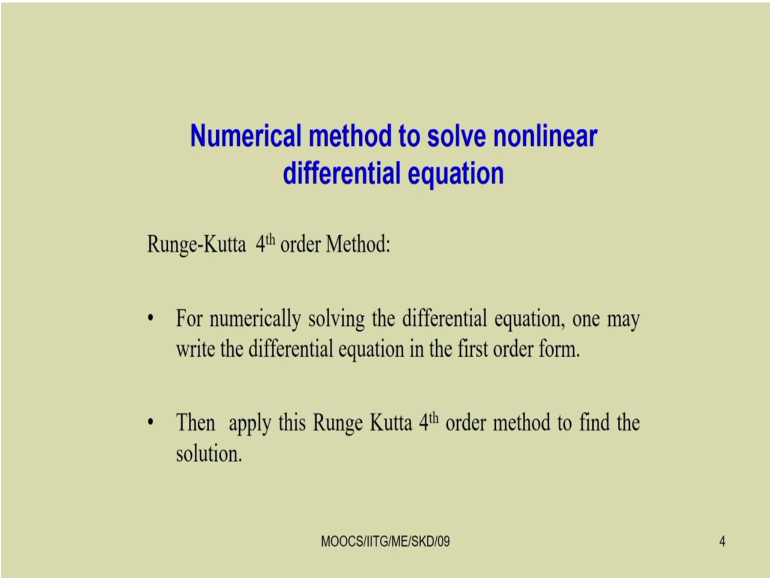
Nonlinear Vibration

Prof. Santosha Kumar Dwivedy
Department of Mechanical Engineering
Indian Institute of Technology, Guwahati

Lecture - 09
Lindstd-Poincare' method

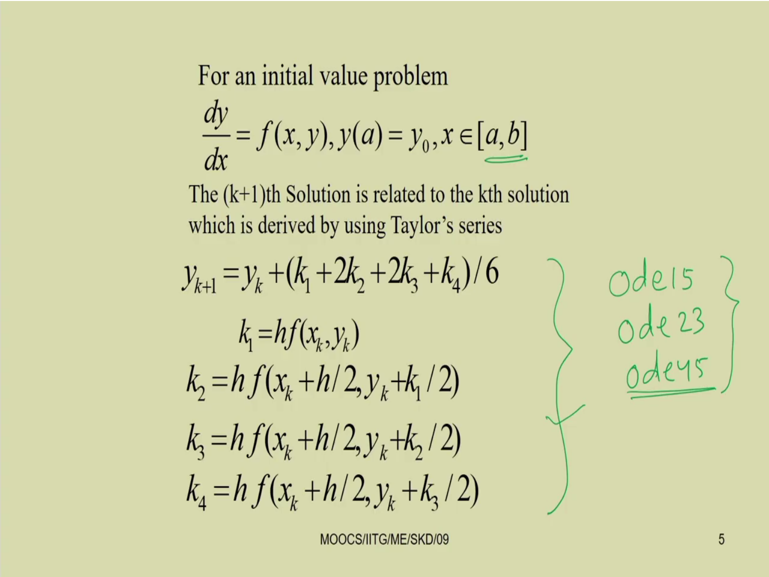
So, welcome to today's class of Non-Linear Vibration. So, in this module, we are studying how to solve this non-linear differential equations. So, initially we have gone for the qualitative analysis.

(Refer Slide Time: 00:41)



And then I told you regarding these numerical methods, so where we can use this Runge Kutta method.

(Refer Slide Time: 00:51)

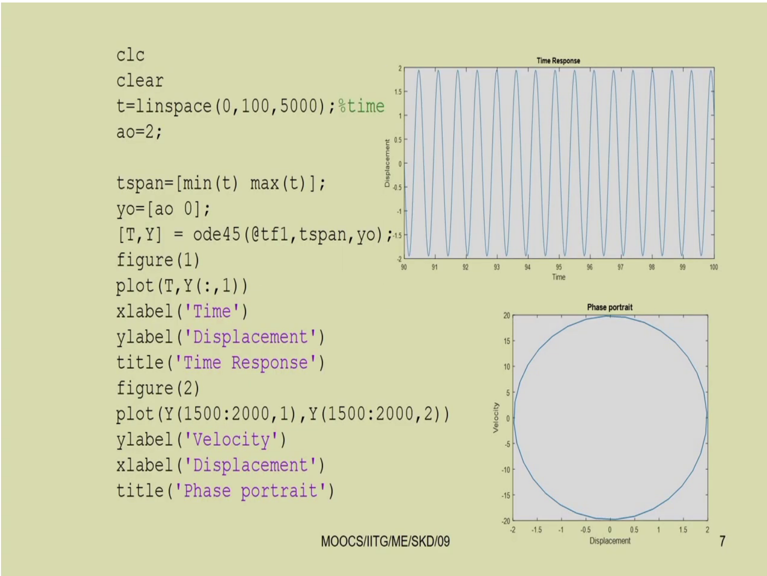


So, this 4th order Runge Kutta method can be conveniently used by writing or solving this first order differential equation. So, if you have second order differential equation, which is generally written for the dynamical systems. So, you can first convert those second order differential equation to a set of two first order differential equation. So, after getting the differential, first order differential equation; so you can use this procedure to write down your own code or you can use these standard codes available in different software's.

For example, in case of MATLAB, so you have several functions available; for example, you can take this ode, you can find ode 15, ode 23, ode 23, then ode 45. So, the purpose of all these functions are different; so you can go through the purpose of all these functions, generally you can use these ode 45; this is 4th and 5th order Runge Kutta method to find the solution of a differential equation.

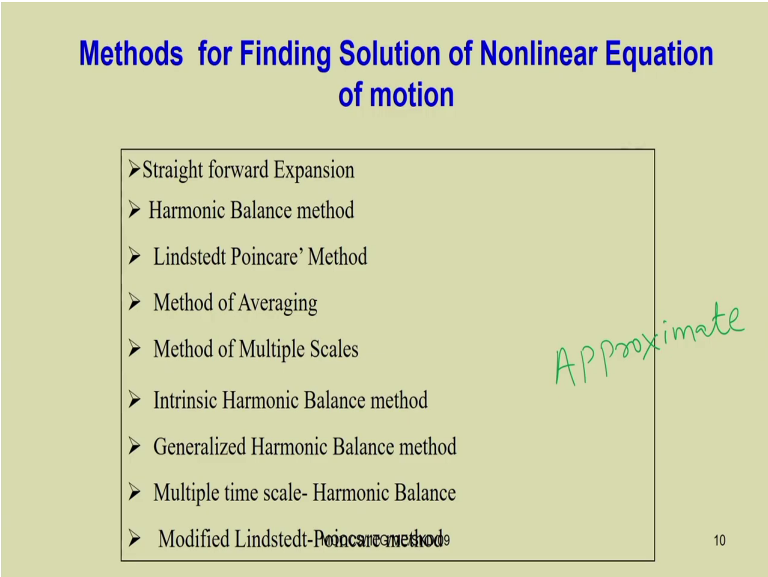
So, here this is the 4th order differential, 4th order Runge Kutta method, so here we are using four constants that is k 1, k 2, k 3, k 4. And by writing down these equations, so you can easily solve for a given value of initial condition. So, taking an initial condition of x 0, y 0, so you can solve this differential equation of the form d y by d x equal to f x, y by using these numerical methods. So, this numerical method is a very powerful method for solving the differential equation.

(Refer Slide Time: 02:47)



So, we have seen some examples also and you know how to plot these. So, let us skip all this part.

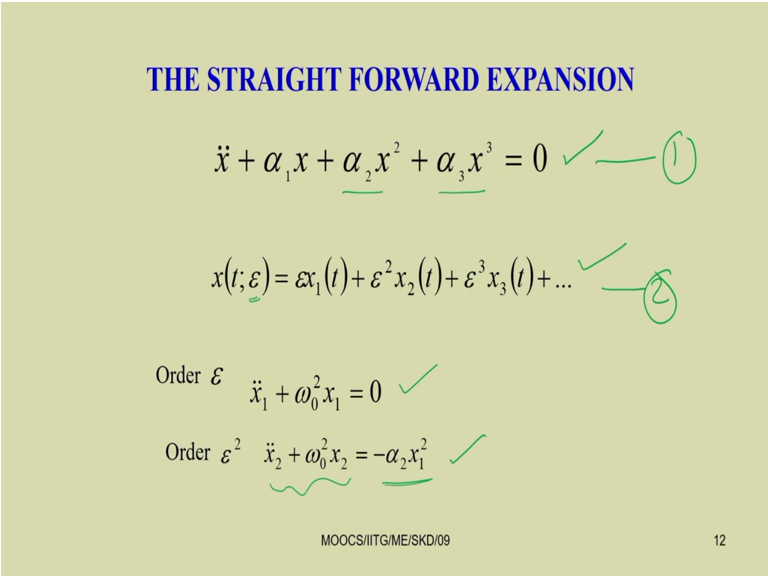
(Refer Slide Time: 02:52)



And here also I told you differential some other methods, perturbation methods or some approximate methods. So, you can use some approximate methods to find the solution. So, these are some of the approximate methods actually to finding the solution. So, you can do the straight forward expansion; then these harmonic balance methods, Lindstedt-Poincare method, method of averaging, method of multiple scale.

So, in this week, we are studying these straight forward expansion method, then method of Lindstedt Poincare method, Lindstedt Poincare method. So, this is modified Lindstedt method, we will study later; so straight forward expansion, Lindstedt Poincare method and harmonic balance method. So, already we have seen the straight forward expansion; today I will tell regarding this Lindstedt Poincare method and this harmonic balance method. So, in straight forward expansion methods, you have seen by directly expanding this equation.

(Refer Slide Time: 04:07)



For example, you have taken the straight forward expansion; we are led this is the non-linear governing equation having quadratic and cubic nonlinearity. So, just now you have expanded this x t, epsilon, where epsilon is a bookkeeping parameter; so which is less than 1, very very less than 1.

