Viscous Fluid Flow
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Module - 05
Steady, Two-dimensional Rectilinear Flows
Lecture - 01
Flow through Rectangular Duct

Hello everyone. So, in today's class, we will consider steady pressure driven flow of incompressible Newtonian liquid through rectangular duct. So, in this case, we will consider the velocity is function of two spatial variable and to solve this problem, we will use separation of variables method. So, we have already learnt when and how to use separation of variables method to solve a partial differential equations.

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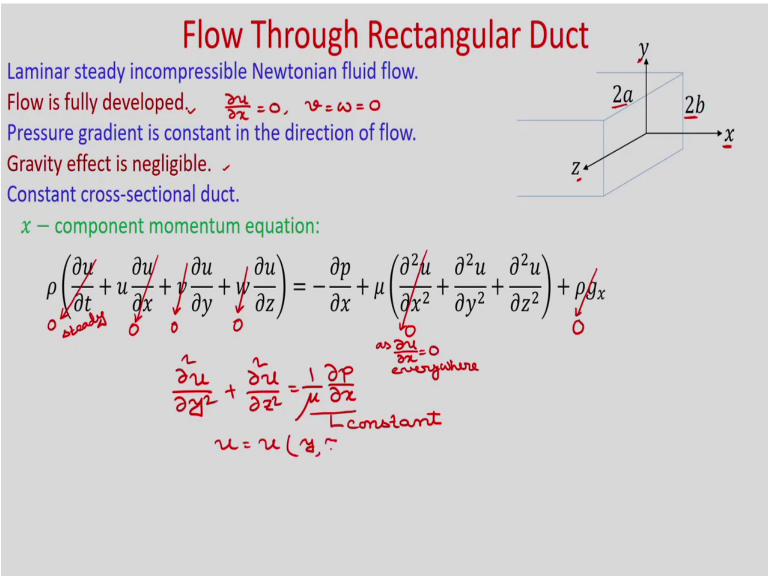

So, let us consider this rectangular duct of constant cross section, x is the axial direction, this is the y direction and this is the z direction and the length in z direction it is 2a and in y direction it is 2b. We are considering flow as fully developed that means, if x is the axial direction, then del u by del x will be 0 and the velocity v and w will be 0.

We are considering a constant pressure gradient with negligible gravity effect. So, you can see this is our x component momentum equation. So, considering these assumptions, you can simplify this equation, so obviously, it is a steady flow so, this is 0 as it is a steady flow, it is a fully developed flow so, del u by del x is 0 ok, v is 0 and w is 0. As del u by del x is 0 everywhere so, del 2 u by del x square is also 0 ok. As del u by del x is 0 everywhere ok and negligible gravity so, this is also 0.

So, you can see now you can simplify this equation as del 2 u by del y square plus del 2 u by del z square is equal to 1 by mu del p by del x and as we have constant pressure gradient so, this term is constant ok. So, you can see in this case, u is function of two spatial variable y and z ok.

So, now, how to solve this problem? So, as we told that we will use separation of variables method. Now, let us check whether we can use directly separation of variables method for this partial differential equation. So, we can see this equation is linear, but non-homogeneous because we have a constant pressure gradient term in the right-hand side so, obviously, we cannot use directly separation of variables method.

So, what we will use first? The superposition technique. So, we will split this problem into two sub problems such that one problem will be two-dimensional problem where u is function of y and z and we will be able to use separation of variables method for that particular problem.

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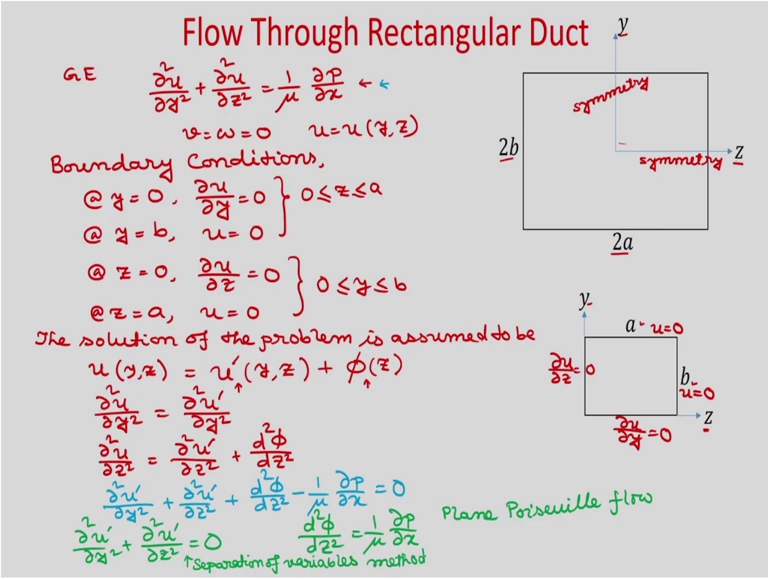

So, first let us discuss about the domain. So, you can see this is our domain of 2a by 2b cross section, this is the z direction, this is the y direction and obviously, x is the axial direction, it is a fully developed flow so, at a different cross section obviously, we will have the same velocity profile.

Now, you can see that the geometry is symmetrical and also the boundary conditions are no slip condition right. So, you have no slip boundary condition at the wall. So, obviously, considering the symmetry of the problem and the boundary condition, we can solve the one quarter of this problem so, you can see one-fourth of this problem, we can consider and this is the domain we can use because it will be symmetry about this z axis as well as y axis ok. So, this is the symmetry line and this is also symmetry.

