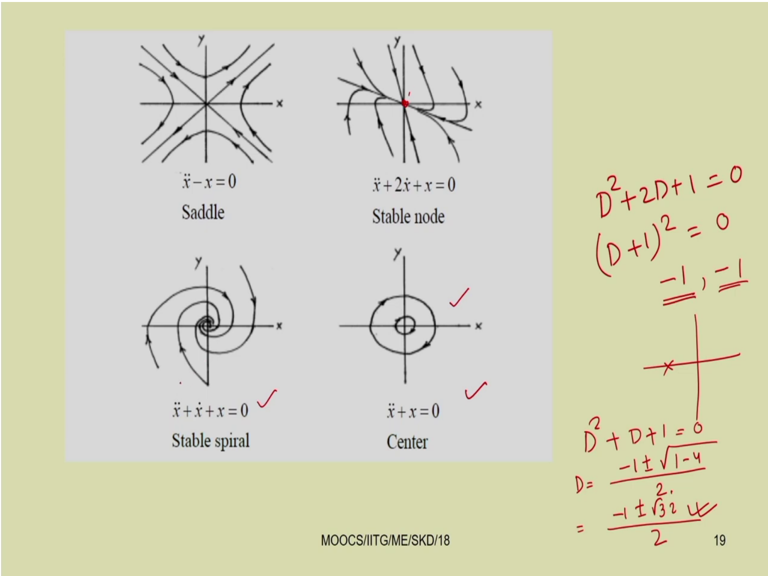

Already we have seen saddle. So, this is the saddle point and this is the saddle x double dot minus x equal to 0. So, you just see the roots of the eigenvalue or roots of this equation. So, x double dot minus x equal to 0. So, x double dot will be or this auxiliary equation will be D square minus 1 equal to 0 or D square equal to 1. So, D equal to plus minus 1.

So, the solution will be c 1 e to the power t plus c 2 e to the power minus t. So, the c 1 e to the power t term will exponentially grow the response as t tends to infinite. So, that is why so the response so you can get. So, one part we will pull it off and the other part c 1 c 2 e to the power minus t. So, that will give rise to. So, as t tends to infinity it tends to 0.

So, these type of response are known as saddle, so this is a saddle point. So, in the saddle you can if you draw these x verses x dot x verses x dot. So, you can see at different region. So, the flow will be like this thing. So, the flow will be, so here the flow will going away.

So, here the flow is this way this direction left side it is in you just see in the left side it is moving in clockwise direction, in right side it is moving in anti clockwise direction and here in the top portion here you just see it is moving in.

So, it is moving in anti clockwise direction and here it is moving in clockwise direction. So, similarly so the stable node you can get x double dot plus 2 x dot plus x. So, this is similar to that of a this damped. So, this is that of a single degree of freedom system with damping.

So, in case of damping as t tends to infinite the response tends to 0. So, it tends to, so all the responses you just see all the response to the left hand side is closing to this point and similarly from right hand side also they are closing to the same point.

So, all the in the phase portrait, so all of them are moving to the same point and here. So, this is a stable node point. So, similarly you have a center. So, in this case x double dot plus x equal to 0. So, the solution becomes x double dot minus x equal to 0 here the auxiliary equation d square equal to minus 1. So, as D square equal to minus 1, so D equal to plus minus i.

So, the solution becomes a cos omega t plus b sin omega t. So, where this omega equal to one in this case. So, the response is harmonic. So, as the response is harmonic if the displacement is written in terms of for example, in sin. So, the velocity will be in cos and if you plot this displacement versus velocity, so it will be circle only.

So, x square plus y square will be equal to 1. So, this way so it becomes a circle. So, this is a center in this case this is a center and this is a. So, in this case you can see that a stable spiral you can get a stable spiral. So, it is spiraling down to, so this is a stable node.

So, stable node means so in this case, so you have in this case you just see the auxiliary equation and you can verify or you just find the roots. So, from the roots you can find that the roots are. So, the roots will be a real root here. So, because you just see. So, your equation becomes. What is the equation? D square plus 2 D.

So, here the equation is D square; D square plus 2D d square plus 2D plus 1. So, this is the auxiliary equation. So, it should be equal to 0. So, then this is D plus 1 whole square D plus 1 whole square equal to 0. So, you have two roots and they are equal roots and the value is minus 1 roots are minus 1 minus 1 and minus 1.

So, these are real roots only real roots and they lie in the left hand side of the x plane. So, as they are lying in the left hand side of the x plane, the solution must be stable. So, this is a stable solution you are getting, so it is a node. So, if you have imaginary parts for example, in this case what is the auxiliary equation? Auxiliary equation this is D square plus D square plus D plus 1.

So, this is the thing. So, this should be equal to 0. So, the roots of this equation you can write for example, so D will be equal to minus b minus 1 plus minus b square minus 4 ac by 2 a b square becomes 1 minus 4 ac this is 4 by 2 a that is 2. So, this becomes minus 1. So, here the root you just see this becomes minus 3.