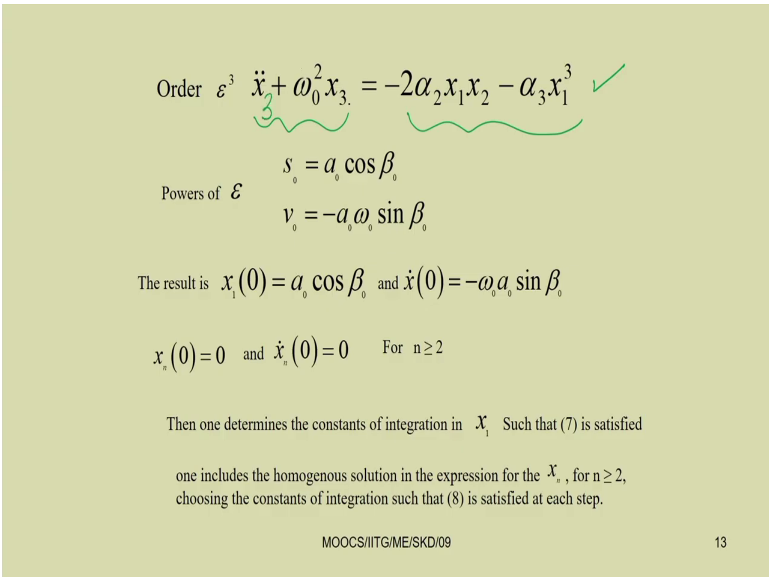
So, t epsilon you can write in this form epsilon into x 1 plus epsilon square into x 2 plus epsilon cube into x 3 and you can add the higher order terms also by substituting the 2nd equation. So, in 1st equation and ordering writing different order of epsilon; so you can get several equations.

And so, for example, you got for this case the first case the order of epsilon is x 1 double dot plus omega 0 square x 1 equal to 0; the solution of this thing you know. So, now, the 2nd equation we will write in such a way that, so in the left hand side you will have the unknown parts, and right hand side you will have the known part.

For example, already you got the solution of x 1. So, this x 1 terms will be in the right hand side. So, your order for epsilon square this is equal to x 2 double dot plus omega 0 square x 2 equal to minus alpha 2 x 1 square.

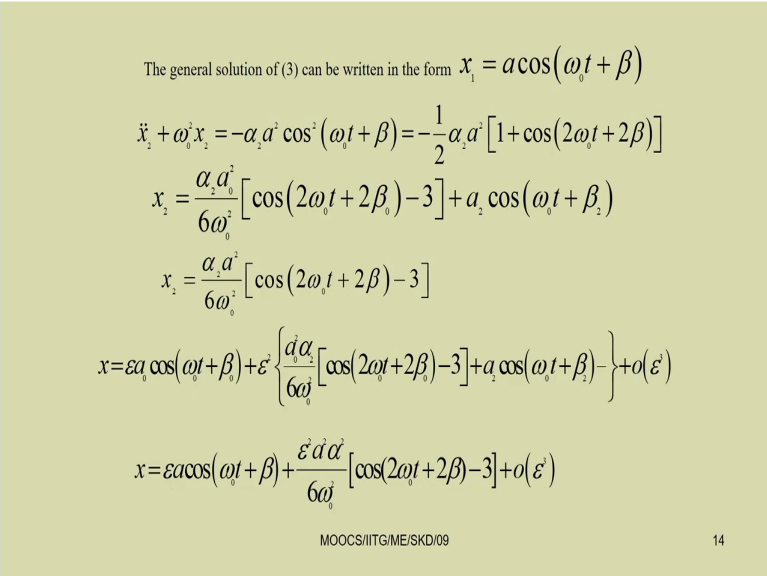
So, you are substituting the solution of x 1 in this equation. So, now, you got the solution of x 2. So, you can write the. So, the solution can be in two parts; one will be the particular integral, and other one will be the complementary function. So, in the complementary function, so you will have the constants which depend on the initial conditions.

(Refer Slide Time: 05:55)



So, similarly now you can, after getting this x 1, x 2 expression; so you can write down this 3rd equation. So, as you have taken order of epsilon cube. So, in the 3rd equation also in the left hand side unknown terms with x 3 will be there, and in the right hand side you have the terms with the known term that is x 1 and x 2. So, now by solving, so you know your x 1 and x 2. So, knowing these x 1 x 2, you can substitute in this equation to find the expression. So, this is x 3, expression for x 3.

(Refer Slide Time: 06:35)



So, you can see by putting these initial conditions. So, you can get the solution of this equation.

(Refer Slide Time: 06:40)

$$a = A + \varepsilon A_2 + \ldots\ldots, \qquad \beta = B_0 + \varepsilon B_1 + \ldots.$$

Then

$$\varepsilon a\cos(\omega_0 t+\beta) = (\varepsilon A_1 + \varepsilon^2 A_2 + \ldots.)\left[\cos(\omega_0 t+\beta_0)\cos(\varepsilon B_1+\ldots) - \sin(\omega_0 t + B_0)\sin(\varepsilon B_1 + \ldots)\right]$$

$$= \varepsilon A_1\cos(\omega_0 t + B_0) + \varepsilon^2\left[A_2\cos(\omega_0 t + B_0) - A_1 B_1 \sin(\omega_0 t + \beta_0)\right] + o(\varepsilon^3)$$

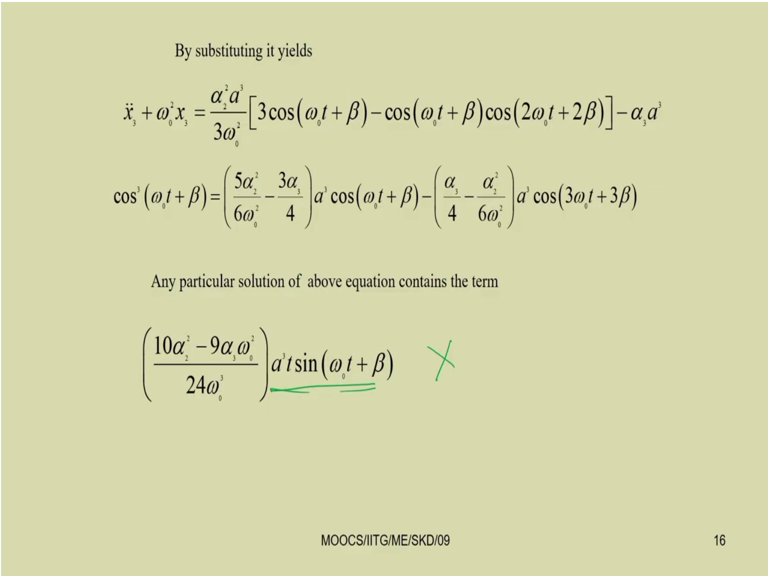
$$= \varepsilon A_1\cos(\omega_0 t + \beta_0) + \varepsilon^2\left(A_2^2 + A_1^2 B_1^2\right)^{1/2}\cos(\omega_0 t + \theta_2) + O(\varepsilon^3)$$

Where $\theta_2 = B_0 + \tan^{-1}\left(\dfrac{A_1 B_1}{A_2}\right)$ We can choose $A_1 = a_0, B_0 = \beta_0$

A_2 And B_1 Such that $\left(A_2^2 + A_1^2 B_1^2\right)^{1/2} = a_2$ and

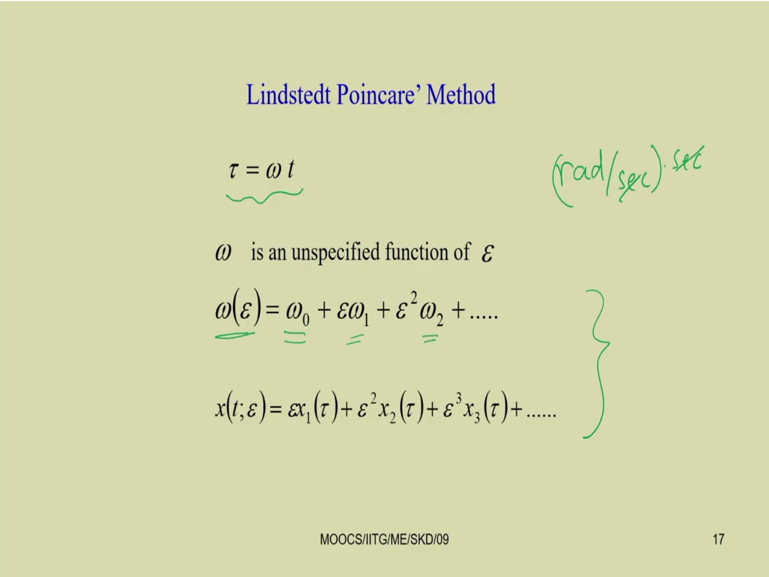
$$\beta_0 + \tan^{-1}\left(\frac{A_1 B_1}{A_2}\right) = \beta_2$$

(Refer Slide Time: 06:45)



So, here you are observing that finally, you are getting a solution, so which is a function of t. So, as t tends to infinite; so this expression that is x 3 tends to infinite also. But are the solutions are bounded, so this type of solution is not feasible. And so, that is why one has to modify this equation or modified this procedure, straight forward expansion procedure to find the solution. In actual case, these terms which involving these t are known as the singular term. So, one has to circumvent or one has to work in such a way that, this term has to be eliminated.

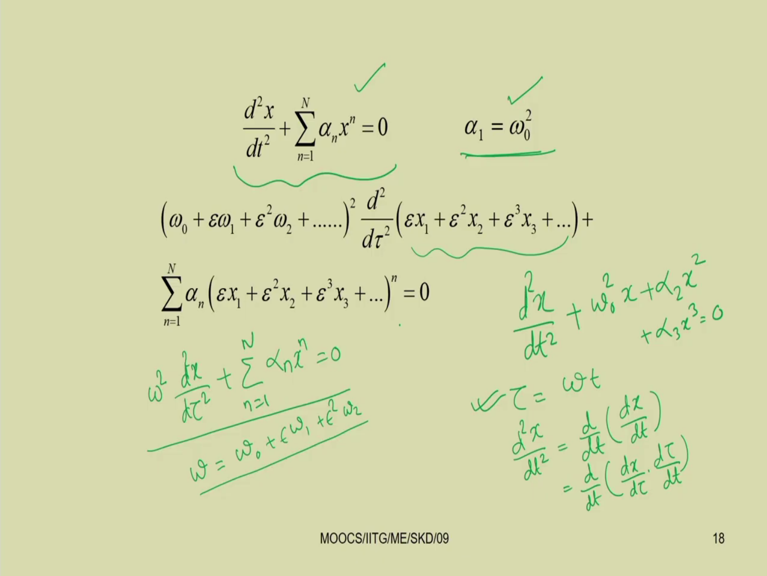
(Refer Slide Time: 07:35)



So, this modified to modify these methods, so one has used this Lindstedt Poincare method. So, in this Lindstedt Poincare method. So, the time is written by using another non dimensional time parameter that is tau. So, tau is the non dimensional time. So, you know these omega is in radian per second, omega is in radian per second, and this time in second. So, the unit of tau is in radian that is non-dimensional. So, by using a non-dimensional time tau, so we can first let us write tau equal to omega t.