So obviously, we can if we can solve one quarter of the problem, then we can actually have the solution of the full domain because in every half, it will be symmetrical solution. So, obviously, for this one quarter problem, you can see so, this is the z direction, this is the y direction and this is the length a and this is the length b.

Now, if you discuss about the boundary condition for this problem, then you can see we have the governing equation del 2 u by del y square plus del 2 u by del z square is equal to 1 by mu del p by del x subjected to the boundary conditions so, obviously, you can see here v and w are 0 and u is function of y and z only.

So, boundary condition will be u is equal to 0 because it is a wall, here also it will be 0 as it is wall and this is the symmetry boundary condition so, we will be using del u by del y is equal to 0 and this boundary is also symmetry so, we will use del u by del z is equal to 0. So, you can see the boundary conditions.

So, we can use at y is equal to 0 ok, we have del u by del y is equal to 0 ok for z less than equal to a greater than equal to 0. At y is equal to b, u is equal to 0 ok and at z is equal to 0, this is a symmetry line so, del u by del z is equal to 0 and z is equal to a we have no slip condition, u is equal to 0 and y less than equal to b greater than equal to 0.

Now, let us use superposition technique. So, we will split this problem into two sub problems. So, we will consider u which is function of y and z ok. So, the solution of the problem is assumed to be u as u prime which is function of y and z plus another solution phi which is function of z only.

So, now, we will split this problem such a way that this we will consider the flow inside a duct without pressure gradient and this we will consider one-dimensional problem with pressure gradient. So, now, if we substitute these here so, what we will get? Del 2 u by del y square so, you can see from here phi is function of z only so, only you will get del 2 u prime by del y square ok. And if you consider del 2 u by del z square, then we will get del 2 u prime by del z square plus d 2 phi by dz square right because phi is function of z so, we can write ordinary derivative.

So, now, if you substitute this in this equation so, what we will get? Del 2 u prime by del y square plus del 2 u prime by del z square ok plus d 2 phi by dz square minus 1 by mu del p by del x is equal to 0. So, you can see, we can now use two solutions; one problem is del 2 u prime by del y square plus del 2 u prime by del z square is equal to 0. So, you can see this is the problem without pressure gradient and another problem we will use d 2 phi by dz square is equal to 1 by mu del p by del x.

So, from here, you can see that this is the one-dimensional problem right with pressure gradient that mean it is plane Poiseuille flow ok so, it is a plane Poiseuille flow. So, one-dimensional problem with constant pressure gradient and you can see this is a two-dimensional problem without pressure gradient ok. So, it will now become linear and homogeneous equation because there is no constant term in the right-hand side. So, you will be able to solve this problem using separation of variables method.

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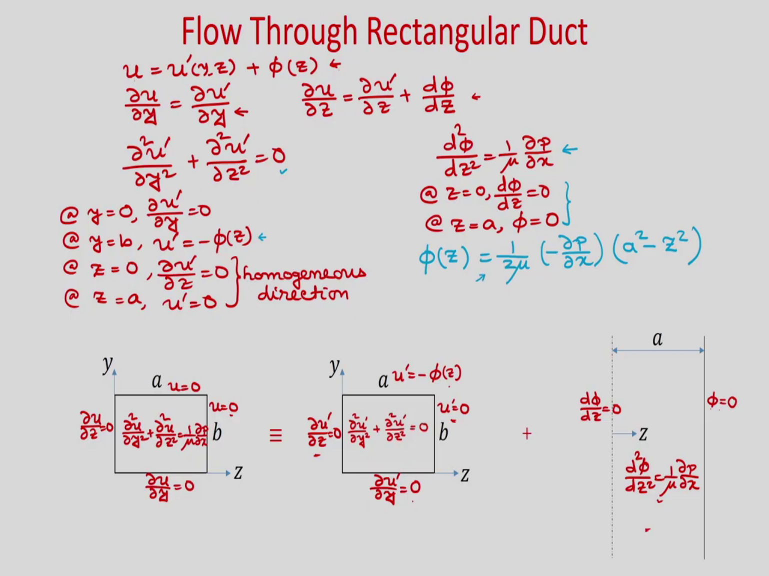

So, now, we have actually used superposition technique and divided the original problem into two sub problems; one problem is two-dimensional steady state problem without pressure gradient and another problem is one-dimensional flow in a plane channel with constant pressure gradients. So, that is plane Poiseuille flow ok.

So, now, discuss about the boundary conditions. So, we have boundary conditions. So, u is equal to u prime which is function of y and z plus phi, which is function of z only and now, we know the boundary condition in terms of u. So, now, let us find the boundary condition in terms of u prime and phi. So, you can see we can write del u by del y is equal to del u prime by del y and we can write del u by del z is equal to del u prime by del z plus d phi by dz ok.

So, now, let us discuss about the boundary conditions. So, you can see in this case, this is the original problem where we need to solve del 2 u by del y square plus del 2 u by del z square is equal to 1 by mu del p by del x. So, this is the equation we need to solve inside the domain and boundary condition was u is equal to 0; u is equal to 0, here del u by del z is equal to 0 and here del u by del y is equal to 0 because these are symmetry lines.

Now, if you consider two problems so, another problem now, we have splitted into two problem so, this is the problem del u prime by del y square plus del u prime by del z square is equal to 0 right, this we will solve and this is the problem where phi is function of z only right. So, the equation will solve d 2 phi by d z square is equal to 1 by mu del p by del x ok.