So, inside this square root we have minus 3. So, it will be root 3 i plus minus. So, it will be plus minus root three i by 2, so here it is imaginary. So, we have the imaginary root. So, if we have imaginary root. So, the response will be oscillatory. So, that is why you are getting the spiraling of the solution. So, in this case it will not be spiral. So, it is a stable node, but in this case it will spiral down and finally, it will be at 0.

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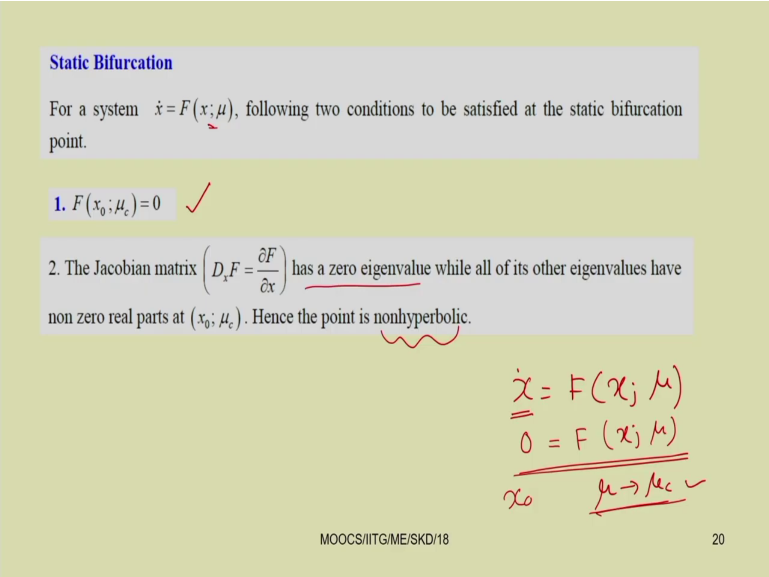

Finally it will be at 0. Last class we have studied regarding the static bifurcation. So, we can have further discussion on the static and dynamic bifurcation today, given a equation x dot equal to F x mu. So, you just see here we are using semicolon instead of comma as mu is a parameter that is why we are using the semicolon here, x is a variable, but mu is a parameter which we are varying or we are changing to get different response, for a given equation x dot equal to.

So, if we have a equation x dot equal to F x mu x is the variable and mu is the parameter. So, for steady state or for equilibrium position or for fixed point position, so this x dot must be equal to 0. So, we can get all the fixed point response. So, after getting these fixed point response. So, we can study whether the fixed point is stable or unstable or there is some bifurcation points also to have a bifurcation point late for mu equal to mu c mu critical value let there is a bifurcation point.

So, to find this bifurcation point the first condition must be if x 0 let x 0 is the corresponding value. So, F x 0 mu c equal must be equal to 0. So, this is the first condition and second condition the Jacobian matrix that is D x F that is del F by del x has a zero eigenvalue while all other eigenvalues have nonzero real parts at x 0 at mu c. So, has a zero, so it must have a zero eigenvalue.

So, the Jacobian matrix D x F del F by del x has a zero eigenvalue while all of its other eigenvalues have nonzero real parts at x 0 mu. Hence this point is non hyperbolic equation at bifurcation point for a hyperbolic. So, in case of a hyperbolic fixed point all the real parts of the roots must be must not be 0.

So, it should be nonzero real part of the eigenvalue for hyperbolic fixed point, but if one of these eigenvalue is zero. So, then this becomes non hyperbolic. So, for the static bifurcation these two condition must be satisfied. So, 1 F x 0 mu c equal to 0 and this Jacobian matrix must have a zero eigenvalue in case of dynamic bifurcation. So, we can see, so it must have a eigenvalue.

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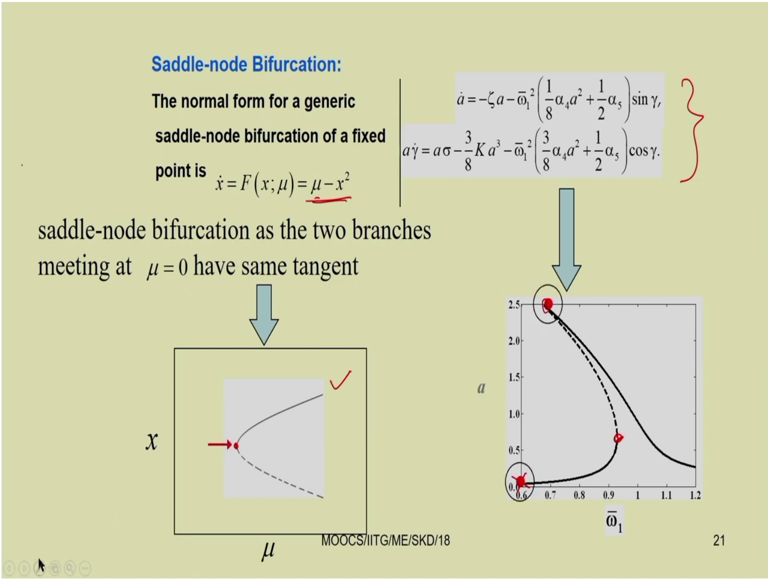