So, here you just note that omega is an unspecified function of epsilon. So, now, we can write this omega as a parameter of epsilon. So, omega can be written as omega 0 plus epsilon omega 1 plus epsilon square omega 2. So, up to higher order terms also can be written. And x like previous case, we can write this x equal to epsilon x 1 plus epsilon square x 2 plus epsilon cube x 3.

(Refer Slide Time: 08:55)



So, now substituting these two in our original equation that is x double dot plus omega 0 square x plus alpha x square plus beta x cube, so in that form. Or we can write this equation in a simplified way like this, that is d square x by d t square plus alpha n x to the power n equal to 0 summation n equal to 1 to N alpha n x to the power n equal to 0, so where this alpha 1 equal to, so you can take alpha 1 equal to omega 0 square.

So, this equation if you take n equal to 3 will reduce the previous equation what we have studied, otherwise you can take higher order terms also. So, now we can by substituting this previous equation; so first we have to write. So, if we are substituting for example, this is d square x by d t square plus; for example, let me write only three terms that is omega 0 square x plus, second term will be alpha. So, this is alpha 1, then alpha 2 x square plus alpha 3 x cube equal to 0, so alpha 3 x cube equal to 0.

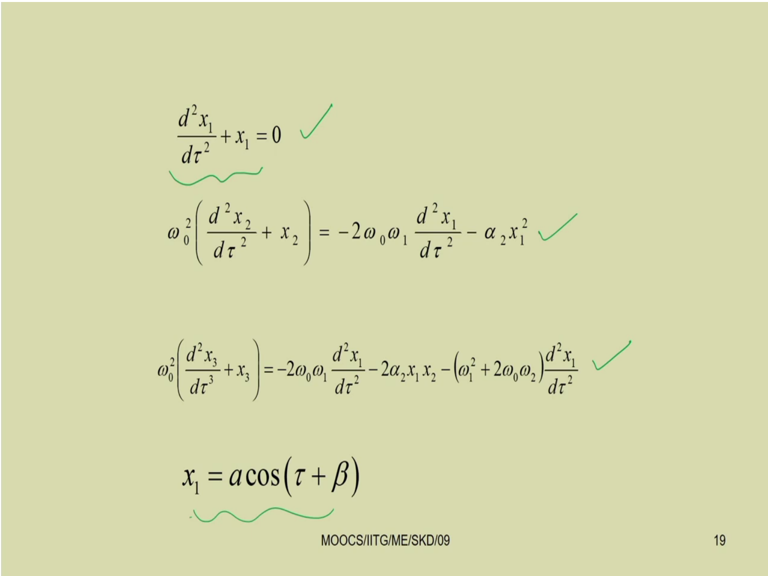
So, we are substituting tau equal to. So, tau equal to omega t, tau equal to omega t if you are substituting; then this d square x by d t square can be written as d by d t of d x by d t. So, again, so this is can be written by d by d t of d x by d t can be written as d x by d tau into d tau by; d tau by d t. So, from this equation, d tau by d t equal to omega. So, again this d by d t can be written by d by d tau into d tau by d t; so omega into omega, so that will give you omega square.

So, first term will be equal to omega square d square by d tau square x. So, this becomes omega square. So, this is omega square d square x by d tau square. So, you will get this term plus summation n equal to 1 to capital N alpha n x to the power n equal to 0. So, now we can substitute the expression for omega what you have written in the previous slide. So, omega equal to omega 0. So, omega 0 is known to us; omega 0 is nothing, but this omega 0 square equal to alpha. So, omega 0 square equal to, so omega 0 square equal to alpha 1; so omega equal to omega 0 plus epsilon omega 1 plus epsilon square omega 2.

So, higher order terms also can be written. So, in this way we can write this expression for omega. So, this becomes omega square plus omega square into. So, this is d square by d t square into, for x we are substituting this term; that is epsilon x 1 plus epsilon square x 2 plus epsilon cube x 3. So, we kept up two third term. So, this is the thing plus this n equal to 1 to N alpha n epsilon x 1 plus epsilon square x 2 plus epsilon cube x 3 plus. So, higher order terms depends on up to what term you want to take, so to the power n.

So, generally we will take these n; so to a finite number, so that we can easily expand this thing. So, if it is n equal to small that is 1 or 2 you can manually, so you can manually expand this thing; but if it is not small, so it is difficult to expand. So, in that case, you must have to use the symbolic software or symbolic method to expand these things. So, in I will show you some example. So, how we can use this Mathematica or these MATLAB to symbolically expand these parameters and then call it the coefficients.

(Refer Slide Time: 13:28)



So, this way by substituting these two here, so you can write down and collecting. So, the order of epsilon, so like in the previous straight forward expansion case, so you can collect the order of epsilon, so that you can write these d square x x 1 by d tau square plus x 1 equal to 0. So, this is the first equation. So, the solution of this is known to you, that is x 1 you can find and then. So, order of epsilon.

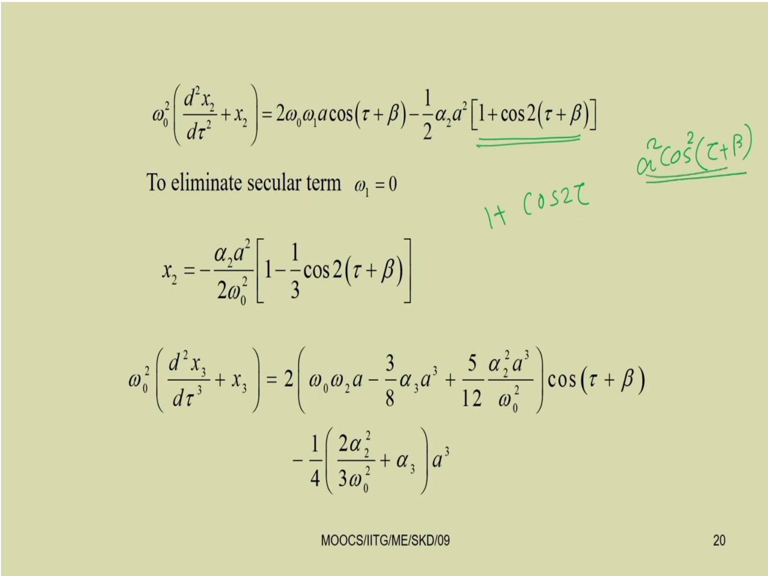
So, if you collect all the coefficient, this is one coefficient you got; the second coefficient you will get this way omega 0 square into d square x 2 by d tau square plus x 2 equal to minus 2 0 2 omega 0 omega 1 d square x 1 by d tau square minus alpha 2 x 1 square. Similarly, you can get another term that is omega 0 square into d square x 3 by d tau square plus x 3 minus equal to minus 2 omega 0 omega 1 d square x 1 by d tau square minus 2 alpha 2 x 1 x 2 minus omega 1 square plus 2 omega 0 omega 2 d square x 1 by d tau square.

So, what we are doing? So, in the left hand side, we are keeping the unknown parameter and in the right hand side, we are putting all the known parameters. So, by solving this first equation, so you can get the solution to be in this form; that is x 1 equal to a cos tau plus beta. So, here this is a second order differential equation, here your omega equal to 1.

So, that is why. So, the solution is a cos omega tau plus beta and that omega equal to 1, so this becomes tau. So, a and beta are constant; otherwise you can write it in this form also a cos tau plus b sin tau or in this form or you can write using the e term also. So, now, knowing these x 1 equal to a cos tau plus beta, where a and beta till now is unknown to us; so which can be determined at a later stage by using the initial condition.

So, substituting this x 1 in the 2nd equation, 2nd equation as you see; so this is d square x 1 by d tau square minus alpha 2 x 1 square. So, easily we can substitute it in this equation and we can write down this equation, the 2nd equation can be written.

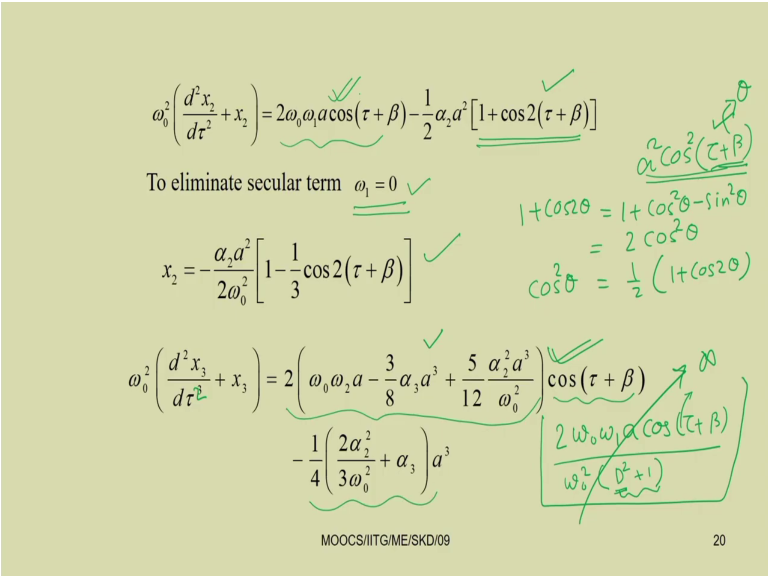
(Refer Slide Time: 15:55)



So, we just see the 2nd equation is written in this form that is equal to omega 0 square d square x 2 by d tau square plus x 2 equal to 2 omega 0 omega 1 a cos tau plus beta. And in the second part, so you have a cos square beta term. So, that thing can be written by using these cos 2 beta. So, as you know. So, these cos square 1 plus cos 2 tau, so you just see this term. So, cos a cos tau plus beta; so here we have x 1 square, so x 1 square will give you a square cos square tau plus beta.