So, now, let us discuss about the boundary conditions ok. If you consider the left boundary ok so, left boundary is del u by del z is equal to 0 ok so, that means, from here you can see del u by del z is equal to 0 left-hand side so, obviously, individual terms also will be 0. So, here you can write del u prime by del z is equal to 0 ok and here, we can write d phi by dz is equal to 0 ok.

If you consider the right boundary. So, right boundary you can see that we have u is equal to 0. So, here in this equation, if u is equal to 0 so, for this problem, we can write u prime is equal to 0 and here phi is equal to 0. So, now, boundary conditions are also satisfies this equation.

Now, what about the other boundary condition? So, you can see del u by del y is equal to 0 ok. So, del u by del y is equal to 0. If you del u by del y is equal to 0, you can see from this equation ok only we can write del u prime by del y is equal to 0 right. So, this will be del u prime by del y is equal to 0. For phi, we do not need any boundary condition.

And we know now, for the top boundary. So, if top boundary, you can see that u is equal to 0 here right, in this equation, u is equal to 0 so, phi we do not need any boundary condition right so, obviously, then u prime will be minus phi right. You can see from this equation, we can write u prime is equal to u is 0 on the top wall, so it will be become minus phi. So, we need to solve this equation and find the phi and this phi, this u prime we will write the boundary condition in terms of minus phi ok.

So, can you see this problem? So, this problem obviously, you can see, this is the symmetry line flow is occurring due to the constant pressure gradient and obviously, you can see that it is kind of plane Poiseuille flow right with a boundary condition at x equal to a, phi is equal to 0 and at x equal to sorry z is equal to a, phi is equal to 0 and z is equal to 0 this is a symmetry line so, d phi by dz is equal to 0.

And this equation you can see this is linear and homogeneous equation, now we can use separation of variables method with applied boundary condition and if you see the boundary conditions, in the z direction, we have two homogeneous boundary condition because del u prime by del z is equal to 0 and u prime is equal to 0 so, z is the homogeneous direction. So, we can apply separation of variables method for the linear and homogeneous governing equation and y direction you can see one is homogeneous, one is non-homogeneous so, obviously, we can use separation of variables method.

So, if we apply the boundary conditions for the problem del 2 u prime by del y square plus del 2 u prime by del z square is equal to 0, so you can write at y is equal to 0, del u prime by del y is equal to 0 and at y is equal to b, u prime is equal to minus phi z and we have at z is equal to 0, del u prime by del z is equal to 0 and at z is equal to a, u prime is equal to 0. So, you can see this is the homogeneous direction ok.

And the for the other problem, d 2 phi by d z square is equal to 1 by mu del p by del x so, boundary conditions at z is equal to 0, d phi by d z is equal to 0 and at z is equal to a, phi is equal to 0. So, for this problem, you know the solution already we have solved. So, I will just write the final solution for this problem phi is equal to 1 by twice mu minus del p by del x a square minus z square ok.

So, this is the solution for this equation subjected to these boundary conditions. So, now, we know phi z right. So, when we will solve this equation, we can just put phi z value from this expression.

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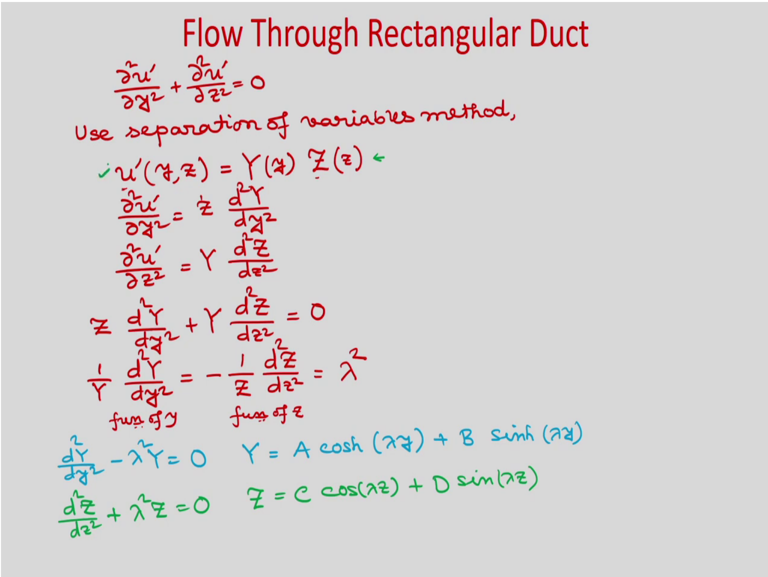

Now, let us consider the two-dimensional steady problem which is actually linear and homogeneous and we will apply separation of variables method. So, you know that this is the Laplace equation. So, del 2 u prime by del y square plus del 2 u prime by del z square ok.

So, now, use separation of variables method. So, now, what we will do? That we will seek the solution of this u prime as product of two solutions such that each solution will depend on only one spatial variable. So, in this case, we will write u prime which is function of y and z is equal to product of two solutions ok, let us say one solution is y which is function of y only and another solution z which is function of z only.

So, this using separation of variables method, we will just convert this partial differential equation to two sets of ordinary differential equation and applying the boundary condition, we will find the solution for y and z and we can find the final solution u prime as product of these two individual solutions ok.

So, now you can write del 2 u prime by del y square is equal to z d 2 y by dy square right because now it is a ordinary derivative because y is function of y only. Similarly, del 2 u prime by del z square you can write y d 2 z y dz square ok. So, now if you put it here so, you can see, we can write z d 2 y by dy square plus y d 2 z by dz square is equal to 0.