So, which will have the imaginary parts also, so already we have discussed regarding the saddle node bifurcation and the in case of saddle node bifurcation the generic form of this thing equal to mu minus x square. And so, by putting these x dot equal to 0. So, we have the fixed point equal to x square equal to root over mu. So, from that thing if you plot this x versus mu, so we have this curve.

So, now if we can find the slope at each point, so you can see the slope at this bifurcation point. So, before mu equal to 0, so all the points mu less than 0. So, root over mu is imaginary. So, it is not real, that is why the solution does not exist solution does not exist before mu equal to 0. So, after mu equal to 0 we have two solutions. So, out of these two solutions the upper branch you can see. So, it is stable because its eigenvalue will be negative and the lower parts is unstable.

So, if you plot the plot the slope at each point. So, you can see at this bifurcation point or at this point they have the same tangent. So, saddle node bifurcation are the two branches meeting at mu equal to 0 have the same tangent. So, here also by taking some other examples. So, you can see, so this for, so we have a saddle node bifurcation point here. So, it is not here. So, we have a saddle node bifurcation point here and also one saddle node bifurcation point here.

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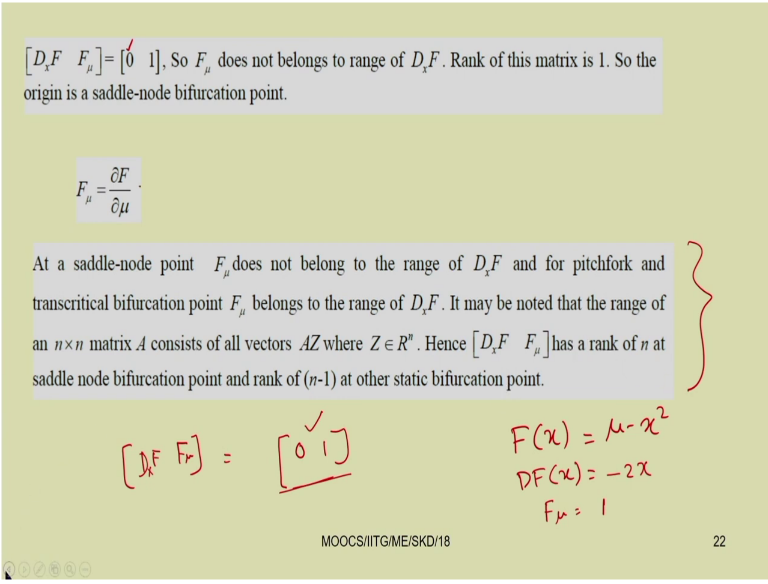

In this case further you can construct a matrix that is D x F and F mu matrix. So, where F mu equal to del F by del mu and D x F is nothing but the Jacobian matrix and you just construct a matrix with the Jacobian matrix and F mu and in this particular case. So, in this particular case for example, this is F equal to mu minus x square F equal to. So, as our F equal to F x equal to mu minus x square.

So, our D x F, so if you differentiate this thing with respect to x DF x equal to minus 2 x and F mu equal to F mu you differentiate this thing with respect to mu. So, this becomes 1. So, if I will construct a matrix. So, for the equilibrium position. So, for the equilibrium position D x F equal to 0. So, this part equal to 0 and F mu equal to 1.

So, we have a matrix now we have construct a matrix that is DF x D x F D x F and F mu. So, these matrix equal to 0 and 1 in this particular case. So, here you can see this F mu does not belongs to the range of DF x. So, rank of this matrix is one the origin of the saddle node bifurcation point. So, the origin of those this is the. So, the origin is the, so at x equal to 0. So, this origin is the is a bifurcation point.

So, here at this point it satisfy both the conditions and at the same times. So, to distinguish the saddle node bifurcation point with other different static bifurcation point you must have to do these test, just find the rank of this matrix. So, this rank you can see in this case. So, the rank of this matrix is 1 this origin is a saddle node bifurcation point. So, we will see in other cases and a saddle node point F mu does not belong to the range of F mu. So, here F mu equal to 1.

So, this does not belong to the range of D x F. So, in case of D x f, so you have only 0. And for pitchfork and trans critical bifurcation point F mu belongs to the range of D x F. So, we can see later in this example with the example for this pitchfork and trans critical bifurcation.

So, we will construct this matrix and check. So, what is the range of this matrix? So, it may be noted that the range of an n cross n matrix A consist of all vectors AZ where Z belongs to R to the power n.