So, a square, so you have to find how we can write a square cos square tau plus beta. So, many students used to do mistake here; so they must have to expand this thing and write that thing by using this cos 2 theta form.

(Refer Slide Time: 17:03)



So, as you know this 1 plus cos 2 theta equal to 1 plus cos square theta minus sin square theta. So, this 1 minus sin square theta equal to cos square theta, so this becomes 2 cos square theta. So, cos square theta can be written, cos square theta can be written equal to half 1 plus cos 2 theta.

So, here, so in this equation if you equate to this equation; then this cos square tau plus beta, so this tau plus beta can be written as theta tau plus beta is theta. So, this cos square theta becomes up into 1 plus cos 2 theta, so that is written in this here. So, minus half alpha 2 a square; a square a square is kept here 1 plus cos 2 theta is tau plus beta. So, this way you can expand this second term and write down this one.

So, now you can observe that in this equation omega 0 square d square x 2 by d tau square plus x 2 equal to 2 omega 0 omega 1 a cos tau plus beta minus half alpha 2 a square 1 plus cos 2 tau plus beta. So, here the solution; so you can find the solution of this equation, ok. So, now the solution of these equation, can the particular solution you can write the particular solution of this equation. So, that will be equal to 2 omega 0 omega 1 a cos tau plus beta by d square plus 1.

So, these term can be, so this term will lead to a term a secular term. So, to eliminate the secular term, as the coefficient we just see. So, here the particular integral if you want to write, so the particular integral will be 2 omega 0 omega 1 a cos tau plus beta. So, by, so this omega 0 square and so, here we will write this is equal to d square plus 1. So, here are the coefficient of this thing equal to 1. So, here in place of d square, we have to put minus 1. So, this minus 1 plus 1, so this becomes 0. So, as this is becoming 0, so this whole term stands to infinite. So, as this whole term stands to infinite, so this is known as a secular term. So, this term is known as a secular term.

So, as the solution is bounded, so these term must has to be eliminated. And so, as you know these cos term, cos tau plus beta, so it will have a maximum value of minus 1 to plus 1; so these cannot be, so this term always is not 0. Similarly, a if it is not trivial state; a is also not equal to 0 and omega 0 equal to root over alpha 1, so that is also not equal to 0.

So, to eliminate the secular term, we must have omega 1 equal to 0. So, you may note that while we have written this term omega, so this term omega. So, only omega 0 term is known to us. So, we have to determine the value of omega 1, omega 2 and by eliminating secular terms actually, we can get all those terms. So, here we have seen. So, while eliminating this term.

So, we know that this term can be eliminated only if omega 1 equal to 0. So, in this way we got the value of omega 1. So, now, this omega 0 square d 0 square x 2 by d tau square plus x 2 becomes minus half alpha 2 a square into 1 plus cos 2 tau plus beta. So, from these things, so this particular solutions can be found, which can be written in this form.

For the cos 2 tau plus beta value, so similar way you can find the particular integral. So, in place of d square, you have to substitute minus 4. So, minus 4 plus 1, so this becomes 3. So, these 3 is coming due to that one and for this 1 also, so this is a constant. So, constant by d 0 square, constant by omega square into d square plus 1. So, you can take this to the top and then expand that thing by using binomial theorem and you can get the particular solution.

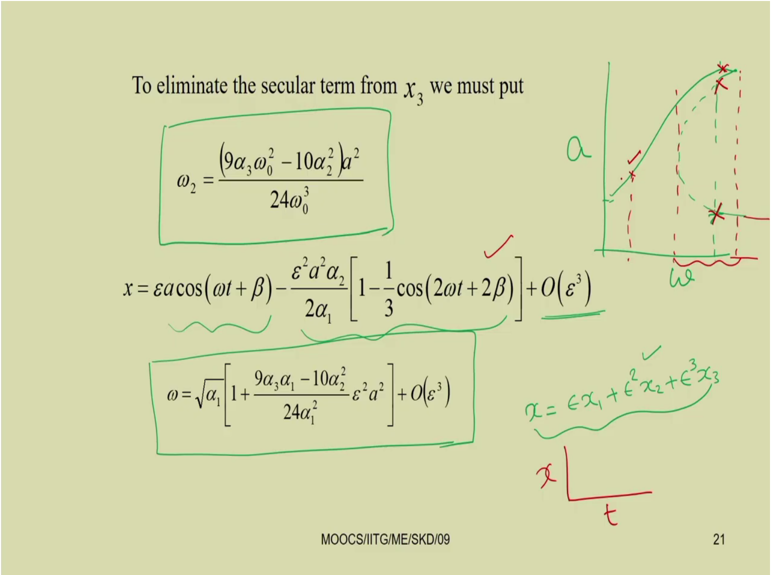
So, by doing that way, you can get the expression for x 2 equal to minus alpha 2 a square by 2 omega 0 square 1 minus, this 1 by 3 cos 2 tau plus beta. So, now, so already you got the expression for x 1; now you got the expression for x 2.

So, we can substitute this expression for x 1 and x 2 in the 3rd equation; that is in the expression for x 3. So, omega 0 square d square x 3 by d t d tau square tau square, so this is square, tau square plus x 3 equal to; now you just substitute other terms. So, you can get these term, the first term we just see; the first term is the coefficient of cos tau plus beta and then minus 1 by 4 2 alpha 2 square 3 omega 0 square alpha 3 into a cube.

So, here you just note, you just see this will lead to like in the previous case here; these term also leads to the secular term as the coefficient of tau equal to 1 here. So, doing this particular integral, so you can divide this by d 0 square plus 1 and here the d 0 you have to substitute by minus 1, so that leads to a secular term.

So, we must have to eliminate the secular term. So, to eliminate the secular term; the coefficient of cos tau plus beta must be equal to 0, so we must have to put this part equal to 0. So, already you just see, we know the expression for omega 0; omega 1 we have found to be 0 and from these things, so we can know the expression for omega 2.

(Refer Slide Time: 24:09)



So, now by substituting this part equal to 0, so we can write this omega 2 equal to 9 alpha 3 omega 0 square minus 10 alpha 2 square by 24 omega 0 cube into a square. So, you just see. So, now, we got one relation between omega 2 and a.

So, a, where a is the amplitude and omega 2. So, you have seen, so omega 2 is part of these omega. So, now the expression for x can be written in this form. So, already we know that expression we have written, x equal to; x equal to epsilon x 1 plus epsilon square x 2 plus epsilon cube x 3. So, that is the expression we have written.

So, now, keeping of two epsilon square; so we can write this thing as x equal to epsilon x 1. So, already we know x 1 expression is a cos omega t plus beta and epsilon, x 2 expression is this expression for x 2. So, now one part we have eliminated and the remaining part is this. So, the expression for this thing we can find from the particular integral of that part. So, from that thing, so we can find this is the solution of the, that differential equation. So, the higher order the term up to cubic order, cubic order terms we have neglected.

Now, this expression for omega, so we have written omega equal to omega 0 plus epsilon omega 1 plus epsilon square omega 2 as omega 1 equal to 0. And already we got the expression for omega 2, so where this omega 2 we can write here. So, then this omega can be written in this form.

So, you just see or like in case of linear system, where the frequency does not depends on the amplitude. So, here the amplitude there is a relation between these amplitude and amplitude a and this omega. So, this will actually give rise to the frequency response plot, where you can always plot this omega versus a, a versus. So, you can get different type of response, for a typical case you may get a response like this also.

So, here the response amplitude goes on increasing, then you can have. So, you can see you, you have multiple solutions. So, for example, this is one solution. So, you can see multiple solution; so you have one solution here, this is another solution, this is another solution. So, you have for example, in this particular case up to these things, so you have a single solution.

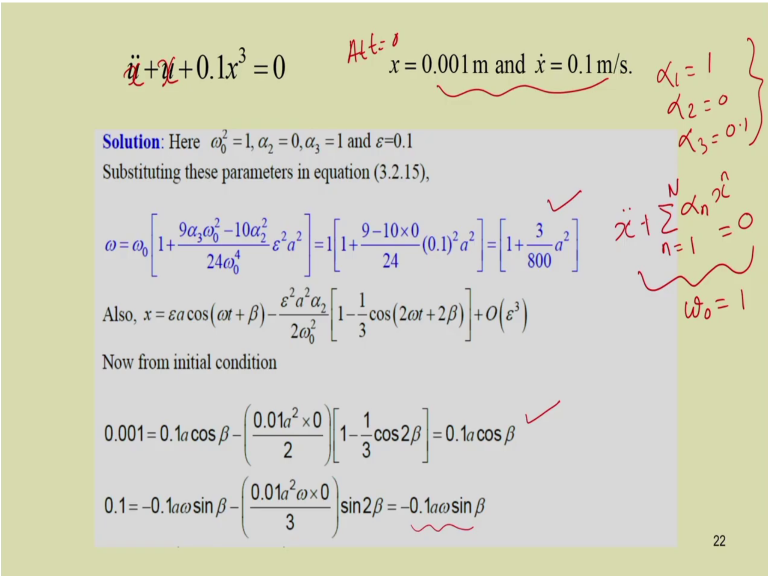
So, for one value of omega, you have one value of a and after that thing; so you can see, you can have. So, up to this point, up to this point you can have multiple; that means three value three value of, a for a particular value of omega and again after this thing, you have a single value of a for a particular value of omega.

So, in this region, so in this region, so you have multiple value of a for a particular value of omega. So, out of all these solutions or equilibrium points; so some of them are stable and some of them are unstable. So, these things will that stability analysis; when we will study the stability analysis, that time we will know more regarding this thing.

Now, you can plot. So, as an assignment, so you can plot these x versus; so you can plot these x, variation of x with time. So, taking different initial condition, so taking different initial conditions; so you can plot x versus time. So, by using this Runge Kutta method, you can plot and by using this method; that is this Lindstedt Poincare method also you can plot this thing. So, here you can find different value of a with this omega and you can plot this response.

So, for example, for a particular value of omega, so let this is the value of a. So, from this curve, you can compare by plotting this time response and this frequency response, you can compare this result. So, you just check, whether for this value of omega, you are getting this value of a or not.