Now, we will rearrange and in the left-hand side, just we will write the terms which is function of y only ok and in the right-hand side, now we will use the terms which are function of z only. So, you can see now we have separated the variables. Left-hand side is function of y only right and this right-hand side is function of z only. So, we have separated the variables ok.

So, as you know that these left-hand side is function of y only and right-hand side function of z only and it should be equal to some constant ok and that constant we know how to choose. So, now what sign of this constant we will choose such that that in homogeneous direction, we should get a characteristic value problem that means, its solution will give a periodic solution right. In this case, we know that z direction is homogeneous direction right so, z direction is homogeneous direction. So, in that direction, we should get a characteristic value problem.

So, you can see that to make the solution of z periodic, we should choose the value of lambda square as plus ok. So, you can see from here, you can write two ordinary differential equations so, one is d 2 y by dy square minus lambda square y is equal to 0 and you know its solution so, you can see this is a finite domain so, we will write the solution of this equation in hyperbolic form so, we will write A cos hyperbolic lambda y plus B sin hyperbolic lambda y ok.

And the other problem we will get d 2 z by dz square plus lambda square z is equal to 0. So, you can see this solution obviously, we can write in C cos lambda z plus D sin lambda z ok.

So, these are periodic solution. So, now, we have found the solution of y and z. So, if we put it here so, we will be able to find the value of u prime.

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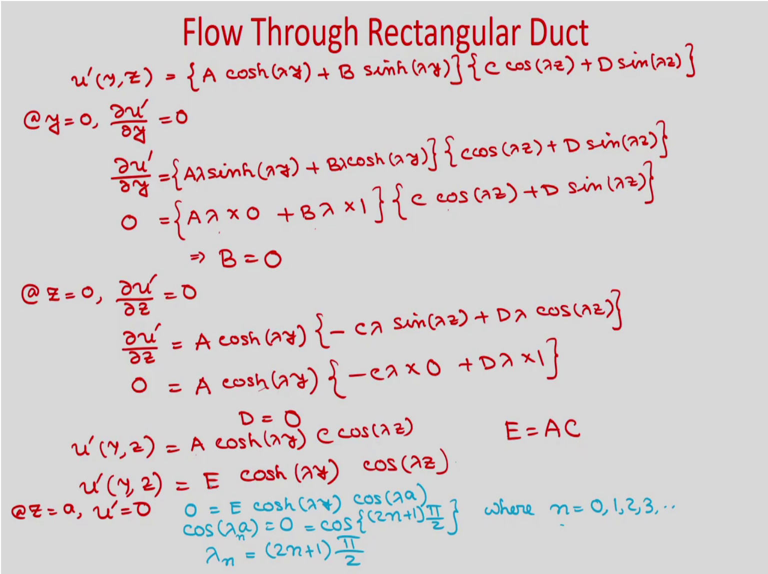

So, now let us write the final solution u prime as product of these two solutions y and z and we will apply the boundary conditions and we will find the integration constants A, B, C, D. So, we can write the velocity profile u prime y, z as product of two solutions so, first solution is A cos hyperbolic lambda y plus B sin hyperbolic lambda y ok and the solution z is C cos lambda z plus D sin lambda z ok.

So, now, apply the boundary conditions, at y is equal to 0, del u prime by del y is equal to 0 ok. So, if you find del u prime by del y, what you will get? So, you can see this will be A sin hyperbolic lambda y and one lambda will be there plus B cos hyperbolic one lambda will be there, lambda y ok and this will be as it is C cos lambda z plus D sin lambda z ok.

So, now, you can see at y is equal to 0, del u prime by del y is equal to 0 so, this will be 0 then this obviously, A lambda into 0 ok plus B lambda into 1 right; B lambda into 1 because cos hyperbolic 0 will be 1 so; so into this term ok C cos lambda z plus D sin lambda z. So, from here, you can see that first term is 0; second term is B lambda into this term. So, these terms cannot be 0, then you will not get any solution. So, to have the left-hand side is equal to 0, B must be 0. So, B is equal to 0.

Now, another boundary condition at z is equal to 0, del u prime by del z is equal to 0 ok. So, del u prime by del z what you will get? So, you can see from here B is equal to 0 so, you can write A cos hyperbolic lambda y and we will have minus C lambda sin lambda z plus D lambda cos lambda z ok. So, if we apply z is equal to 0, del u prime by del z is equal to 0 so, this is 0, A cos hyperbolic lambda y and obviously, sin 0 is 0 so, minus C lambda into 0 plus D lambda into 1 ok.

So, to have the solution, we cannot have this A cos hyperbolic lambda y is equal to 0 so, D must be 0 ok. So, B 0, D 0 so, you will get u prime y, z as A cos hyperbolic lambda y and C cos lambda z. Now, let us write another constant E as product of A and C. So, E is equal to A into C. So, you can find u prime y, z is equal to E cos hyperbolic lambda y and cos lambda z ok.

So, now, let us apply another boundary condition ok where you have at z is equal to a, u prime is equal to 0 ok. So, you can see if at z is equal to a, use prime is equal to 0 so, you will get this E cos hyperbolic lambda y cos lambda a. So, obviously, you can see E cannot be 0, then you will not get any solution, cos hyperbolic lambda y cannot be 0 so, only possible thing is that cos lambda a will be 0.

And you can see for different values of lambda, you will get cos lambda is equal to 0 so, we can write this as cos twice n plus 1 pi by 2 where n is equal to 0, 1, 2, 3 to infinity. So, you can see for different values of n, you will get this as 0 so, we can find the lambda n from here. So, lambda n will be twice n plus 1 pi by 2.