Hence D x F F mu has rank of n at saddle node bifurcation point in this previous case. So, it has a rank of 1 at the saddle node bifurcation point and the rank of n minus 1 at other static bifurcation point. So, if you take other static bifurcation point. So, you can find the rank is less than 1. So, that is n minus 1.

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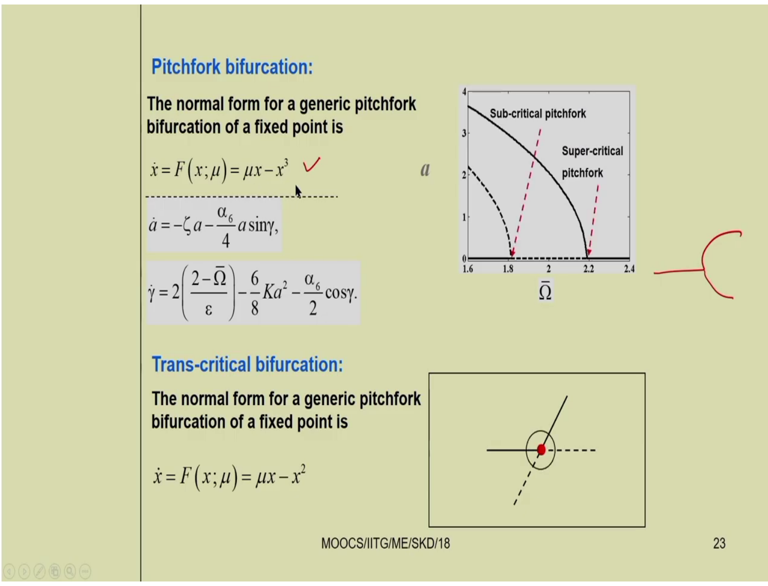

This point has to be remember this point has to be taken care while distinguishing between the saddle node bifurcation point, trans critical bifurcation point and these pitch fork bifurcation point. Particularly from the shape also we can decide that thing. So, in case of a saddle node bifurcation point at the saddle point. So, the two branch are the meeting point of the two branch.

So, they will have the same tangent what in case of pitchfork bifurcation the from the shape itself it can be known that this is pitchfork. So, it is starting, so like a pitchfork. So, pitchfork we know. So, the pitchfork type of bifurcation and here also the generic form is mu x minus x cube.

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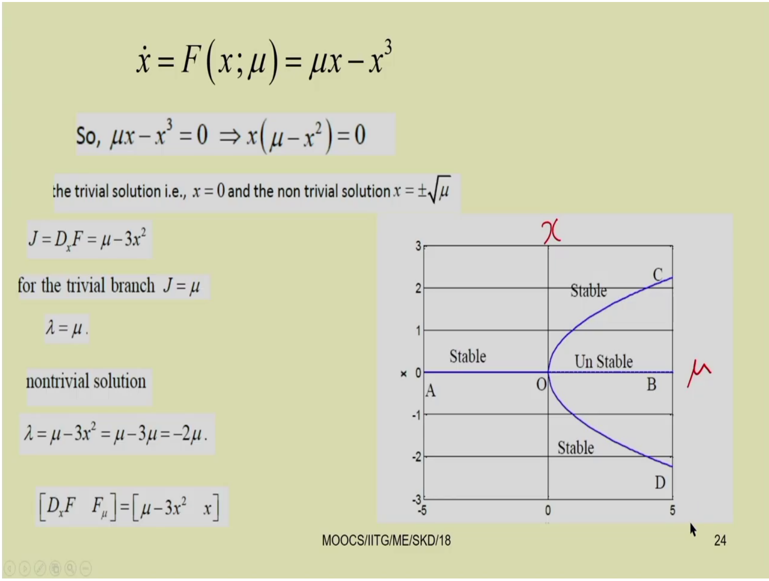

So, x dot equal to mu x minus x cube. So, in that case let us see. So, x dot equal to F x mu. So, it is equal to mu x minus x cube. So, by putting these x dot equal to 0. So, we will get the equilibrium position. So, mu x minus x cube equal to 0. So, by taking these x common. So, x into mu minus x square equal to 0. So, x equal to 0 is an equilibrium point and also x square equal to mu is also an equilibrium point.

So, you have three equilibrium points one is x equal to 0 and other 1 plus minus root over mu will be another two equilibrium points. By plotting these things we can plot o to a that is equal to 0 o b also a to b the trivial state that is x equal to 0 and this non trivial state.

So, this is mu verses x you can plot mu versus x you can this is x this is mu x axis a y x axis is mu x axis is mu and y axis is x. So, by plotting these thing, so now, you can see the

Jacobian matrix. So, differentiating these thing mu x minus x cube. So, we will get the Jacobian matrix. So, this becomes mu minus 3 x square, so for the trivial state.