(Refer Slide Time: 29:16)



So, let us take one example. For example, this let us solve this equation u double dot plus u plus 0.1 x cube equal to 0. So, taking at t equal to 0, let us at t equal to 0 or x equal to 0.001 and x dot equal to 0.1. So, these are the initial conditions given to us and now by using this Lindstedt Poincare method, so we want to solve it. So, now from this equation, so you can you know your alpha 1 equal to 1 and u. So, it should be, let us take this equation in terms of x.

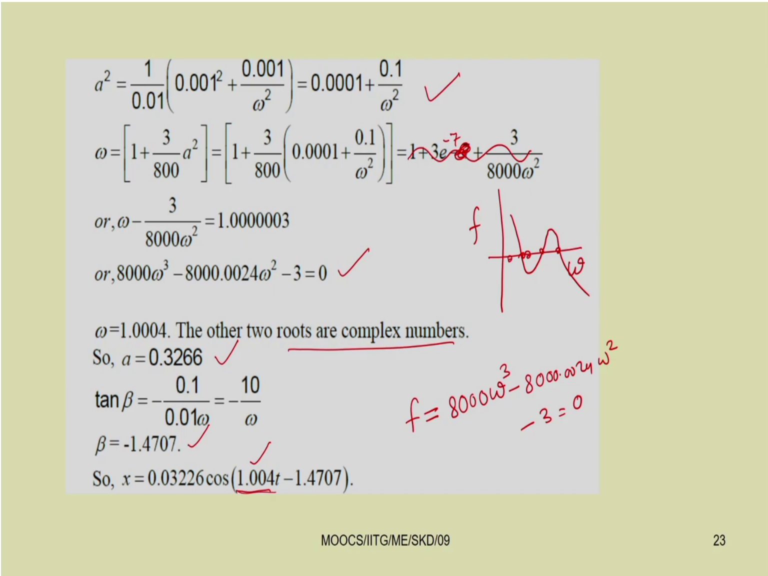
So, this equation can be written x double dot plus x, x double dot plus x plus 0.1 x cube equal to 0. So, here alpha equal to 1 and then x square as x square is missing, so your alpha 2 equal to 0. So, alpha 1 equal to 1, alpha 2 equal to 0, and alpha 3 equal to 0.1. If you compare this thing with your original equation, that is x double dot plus summation n equal to 1 to capital N alpha n x to the power n equal to 0. So, if you compare with this equation. So, if you compare with this equation, now you have seen alpha 1 equal to 1; so omega 0 equal to root over alpha 1 that is equal to 1, alpha 2 equal to 0, and alpha 3 equal to 0.1.

So, from the expression what we have already written, that is omega equal to omega 0 into 1 plus 9 alpha 3 omega 0 square minus 10 alpha square by 24 omega 0 4th into epsilon square a square. So, by taking these for example; let us take epsilon equal to 0.1, epsilon equal to 0.1. So, we can write this omega equal to in this form. So, we got this omega equal to 1 plus 3 by 800 into a square.

So, we can always plot this thing, this omega equal to 1 plus 3 by 800 a square. So, this is an assignment for you to plot this curve. So, this x can also be written x equal to epsilon a cos omega t plus beta minus epsilon a square alpha 2 by 2 omega square omega 0 square into 1 minus 1 by 3 cos 2 omega t plus 2 beta. So, here you just note. So, now, you substitute this equation; so by putting this initial condition at t equal to 0, x equal to 0.001, so 0.001 equal to. So, this expression can be written. So, this is the thing. So, you got this is equal to 0.1 a cos beta.

Similarly, now x dot differentiating this equation; you can write this x dot equal to 0.1. So, 0.1 equal to minus 0.1 a omega sin beta minus 0.01 a square omega into 0 by 3 into sin 2 beta. So, this gives rise to minus 0.1 a omega sin beta. So, 0.1 a cos beta equal to 0.001 and 0.1, 0.1 equal to minus 0.1 a omega sin beta. So, already we know the expression for a omega. So, omega equal to 1 plus 3 by 800 a square.

(Refer Slide Time: 33:29)



So, by substituting these thing here, so in this equation, in these two equations; so we can write this a square equal to 0.0001 plus 0.1 by omega square and omegas equal to we can write this omega equal to 1 plus 3 by 800 a square. So, expression for a square first we have got. So, you just see now by squaring and adding also you can do it.

So, this is cos beta; you can find the expression for a cos beta and a sin beta. So, for a cos beta equal to 0.001 by 0.1; similarly a sin beta equal to 0.1 by 0.1, 0.1 cancels. So, this is equal to 1 by omega minus 1 by omega will be equal to a sin beta. Now, squaring both the sides and adding; so you will get the expression for a square.

So, a square you have got the expression. Now, substituting that thing in this equation. So, we can get omega equal to 1 plus 3 by 800 a square. So, this equal to 1 plus 3 by 800 into 0.0001 plus 0.1 by omega square. So, this way, so this is equal to 1 plus, you can expand these things. So, e to the power minus 7; so this is e to the power minus 7, so 1 plus e to the power 1 plus 3 by this one, no problem.

So, you can avoid this part also. So, you can write the expression for omega and then or simplifying more simplification way; if you write this is a cubic order expression in omega you will get. So, you can solve this thing. So, you can solve. So, there are several methods to solve this equation.

So, for example, so you can take these Newton Raphson's method or you can take these method of. So, there are several methods you can use for finding the solution several numerical methods; you can use to find the solution of this thing, this polynomial equation ok, otherwise simply you can plot it. So, let this function f, f equal to, take f equal to. So, let me take f equal to 8000 omega cube minus 8000 0024 omega square minus 3 equal to 0.

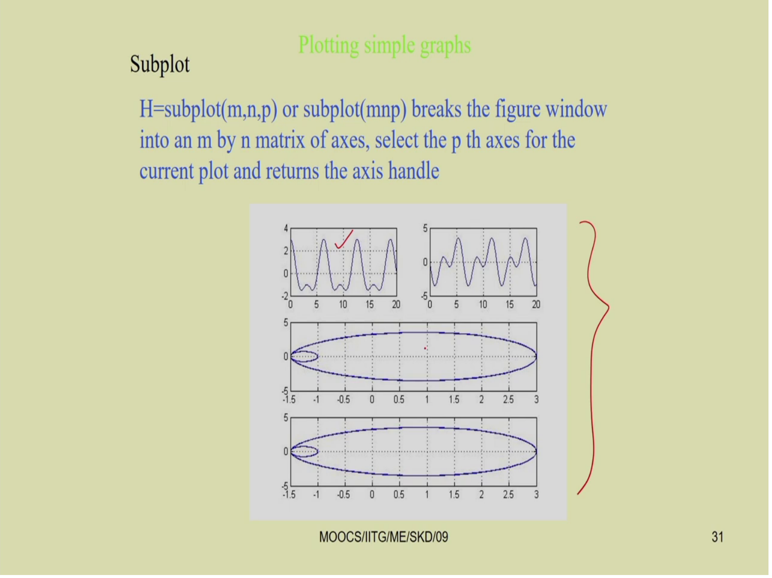
So, you can plot omega versus this f. So, that at these cubic order polynomials, so we can get three roots. So, we can see, so it will caught this zero line at three points. So, one, two and so, I have caught it at more points. So, you can caught it at three point.

So, this is one, two and three points. So, you can getting this value. So, this way you can find this value of omega. So, out of these three root, you can see that one root may be real root and other two are complex; numbers you may get complex number as it is shown here.

So, taking that value of omega, so we can find the expression for a and from that thing you can get a equal to this; then tan beta, you can find tan beta equal to tan beta equal to dividing these cos and sin, sin and cos you can get tan. So, tan beta equal to these, from where you can get beta. So, the x can be written equal to 0.03226 cos 1.004 t minus 1.4707. So, you just see the frequency here equal to 1.004; though your omega 0 equal to 1, the frequency with which it will oscillate equal to 1.004.

So, this way by using method of Lindstedt Poincare method, so you can solve these non-linear equation of motion. So, there is another method. So, modified Lindstedt Poincare method is also, so which we will study at a later stage. So, let us see these things. So, if you want to plot, multiple plot at a using a single figure, so you can use the subplot command.

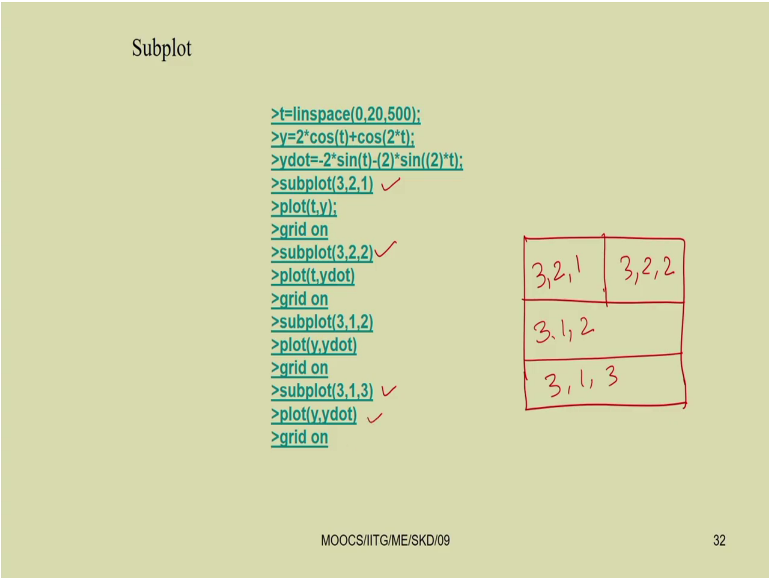
(Refer Slide Time: 38:23)



So, last class I told you regarding that thing. So, in subplot command, so you can use subplot m, n, p, where m is the number of rows, n is the number of columns and p is the position where you want to place this figure.

So, for example, this first one if you want to put. So, you have to write. So, here you just see; we have divided this thing into three row. So, you must write this is equal to subplot. So, this is three row; so three and two columns and you are putting in the highest position.