So, you can see in this equation, now for different values of n, we will get the different values of lambda n and if you get different values of lambda n, you will get different solution that means, as the governing equation is linear, we can superimpose all the solution that means, we can write the summation of all the solutions for different values of lambda n.

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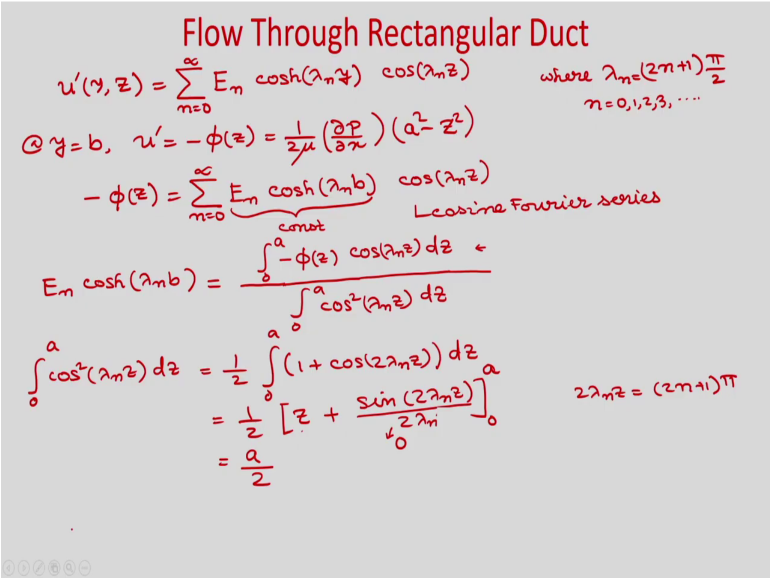

So, if you write that, then u prime y, z will be the summation of all the solutions for different values of lambda n whatever solution you will get. So, you can write n is equal to 0 to infinity E n cos hyperbolic lambda n y cos lambda n z ok where lambda n is equal to twice n plus 1 pi by 2 and n varies from 0 to infinity ok.

So, now, we have to find this constant E n and one boundary condition is left. So, that boundary condition, non-homogeneous boundary condition we will apply, and we will find the value of E n using the orthogonality condition ok.

So, another boundary condition is there at y is equal to b ok, we have u prime is equal to minus phi ok and minus phi you know that this will be just 1 by twice mu del p by del x a square minus z square ok so, if you put it here so, you will get left-hand side as minus phi z is equal to summation over n is equal to 0 to infinity E n cos hyperbolic lambda n b cos lambda n z ok. So, you can see this whole has constant right because at you have y is equal to b so, this is constant. So, this we can find, and this is a cosine Fourier series ok.

So, now, we can find the constant E n cos hyperbolic lambda n b using this orthogonality constant that we are not discussing now, we have already discussed. So, in the numerator, you will have 0 to a ok minus phi cos lambda n z dz and in the denominator, we will have integral 0 to a cos squared lambda n z dz ok.

So, now we have to evaluate these integral. So, first let us evaluate this integral in the denominator. So, 0 to a cos square lambda n z dz. So, it will be just half integral 0 to a 1 plus cos 2 lambda n z dz. So, if you integrate, it will be half z plus sin 2 lambda n z divided by 2 lambda n z limit 0 to a ok.

So, you can see that sin 2 lambda n z ok so, sin 2 lambda n z so, if you put z is equal to 0 so obviously, these two terms will become 0 so, this will become 0 and the sin 0 is 0 and if you put that 2 lambda n z is equal to twice n plus 1 into pi and for n is equal to 0, you can see it will become pi, then sin is sin 2 lambda n z will become 0. If you put n is equal to 1 so, it will become 3 pi ok so, that also will be 0 so, that means, you will get so, these term actually give 0 and for limit 0 z is equal to 0, it will become 0 so, you will get only a by 2 ok.

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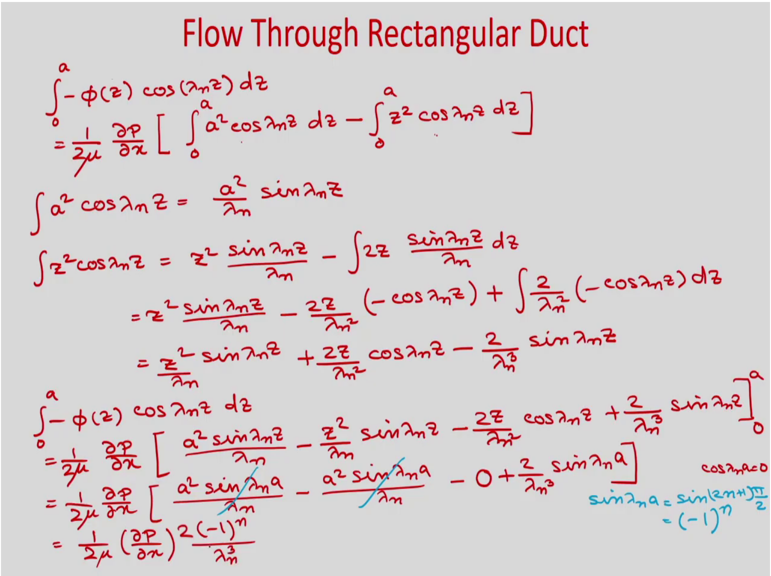

So, now let us evaluate this integral in the numerator. So, we have 0 to a minus phi z cos lambda n z dz and if you put the value of minus phi z and del p by del x is constant so, you can take it outside so, it will become 1 by twice mu del p by del x ok and now, we will have integral 0 to a a square cos lambda n z dz minus integral 0 to a z square cos lambda n z dz ok. So, putting the value of phi z just I have written this.