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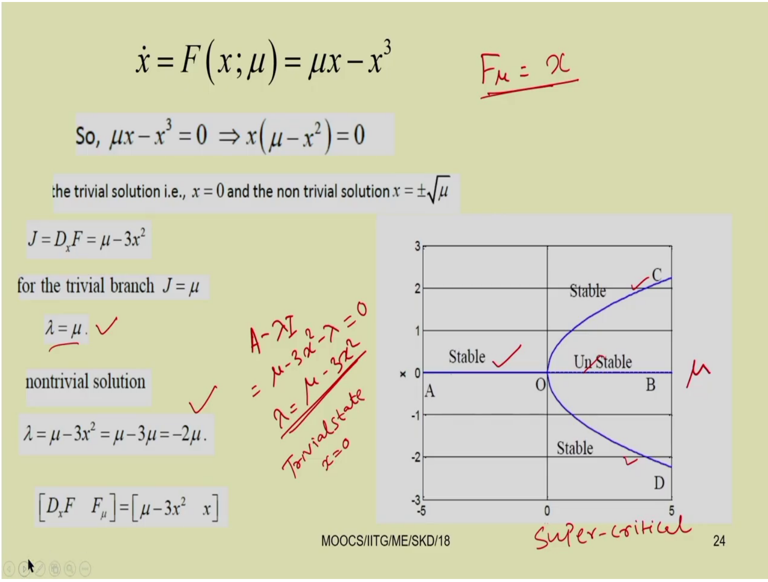

So, we can find this A minus lambda i. So, we can do A minus lambda i. So, in this case, so A is the Jacobian matrix or j minus lambda i. So, this becomes mu minus 3 x square minus lambda making this thing equal to 0. So, we will get lambda equal to mu minus 3 x square. So, mu minus 3 x square is the. So, mu minus 3 x square, so for trivial state as x equal to 0.

So, lambda equal to mu, so for trivial state. So, this is the trivial state. So, here x equal to trivial state x equal to 0, so as x equal to 0, so we have lambda equal to mu. So, for non trivial state, so for non trivial state, now lambda equal to mu minus 3 x square. So, by putting different value of already we have these x square equal to; x square equal to mu x equal to plus minus root over mu.

So, x square equal to mu. So, by putting these things so we can have mu minus 3 mu this is minus 2 mu. So, you can check for non trivial branch. So, the lambda that is the eigenvalue becomes minus 2 mu, now by putting this different value of mu. So, we can find whether the system is stable or unstable, here from A to 0.

So, from A to 0, so that is mu less than, so this is mu value. So, for mu less than 0, so if mu less than 0 for the trivial branch, so mu is negative. So, this part mu equal to negative. So, eigenvalue is negative that is why this is stable and for o to B branch, so mu is positive. So, as mu is positive lambda is positive. So, from this equation lambda is positive.

So, as lambda is positive the response is unstable, so the system is unstable. So, now, for the other two branches let us see. So, this becomes minus 2 mu as to the right hand side for this non trivial branch mu is always positive. So, minus 2 mu is always negative. So, this branch and this branch, so both the branches are then stable these type of bifurcation.

So, from where, so it is moving from a stable trivial state two stable non trivial state is known as super critical pitch fork bifurcation. So, this is super critical super critical pitchfork bifurcation, in this case mu x minus x cube. So, this is super critical pitchfork bifurcation.

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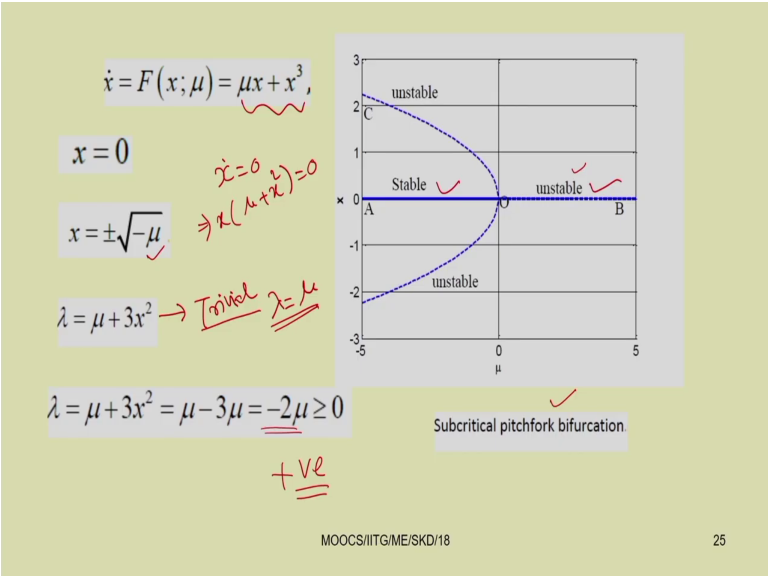

Similarly we can see here again you just see you can check by constructing this DF x and F mu. So, what is F mu in this cases so F mu, so find F mu. So, if you differentiate this with respect to mu. So, you will get F mu equal to x yes or no. So, F mu equal to x. So, for the trivial state x equal to 0. So, F mu equal to 0 already D x F equal to 0.