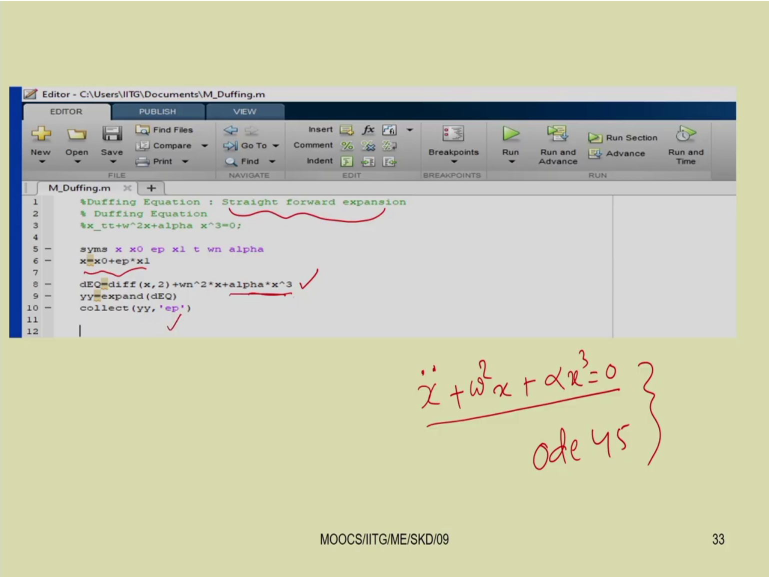
(Refer Slide Time: 39:10)



So, let us see this. So, you have to write subplot 3, 2, 1, so again subplot 3, 2, 2. So, if we are dividing the first part, you have divided into; the first part you have divided into two columns, that is why you are putting 3, 2, 2. Then for the second one, so it will be 3, 1, 2 three row; but in this second thing, so you have made only one column. So, that is why 3, 2, 3, 1, 2; then 3, 1, 3. So, you have divided, so you just note the figure what I have shown you.

So, you have divided the first row into two column, second row only one column and third row only one column. So, that is why the subplot command what you are using. So, here you are writing 3, 2, 1. So, this is 3, 3 row, 2, 1 first position. So, this is 3, 2, 2 and; but in this case you have to write subplot 3. So, you just see 3 row, 1 column and second position. So, this is 3, 1 and 3rd position. So, this way you can write and use the usual plot command in MATLAB to plot the, plot similar to this thing.

(Refer Slide Time: 40:42)

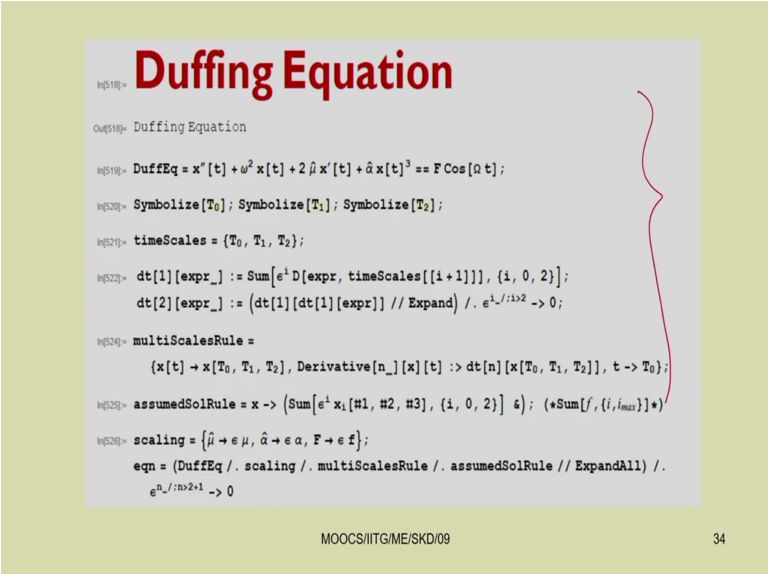


So, here I have shown the Duffing equation; for example, this equation x double dot. So, if you have this equation x double dot plus omega square x plus alpha x cube equal to 0. So, if you want to solve this equation using ode 4 5.

So, the code is written here. So, first here or if you want to use the straight forward expansion method, so here it is written straight forward expansion method. So, in straight forward expansion method, let you take two or three terms if you want to take. So, you can write. So, for example, here if I am substituting x equal to x 0 plus epsilon x 1; so here you can write this differential equation.

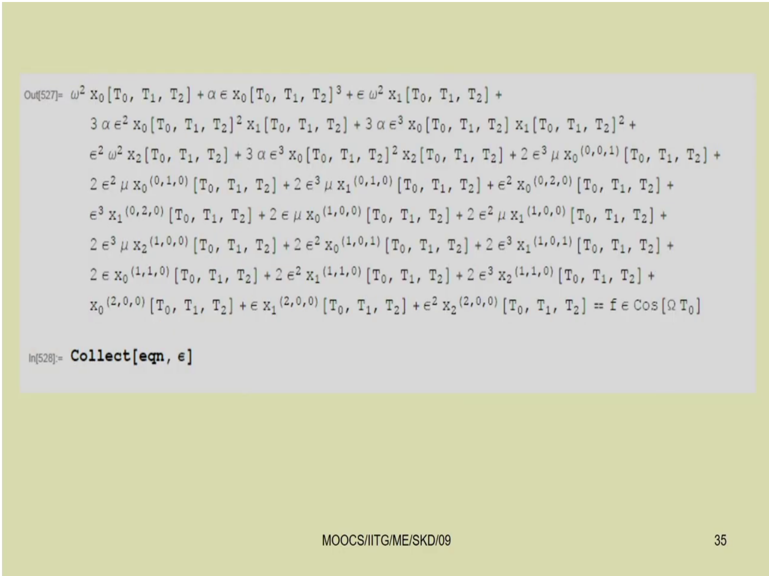
So, in this form you can write; then you can write expand this differential equation; as you have a alpha x cube, x you are substituting by x 0 plus epsilon x 1. So, you can expand that thing; then you collect the terms with the order of epsilon. So, collect. So, you can use this command collect.

(Refer Slide Time: 41:56)



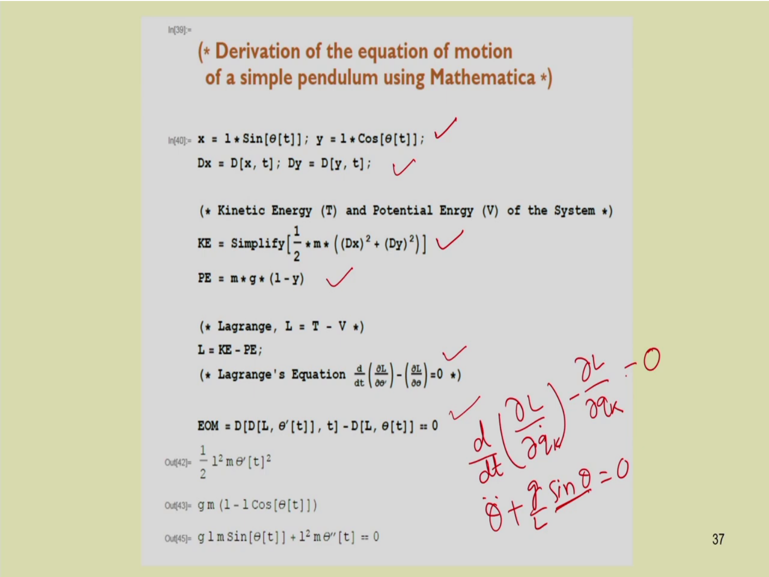
Similarly, by using Mathematica also; so this is the code written by using Mathematica. So, you can see this code, ok.

(Refer Slide Time: 42:05)



So, you can collect the coefficient and of the order of epsilon and then you can find the respective equations, ok.

(Refer Slide Time: 42:15)

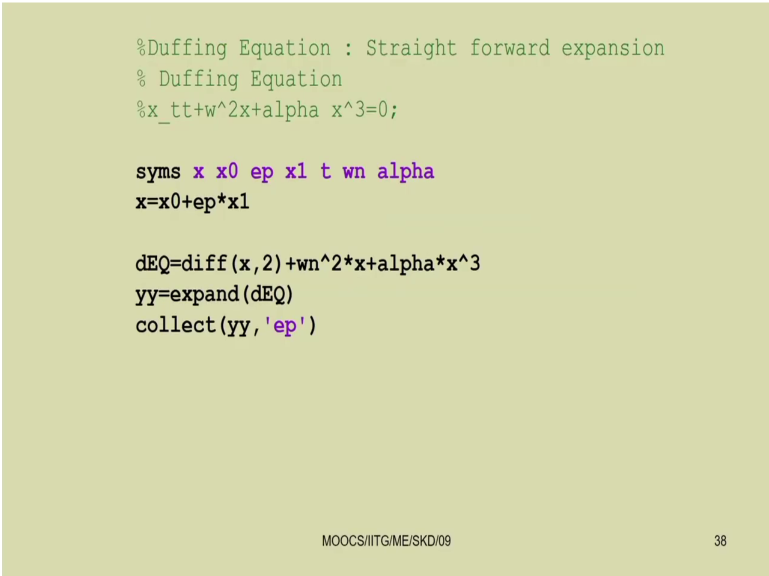


So, for a simple pendulum also similar code is written; for simple pendulum now by using. So, derivation of equation of motion; so using this Mathematica, easily you can derive. So, you have written the expression for kinetic energy, written the expression for potential energy; then Lagrangian equal to T minus V and then you can write down these equations of motion, in this way you can write down this equation of motion. So, you know the equation of motion using Lagrange principle equal to d by d t of del L by del q k dot minus del L by del q k equal to 0.

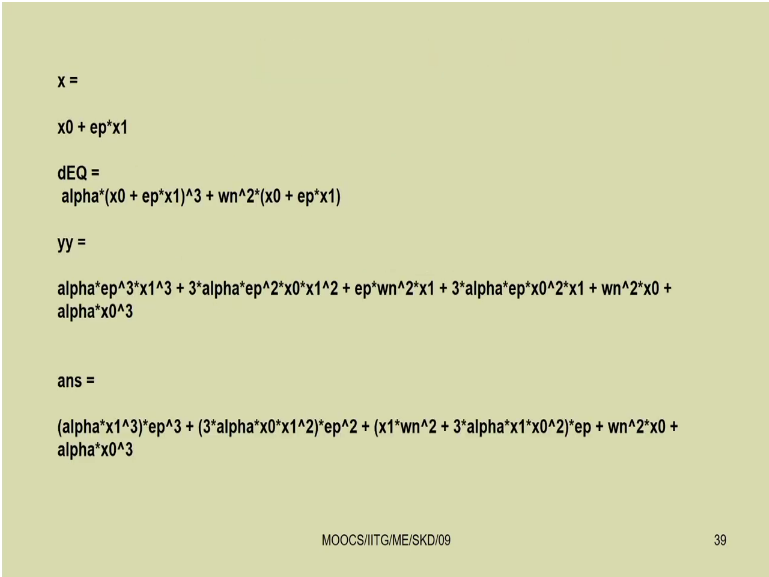
So, using these thing, so you can write. So, first you write the expression for. So, expression for x we have written, then its velocity by differentiating we got; then kinetic energy can be written, potential energy can be written, then this equation of motion can be obtained from the Lagrange equation, this is the Lagrange equation. So, by writing that thing, so you can get by arranging these thing; you can get this expression that is theta double dot plus g by L sin theta equal to 0.