Now, you can see this integral is easy to find, but here you have to use this you have z square cos lambda n z right. So, you have to integrate by parts ok. So, first let us evaluate this term. So, we will first use this integral. So, we will write integral a square cos lambda n z and you know that if you integrate it, you will get a square by lambda n sin lambda n z ok.

Now, let us evaluate this integral. So, you will get integral z square cos lambda n z ok. So, now, we will use integration by parts ok. So, if you use integration by parts so, you can see that you can write z square sin lambda n z divided by lambda n minus integral so, d dz of z square so, you will get 2z, then integral of cos lambda n z so, you will get sin lambda n z by lambda n dz.

So, then you will get z square sin lambda n z by lambda n and now, again integration by parts so, it will be minus 2z by lambda n square and integral sin lambda n z is minus cos lambda n z ok and then, you will have minus minus plus integral so, dz of dz will be 1 so, 2 by lambda n square and integral sin lambda n z, you will get minus cos lambda n z dz ok.

So, now, you can write z square sin lambda n z by lambda n minus minus plus 2z by lambda n square cos lambda n z and now, you will get this as minus and integral of cos lambda n z will be just sin lambda n z by lambda n. So, it will be 2 by lambda n cube sin lambda n z ok.

So, now, we have evaluated the integral. Now, if you put it here and find the value putting the limits so, you will get integral 0 to a minus phi z cos lambda n z dz is equal to 1 by twice mu del p by del x so, this term is a square sin lambda n z by lambda n and this term will become minus z square by lambda n sin lambda n z minus 2z by lambda n square cos lambda n z plus 2 by lambda n cube sin lambda n z and limits 0 to a ok.

So, now you can see that if you put the value of z is equal to 0 ok so, sin 0, 0, this also will become 0, this will become 0 and this will also become 0. Now, let us put z is equal to a. So, in the z is equal to a if you put, you can see, you will get a square sin lambda n a by lambda n minus a square sin lambda n a by lambda n this will become you can see cos lambda n a so, cos lambda n is equal to 0 right so, it will become 0 because cos lambda n a is 0; cos lambda n a is equal to 0 and this we will get plus 2 by lambda n cube sin lambda n a ok.

So, you can see that obviously, these two terms are same so, it will get cancelled. So, you will get now, sin lambda n a is equal to sin twice n plus 1 pi by 2. So, now you can see for n is equal to 0, sin pi by 2 right. So, it is plus 1, n is equal to 1, sin 3 pi by 2 so, it is minus 1 ok and again, n is equal to 2 so, it is phi pi by 2 so, it is again plus 1 so, it will be minus 1 to the power n because n is equal to 0, then it will get plus 1, n is equal to 1 minus 1, n is equal to 2 plus 1 so, it will be minus 1 to the power n.

So, this we can write finally, as 1 by twice mu del p by del x so, you will get minus 1 to the power n divided by lambda n cube ok.

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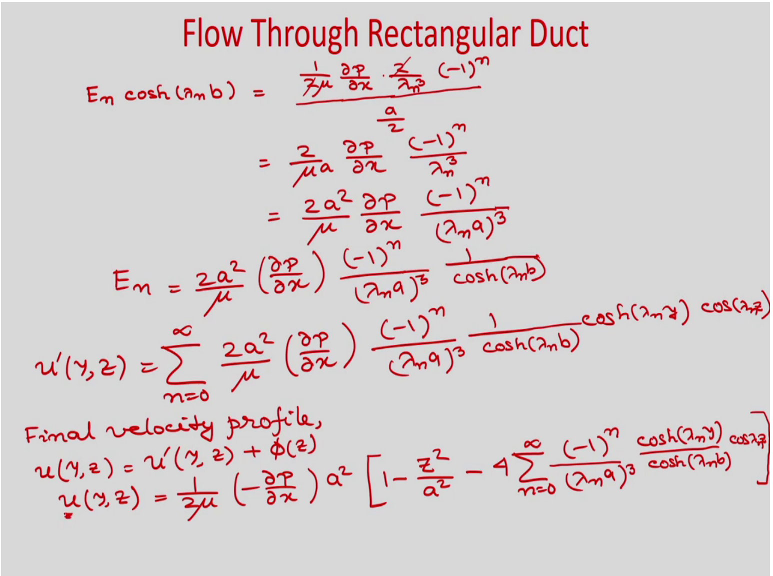

So, now, let us find the constant. So, we have E n cos hyperbolic lambda n b. So, now, this we can write so, in the numerator, we will have 1 by twice mu del p by del x ok and we will have 2 by lambda n cube minus 1 to the power n divided by in the numerator, we have a by 2. So, this will be just twice by mu a del p by del x minus 1 to the power n by lambda n cube ok.

So, now, let us multiply a in denominator and numerator. So, let us multiply a square in denominator and numerator. So, we will get 2a square by mu del p by del x and you will get minus 1 to the power n lambda n a to the power cube ok. So, you can write E n is equal to twice a square by mu del p by del x minus 1 to the power n divided by lambda n a cube 1 by cos hyperbolic lambda n b ok.

So, now, we have found the value of E n. So, now, we will be able to write the velocity u prime as u prime y, z as summation of n is equal to 0 to infinity now, E n value let us put del p by del x minus 1 to the power n by lambda n a cube 1 by cos hyperbolic lambda n b and we have cos hyperbolic lambda n y and cos lambda n z ok.