So, it has a rank of, so this becomes 0 and this becomes 0, so this is 0 0. So, the rank becomes 0. So, this rank is not one here. So, the rank as we have seen there, so rank is n minus 1. So, in this case we can differentiate or from this thing from the saddle node bifurcation point from or by constructing these D F D x F and F mu matrix.

Now, you just take the other one that is x dot equal to F x mu, but you take x dot equal to mu x plus x cube. So, previously we have taken mu x minus x cube. So, here it is mu x plus x cube. So, if you take mu x plus x cube. So, then x dot equal to 0. So, x dot equal putting x dot equal to 0. So, we have this taking this x common. So, x into mu plus x square equal to 0.

So, x equal to 0 is one solution and the other solution becomes x equal to plus minus root over minus mu. So, this is the other two solution. So, we have three solutions here and if we can plot it. So, we can see this one, similarly for trivial state. So, for trivial branch let us find the Jacobian matrix. So, Jacobian matrix from the Jacobian matrix. So, differentiating this thing with respect to x.

So, the Jacobian matrix you will get mu plus 3 x square. So, for trivial state by putting these x equal to 0. So, lambda becomes lambda equal to mu. So, lambda equal to mu for the trivial state and here mu less than 0. So, this branch is stable and mu greater than 0. So, this branch is unstable, similarly for the non trivial state our lambda becomes minus 2 mu lambda becomes minus 2 mu, but in this case this mu is negative minus 2 mu is positive.

So, minus 2 mu is positive as lambda is positive that is the eigenvalue is positive. So, these branches are unstable. So, this is unstable branch. So, here that means, these stable and unstable branch.

So, this one non trivial stable and two non trivial unstable leads to, these unstable branches after the bifurcation. So, already we know the bifurcation point at bifurcation point either it change the response in this qualitatively or quantitatively. So, in this case these three branch merge to have a single branch.

So, that is a number. So, the number is changing. So, that is why it is a bifurcation point also the stability type of stability is also changing. So, here as the resulting response is unstable previously we have a stable branch and two unstable branch. Now this becomes completely unstable.

So, this is known as subcritical pitchfork bifurcation which is a dangerous bifurcation because after this thing as there is no solution available the system response may jump to infinite and the system may tends to an unstable region or it may be; it may be the system may be the system may get damaged.

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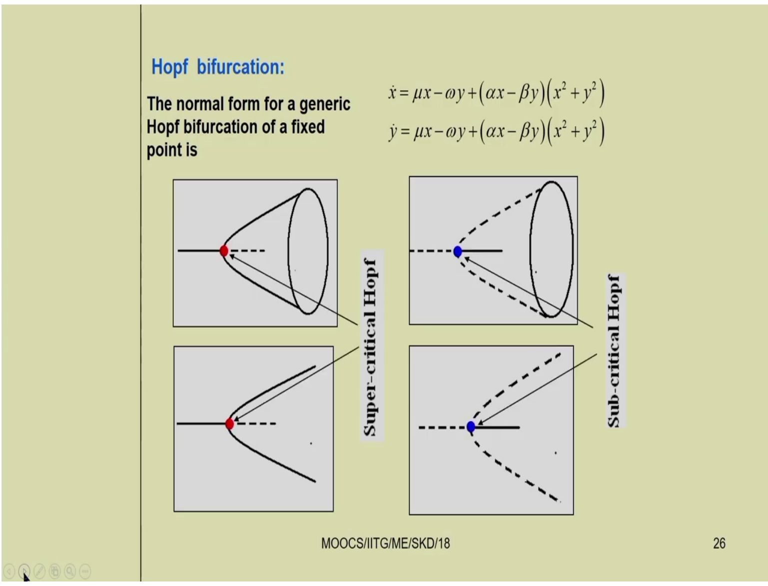

So, let us see the hopf bifurcation this is the dynamic bifurcation.