So, this way by using this Mathematica, so you can write down this equation. Now, the sin theta you can expand equal to theta minus theta q by factorial 3 equal to 0. Now, you can use these straight forward expansion or this Lindstedt Poincare method to find the solution.

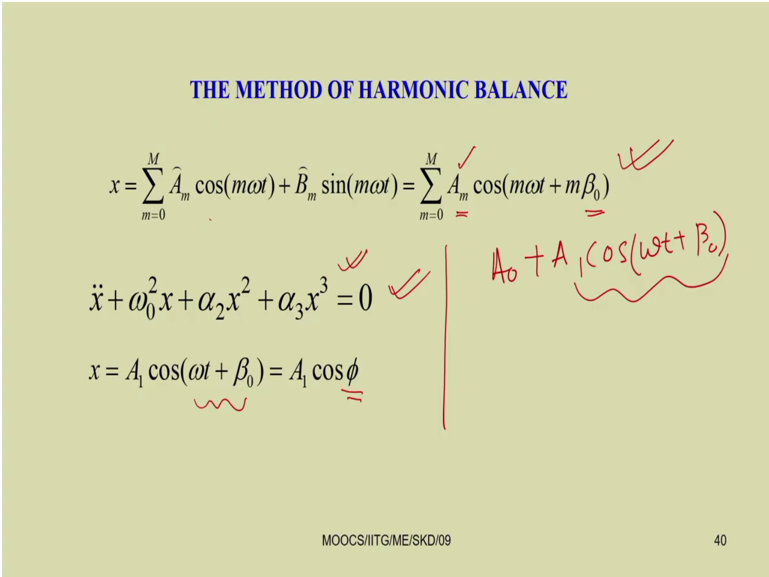
(Refer Slide Time: 44:01)



(Refer Slide Time: 44:04)



(Refer Slide Time: 44:05)



So, let us see another method that is method of harmonic balance methods. So, this is this method already you are familiar with when you are doing this method of variable separation. So, here given equation in terms of x, so always you can substitute a Fourier series in place of this x.

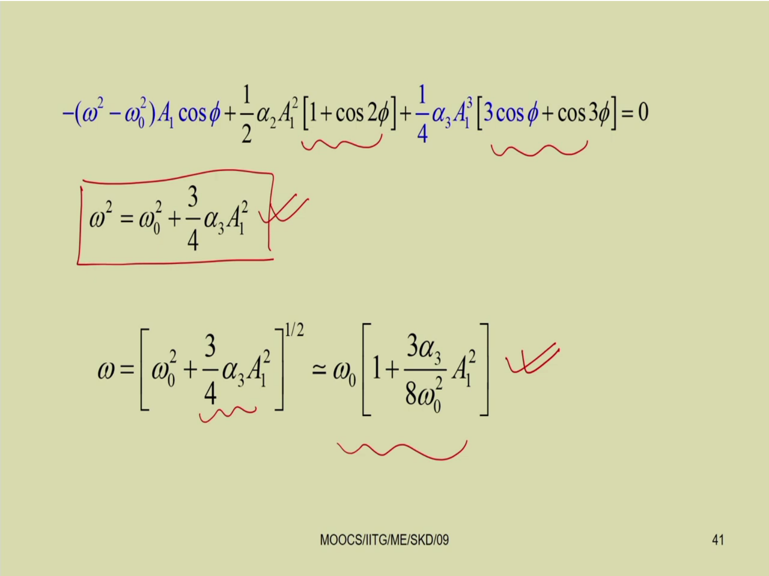
For example, this x is written equal to m equal to 0 to M A m cos m omega t plus B m sin m omega t or in the summation form single sum. So, you can write, either you can write sin and cos, cos and sin or you can use these thing either sin or cos. So, here it is written m equal to 0 to M A m cos m omega t plus m beta 0.

So, we have this beta 0 and A m are; so A m is the amplitude and beta 0 is the phase. So, this way by substituting this equation in the original equation, so let us take this original equation this one. So, if I am substituting for example, let us take me it; let us take only a single term, so that is the. So, you just see the first term if you substitute m equal to 0. So, m equal to 0, so this cos 0; so this becomes A 0 and for B 0, so B 0 equal to 0.

So, the first term becomes A 0 plus, then if I am taking the m equal to 1, this becomes. So, let me write expanding this thing, I can write. So, this becomes A 0, first term becomes A 0 A 0; then plus A 1, A 1, so cos A 1 cos for m equal to 1. So, this is omega t plus m equal to 1 beta 0. So, one can write this way.

So, if you are taking m equal to 0, so this becomes A 0; if you are taking m equal to 1, so you are getting this term A 1 cos omega t plus beta 0. So, let us take x equal to A 1 cos omega t plus beta 0 or this omega t plus beta 0 I can take equal to phi. So, in this way x equal to A 1 cos phi, by substituting x equal to A 1 cos phi in this equation.

(Refer Slide Time: 46:41)



So, it can be written in this form minus omega square minus omega 0 square A 1 cos phi plus half alpha 2 a square. So, you can get a term cos square theta or cos square phi, so that thing is expanded by using these 1 plus cos 2 phi plus 1 by 4 alpha 3 A 1 cube 3 cos phi. So, here you have a. So, here you have 3 cos phi plus cos 3 phi equal to 0. So, you have a term cos cube. So, cos cube theta you can expand in this form.

So, now we just collect the terms with cos phi. So, if you collect the term with cos phi. So, this becomes minus omega square minus omega 0 square A 1 plus 1 by 4 alpha 3 A 1 cube into 3 cos phi. So, and coefficient of cos phi, if we are substituting equal to 0; so then we can get this expression. So, this expression becomes omega square equal to omega 0 square plus 3 by 4 alpha 3 A 1 square.

So, if you just see, these expressions what you have obtained is not same as that what you have caught in case of this Lindstedt Poincare method. So, this omega square is this; now we can write this omega equal to root over this omega 0 square plus 3 by 4 alpha 3 A 1 square. So, by expanding this thing by binomial theorem, so we can write this thing omega 0; as this term is small in comparison to this, so this half will be multiplied with this. So, this omega becomes omega 0 into 1 plus 3 by 8 alpha 3 by omega 0 square A 1 square.

So, you have seen by taking a single term, so we have, we got the expression for omega; but this is not matching with the expression what we got by using this Lindstedt Poincare method.

(Refer Slide Time: 48:49)

$$x = A_0 + A_1 \cos\phi$$

$$\left[\omega_0^2 A_0 + \alpha_2 A_0^2 + \frac{1}{2}\alpha_2 A_1^2 + \alpha_3 A_0^3 + \frac{3}{2}\alpha_3 A_0 A_1^2\right]$$

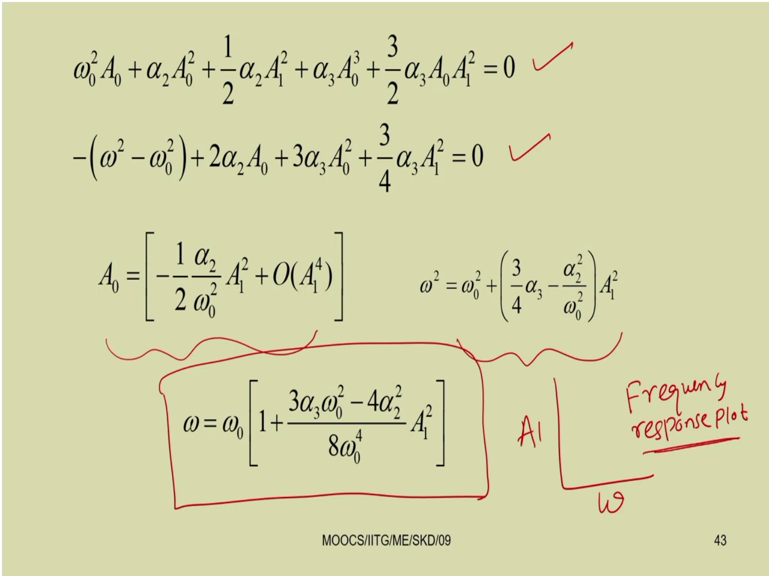
$$+\left[-\left(\omega^2 - \omega_0^2\right)A_1 + 2\alpha_2 A_0 A_1 + 3\alpha_3 A_0^2 A_1 + \frac{3}{4}\alpha_3 A_1^3\right]\cos\phi$$

$$+\left[\frac{1}{2}\alpha_2 A_1^2 + \frac{3}{2}\alpha_3 A_0 A_1^2\right]\cos 2\phi + \frac{1}{4}\alpha_3 A_1^3 \cos 3\phi = 0$$

MOOCS/IITG/ME/SKD/09 42

So, we can take now two term. So, the first term that is A 0, that is a constant term and plus A 1 cos phi. So, by substituting these two in these governing equation, so we can get these expression. So, where, so these terms are coefficient of cos phi. So, this is first term is a constant term; then the second term is coefficient of cos phi and the other terms, so some terms are coefficient of cos 2 phi and cos 3 phi also. So, as we have taken two terms here; so we can equate the constant term equal to 0 and also the coefficient of cos phi equal to 0.

(Refer Slide Time: 49:34)



So, by doing that thing, so we got this expression. So, two expression we got. So, this is the constant term equal to 0 and this is the coefficient of cos phi equal to 0. So, constant by putting this constant of A 0 equal to constant of, constant term equal to 0; we got the expression omega 0 square A 1 plus alpha 2 A 0 square plus half alpha 2 A 1 square plus alpha 3 A 0 cube plus 3 by 2 alpha 3 A 0 A 1 square equal to 0.