So, now, we have u prime, we know the phi value so, if you put it here so, final velocity profile you will get u y, z as u prime y, z plus phi z. So, phi z value if you take and if you write after simplification, we can write as 1 by twice mu minus del p by del x a square 1 minus z square by a square so, it will be minus 4 summation of n is equal to 0 to infinity minus 1 to the power n divided by lambda n a cube cos hyperbolic lambda n y divided by cos hyperbolic lambda n b cos lambda n z ok. So, this is the final velocity u in the constant cross sectional rectangular duct.

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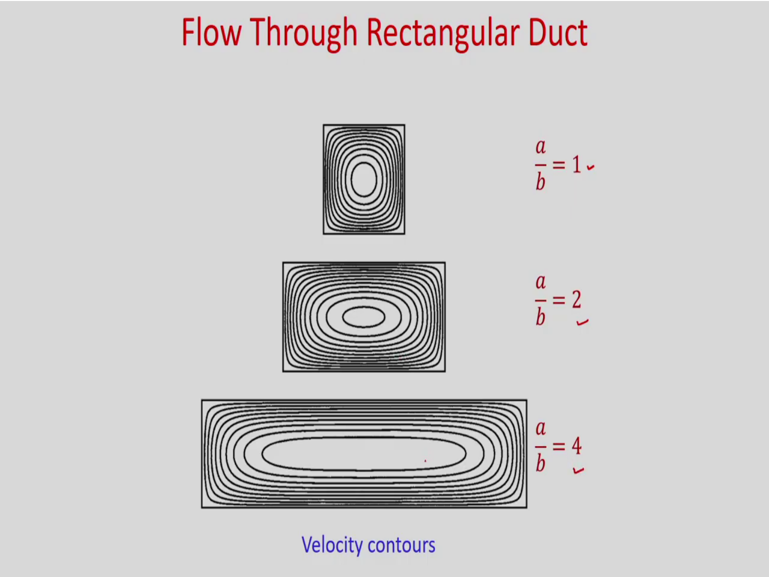

So, now, you have seen that we have derived the velocity profile. Now, if you draw the velocity contours for different a by b ratio, then you can see the velocity contours for different a by b ratio here so, this is a by b is equal to 1 that means, it is square duct, and these are the constant velocity line. So, these are known as velocity contours and this is for a by b is equal to 2 and this is for a by b is equal to 4.

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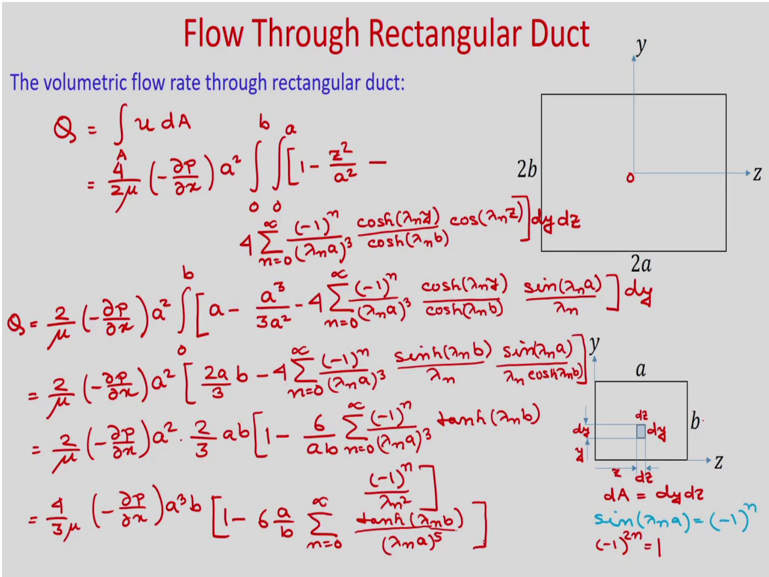

So, now, let us calculate the volumetric flow rate through rectangular duct. So, let us consider one-quarter of the domain. So, you can see we have 2a by 2b, this is rectangular duct and this is the origin and if we take one-quarter of the domain, then you will get this as the domain where the sides are a and b.

Now, let us take one elemental area of dz and dy ok so, this is the area elemental area dA is equal to dy into dz and this is actually at a z distance and from the bottom, it is at y distance. So, this is your dy and this is your dz ok.

So, now, if we want to calculate the volumetric flow rate, then you can calculate Q as area integral u dA and we know what is u for this. So, this we have calculated as 1 by twice mu minus del p by del x a square and now, we have to integrate it right. In z direction if you integrate, then it will be integral 0 to b and integral 0 to a so, then we have 1 minus z square by a square minus 4 summation of n is equal to 0 to infinity minus 1 to the power n divided by lambda n a whole cube cos hyperbolic lambda n y divided by cos hyperbolic lambda n b and cos lambda n z ok. So, now, this actually you have to integrate and you have dA is dy dz ok.

So, now if you write Q is equal to so, this is the u and dA, but you can see this we have written for one-quarter of the domain, but we have rectangular duct. So, we have to multiply this with 4 ok. So, you will write into 4 ok because we have written this expression for one-quarter of the domain. So, we have full domain so, it will be 4 into the this expression ok.

So, now you can write as 2 by mu minus del p by del x a square ok. So, integral 0 to b and this if you integrate, then what you will get? So, you can see it will be y so, y means upper limit if you put it will be a minus this is z cube by 3a square so, it will be a cube by 3a square minus 4 summation n is equal to 0 to infinity minus 1 to the power n divided by lambda n a whole cube.