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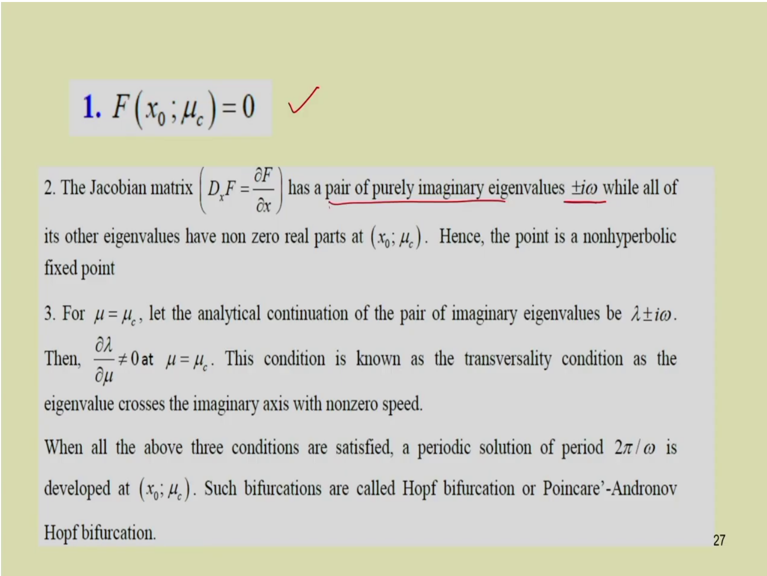

So, in case of the dynamic bifurcation. So, we have this F x mu x 0 mu c must be equal to 0 this is the first condition and the second condition is the Jacobian matrix D x F that is del F by del x has a pair of purely imaginary eigenvalues. So, it must you see the real part is 0, but it must have a pair of imaginary eigenvalues that is plus minus i omega.

So, it will have plus minus i omega while all of its other eigenvalues have nonzero real parts at x 0 mu c. Hence the point is a non hyperbolic fixed point. The third point is for mu equal to mu c let the analytical continuation of the pair of imaginary eigenvalues be lambda plus minus i omega, then del lambda by del mu not equal to 0 at mu equal to mu c. So, this condition is known as the transversality condition as the eigenvalues crosses the imaginary axis with nonzero speed.

As they are crossing the imaginary axis with the nonzero speed, so it will give rise to periodic response. So, when all the above three conditions are satisfied a periodic solution of two pi by omega is developed at x 0 mu mu c such bifurcations are called hopf bifurcation or Poincare Andronov Hopf bifurcation. So, this is hopf bifurcation. So, in case of hopf vibration, so from the fixed point response we are getting the periodic response.

So, if from a stable fixed point we are getting stable periodic response, then it will be known as super critical hopf bifurcation and in case of subcritical hopf bifurcation from a unstable fixed point.

So, we are getting a stable fixed point along with a periodic response we can have supercritical or subcritical hopf bifurcation points. So, the generic form of the hopf bifurcation can be written in this way x dot equal to mu x minus omega y plus alpha x minus beta y into x square plus y square y dot equal to mu x minus omega y plus alpha x minus beta y into x square plus y square.

Now, by making this thing equal to 0. So, we can get the equilibrium conditions or equilibrium points and or the fixed points and finding the eigenvalue. So, we can see how by changing the system parameter mu how the eigenvalue are varying. So, we can see at the critical point. So, it is crossing the imaginary axis with nonzero i with nonzero imaginary parts.

So, if they are crossing the imaginary axis with nonzero imaginary or a nonzero imaginary parts, then we are going to get the hopf bifurcation. So, we will see different examples later how this 3D or all these types of bifurcations are occurring in real systems.

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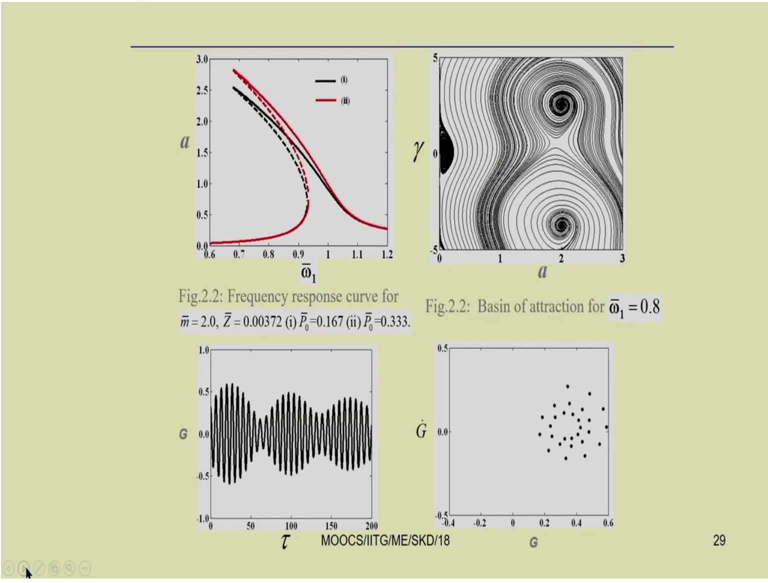