And the second expression by substituting cos coefficient of cos phi equal to 0; so we got these expression minus omega square minus omega 0 square plus 2 alpha 2 A 0 plus 3 alpha 3 A 0 square plus 3 by 4 alpha 3 A 1 square equal to 0.

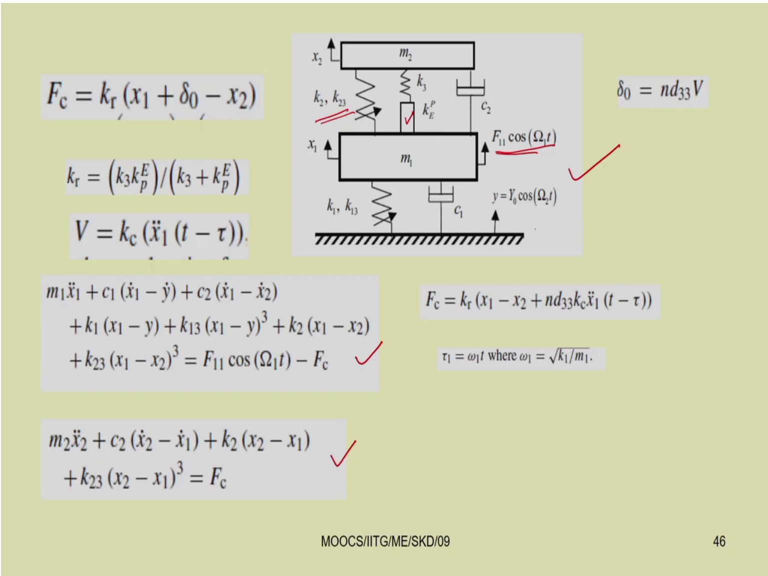
So, from this thing, so you just see; this A 0, A 1, so all these terms are unknown. So, from these expressions; so we can get these A 0 and A 1. So, here we can write this A 0 in terms of A 1. So, here by writing A 0 in terms of A 1; so we can write this way; for example, this is A

0, so this A 0 can be written in terms of A 1 this way and or, ok. So, and then, so from these things; so from these two equation, A 0 can be written in terms of A 1 square and this omega square also can be written in this form.

So, you just see this expression is somehow similar to that what you have observed in case of this Lindstedt Poincare method. So, here omega equal to omega 0 into 1 plus 3 alpha 3 omega square minus 4 alpha 2 square by 8 omega 0 4th A 1 square. So, this way by taking more number of terms actually, you can get more and more accurate expression for omega.

So, now you can plot these A 1 versus omega A 1 versus omega to find the frequency response plot. So, this will give frequency response plot. So, this as an assignment; so you can find the frequency response plot and also by using this Runge Kutta method, you can compare the solution here and you can find the solution of A 0, you can find the solution.

(Refer Slide Time: 52:29)



So, quickly let us see one more example. So, for example, this is a physical system of a vibration absorber. So, in this case, the equation of motion you can write by this expression, so two equation motion you will get; so one for x 1, and one for x 2. So, in this case, so this is a piezoelectric stack actuator and one spring is attached to these things. So, as the spring and this piezoelectric stack actuator, they are in series. And here, so the base is also excited and these mass is also subjected to a force, so here you can see two frequency excitation.

So, the base is excited y equal to Y 0 cos omega 2 t and this force is given F 1 1 cos omega t. So, by using non-linear spring, so here two non-linear springs are used, where the spring expression will be, spring force will be k 1 x 1 plus k; no. So, this is vibrating with y; so this relative displacement of the spring will be x minus y. So, it will be k 1 into x 1 minus y 1 plus k 1 3 into x 1 minus y cube. Similarly, here also you have a non-linear spring. So, one can easily write down this equation of motion, so which are non-linear equation of motion.

(Refer Slide Time: 53:56)

$$
\begin{aligned}
&\ddot{u}_1 + 2\xi_1\dot{u}_1 - 2\xi_2\dot{u}_2 + u_1 + \alpha_{13c}u_1^3 \\
&\quad - (\alpha + \alpha_r)\,u_2 - \beta u_2^3 \\
&\quad = F_1\cos\Omega\tau_1 + Y\cos(\Omega\tau_1 - \gamma) \\
&\quad + \alpha_{13c}\,(Y\cos(\Omega\tau_1 - \gamma))^3 \\
&\quad + 3\alpha_{13c}\begin{pmatrix} u_1^2 Y\cos(\Omega\tau_1 - \gamma) - \\ u_1\,(Y\cos(\Omega\tau_1 - \gamma))^2 \end{pmatrix} \\
&\quad - F_{c1}\ddot{u}_1\,(\tau_1 - \tau) \\
&\mu\ddot{u}_2 + 2\xi_2\dot{u}_2 + (\alpha + \alpha_r)\,u_2 + \beta u_2^3 \\
&\quad = F_{c1}\ddot{u}_1\,(\tau_1 - \tau) - \mu\ddot{u}_1
\end{aligned}
$$

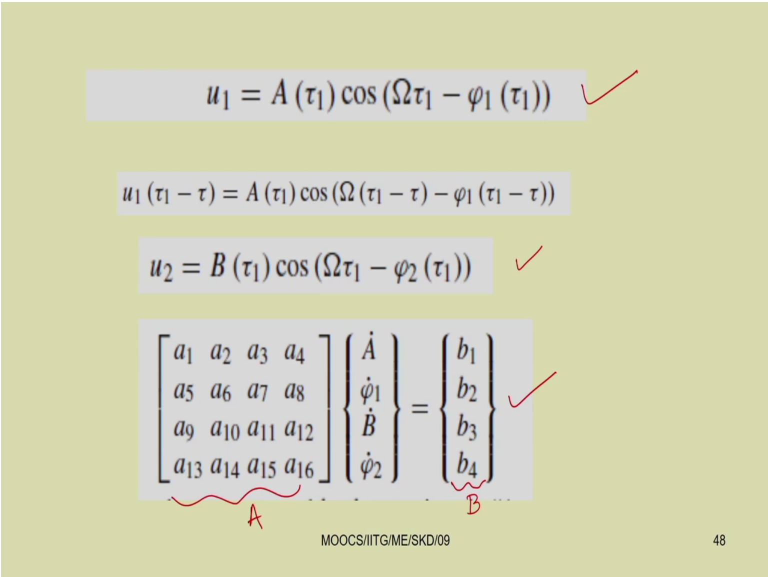
$$
\begin{aligned}
&u_1 = x_1/x_0,\ u_2 = (x_2 - x_1)/x_0, \\
&\mu = \frac{m_2}{m_1},\ \xi_1 = \frac{c_1}{2m_1\omega_1}, \\
&\xi_2 = \frac{c_2}{2m_1\omega_1},\ \alpha = \frac{k_2}{k_1}, \\
&\alpha_r = \frac{k_r}{k_1}, \\
&Y = Y_0/x_0,\ \alpha_{13} = \frac{k_{13}x_0^2}{k_1}, \\
&\beta = \frac{k_{23}x_0^2}{k_1},\ F_1 = \frac{F_{11}}{m_1\omega_1^2 x_0}, \\
&F_{c1} = \alpha_r k_c n d_{33},\ \Omega = \frac{\Omega_1}{\omega_1},
\end{aligned}
$$

$$\frac{\Omega_2}{\omega_1} = \Omega - \gamma,\ \gamma = \text{phase},\ x_0 = \text{ reference length}$$

MOOCS/IITG/ME/SKD/09 47

So, in this non-linear equation of motion; so you can use the harmonic balance method, you can use this harmonic balance method.

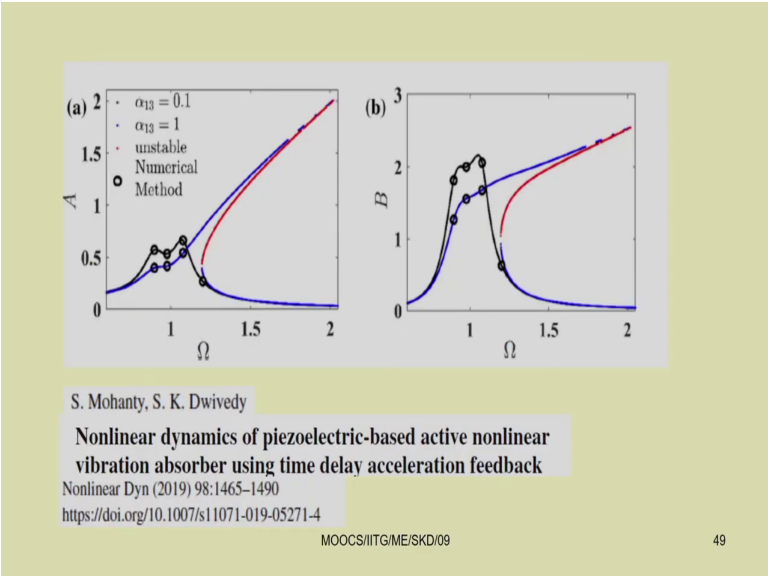
(Refer Slide Time: 54:03)



So, for example, so you can use this way this harmonic balance method; u 1 equal to A tau 1 cos omega tau 1 minus psi 1 tau 1 and u 2 equal to b tau 1 cos omega 1 tau 1 minus psi 2 tau 1. And then by collecting the coefficient of cos, cos and sin terms; so you can get these equations.

So, here you can get these A 1 dot or A 1 dot, psi 1 dot, B dot, and psi 2 dot will be equal to. So, let this matrix is a; so it will be A inverse into this B. So, from this thing, now for steady state this A 1 dot, psi 1 dot, B dot, and psi 2 dot will be equal to 0. So, you will get a set of algebraic equation.

(Refer Slide Time: 55:03)



And by solving those algebraic equation, so you can get these A and beta expression for A and beta. So, if you plot these A versus omega or beta versus, B versus omega; so this will give you the frequency response plot. So, this is a paper published by Mohanty and Dwivedy in non-linear dynamics.

So, for details you can see this paper and this way you can apply this method of harmonic balance to solve this non-linear equation. So, next week we are going to study another three different methods; that is method of harmonic balance, method of averaging, and some new newly developed methods to solve this non-linear equation of motion.

Thank you.