So, now we have this cos hyperbolic lambda n y divided by cos hyperbolic lambda n b and cos lambda n z integral cos lambda n z dz so, what you will get? So, you will get sin lambda n z and if you put the limit so, lambda n a divided by lambda n ok.

So, now, if you again integrate it so, what you will get? 2 by mu minus del p by del x a square, this you will get a minus a by 3 so, it will be 2a by 3 right so, this if you integrate, then 2a by 3 and you will get y so, it will be b minus 4 summation n is equal to 0 to infinity minus 1 to the power n divided by lambda n a whole cube.

Now, this we will get cos hyperbolic lambda n y right so, you will get sin hyperbolic lambda n b divided by lambda n ok and you will get sin lambda n a divided by lambda n cos hyperbolic lambda n b. So, you will get 2 by mu minus del p by del x a square now, let us take outside 2 by 3 ab ok.

So, you will get 1 minus so, if you take 2 by 3 ab so, you will get 6 by ab summation n is equal to 0 to infinity minus 1 to the power n divided by lambda n a whole cube. Now, we can see sin hyperbolic lambda n b and here cos hyperbolic lambda n b so, you will get tan hyperbolic lambda n b ok.

Now, here you have sin lambda n a ok. So, what is that? So, you know sin lambda n a is equal to minus 1 to the power n right. So, if you put it here so, you will get minus 1 to the power n divided by lambda n square ok. So, hence, you will get 4 by 3 mu minus del p by del x a cube b now, we have 1 minus so, here we will write lambda n a whole square. So, we will just multiply in the numerator a square.

So, you will get 6 a by b; 6 a by b summation n is equal to 0 to infinity and here, you can see minus 1 to the power n and here minus 1 to the power n so, you will get minus 1 to the power 2n and for any value of n 0 to infinity, you can see it will return only positive 1 ok.

So, we will write just tan hyperbolic lambda n b divided by lambda n a to the power 5 because lambda n a whole square here and lambda n a whole cube so, you will get lambda n a to the power 5. So, this is the volumetric flow rate through rectangular duct.

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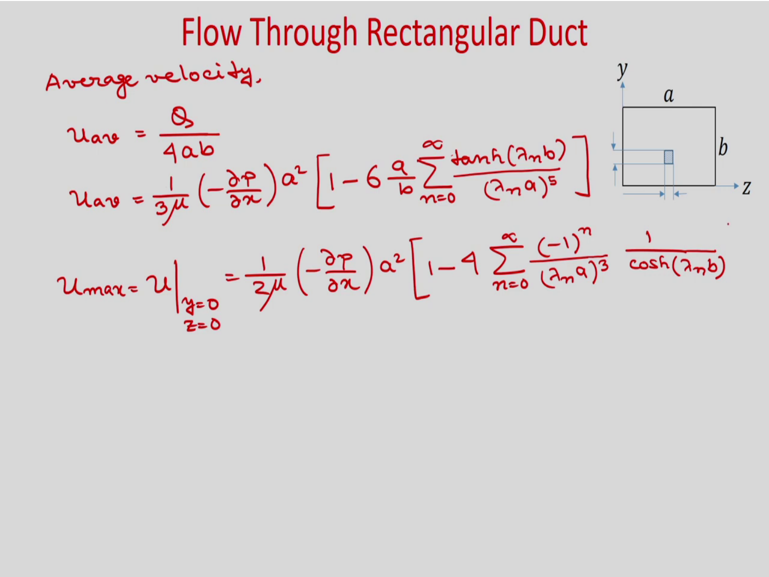

Now, if you want to calculate the average velocity, so, from this expression, you will be able to calculate. So, average velocity; average velocity so, you can calculate u average as Q by area so, what is the area? So, you can see the area is you have twice a into twice b that means, 4ab.

So, u average is now Q you have the expression so, if you divide it by 4ab, you will get 1 by 3 mu minus del p by del x and you will get a square and 1 minus 6 a by b summation n is equal to 0 to infinity tan hyperbolic lambda n b divided by lambda n a to the power 5 ok. So, this is the u average.

And if you want to find the maximum velocity so, obviously, it will occur at the origin right. So, u max will occur at u where y is equal to 0 and z is equal to 0. So, at the origin it will occur. So, if you put it, you will get 1 by twice mu minus del p by del x a square 1 minus 4 summation n is equal to 0 to infinity minus 1 to the power n divided by lambda n a whole cube and you can see cos hyperbolic 0 will be 1 ok. So, for z is equal to 0 and y is equal to 0 in both the places, cos hyperbolic 0 will be 1 so, you will get only 1 by cos hyperbolic lambda n b ok. So, this is the maximum velocity.

So, in today's class, we considered steady pressure driven flow inside a rectangular duct. In this case, we considered it has a fully developed flow and with negligible gravity effect. So, with these assumptions, we could simplify the x momentum equation as a linear non-homogeneous equation where u is function of y and z and x is the axial direction.

As the governing equation is linear and non-homogeneous, we could not use directly the separation of variables method. So, we use superposition technique and we decompose this problem into two sub problems; one is the problem where you have a steady two-dimensional flow without pressure gradient ok that velocity profile is u prime, and another problem is one-dimensional problem with constant pressure gradient which is your plane Poiseuille flow.

So, after splitting this problem, we have seen that the two-dimensional problem has become linear homogeneous problem with z direction as homogeneous direction. So, we use separation of variables method for that problem, and we found the velocity distribution u prime, and you know the solution of plane Poiseuille flow. So, that if you add, then obviously, that will give you the final velocity profile inside the rectangular duct and later, just we found the volumetric flow rate just integrating this velocity over this area.

Thank you.