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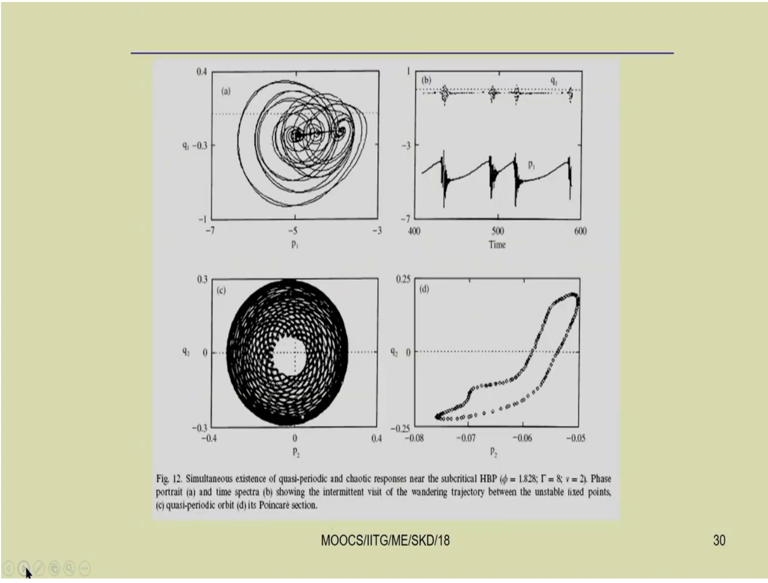

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So, these are some of the examples also you can take. So, where it leads to chaotic response. So, you can see some of the experiments also can be conducted to study all these system for example, let us take a beam. So, this beam can be vibrated to see the response.

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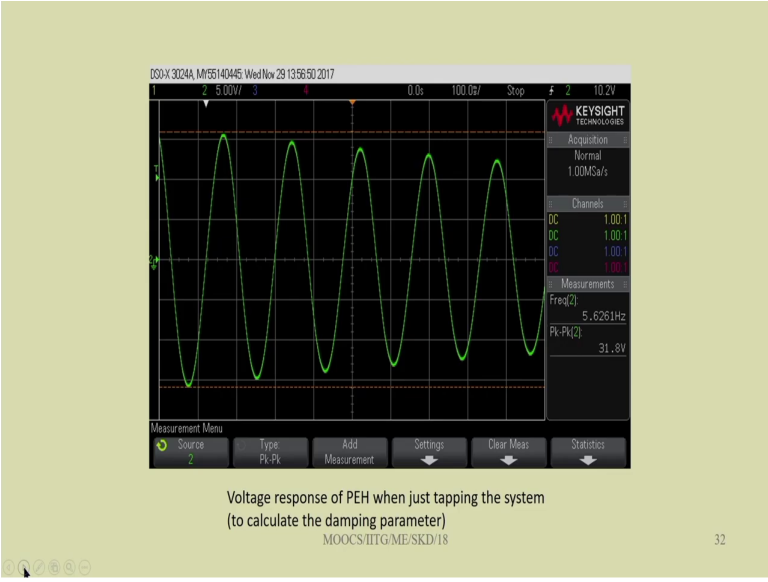

So, some of the response it is. So, by putting this accelerometer or by putting this oscilloscope you can also check the vibration of the system.

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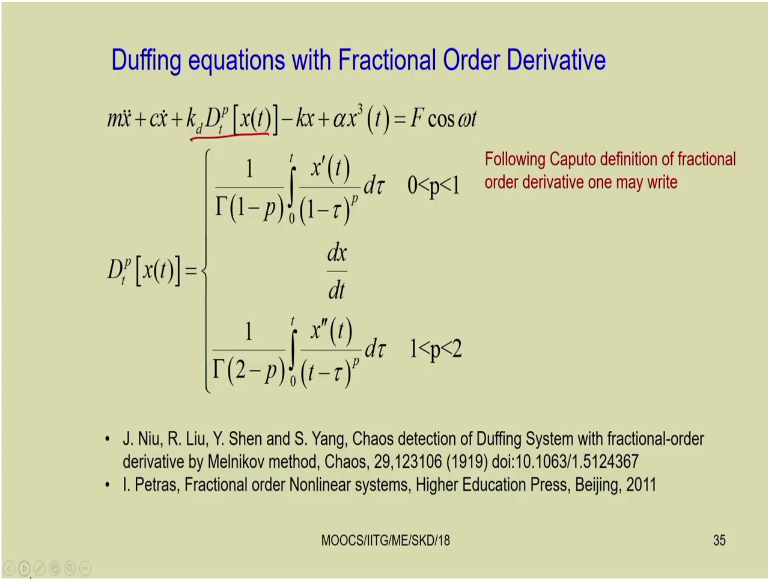

In addition to duffing equation what you have seen before. So, we can have different other type of duffing equation. So, here is one duffing equation duffing equation with the fractional order derivative. So, here these additional term you can see. So, that is known as the fractional order. So, you can study many different type of duffing equations and their response and which will lead to many different type of bifurcation points also.

So, with this thing we will close this class today and we will study the periodic response which generally occur in case of a parametrically excited system. So, we have finished the fixed, we have finished the force vibration or direct excitation system directly excited systems. Next class we will see the systems which are parametrically excited.

Thank you very much.