Viscous Fluid Flow
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Module - 05
Steady, Two-dimensional Rectilinear Flows
Lecture - 02
Flow Through Equilateral Triangular Duct

Hello everyone. So, today we will consider Poiseuille Flow inside a Equilateral Triangular Duct with constant cross section. Obviously, we will consider a Steady Two dimensional Flow such that velocity u is function of y and z where as x is the axial direction.

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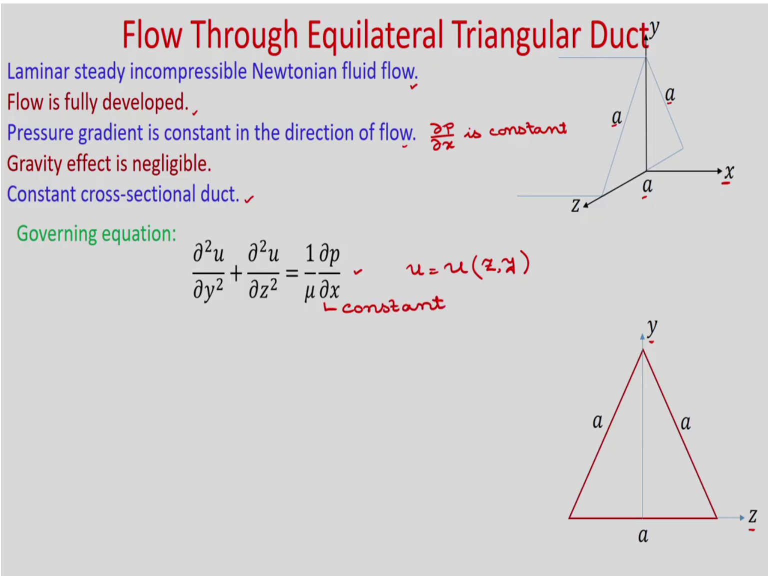

So, let us considered this equilateral triangular duct x is the axial direction, we have consider the origin here; such that x is the axial direction y is measured from this wall and z is this direction and the side of a.

So you can see that in this case if you take one cross section, then obviously it will be z direction, this is y direction and we are considering laminar steady incompressible Newtonian fluid flow and the flow is fully developed; so that del u by del x is 0 and other component of velocity b and w are 0.

Pressure gradient is constant in the direction of flow, so del p by del x is the constant. And gravity effect is neglected and obviously we have considered a uniform cross sectional duct. So, in last class you have seen that you have already derived this governing equation ok.

So, you can see u is function of z and y ok and in right hand side you can see this term is constant. So, in this case we will adapt some different methodology to find the velocity distribution inside this equilateral triangular duct. In this case first let us find the equation for each side. So, obviously whatever velocity profile we will assume it should satisfy the boundary condition; that means, at the boundary that the velocity should become 0.

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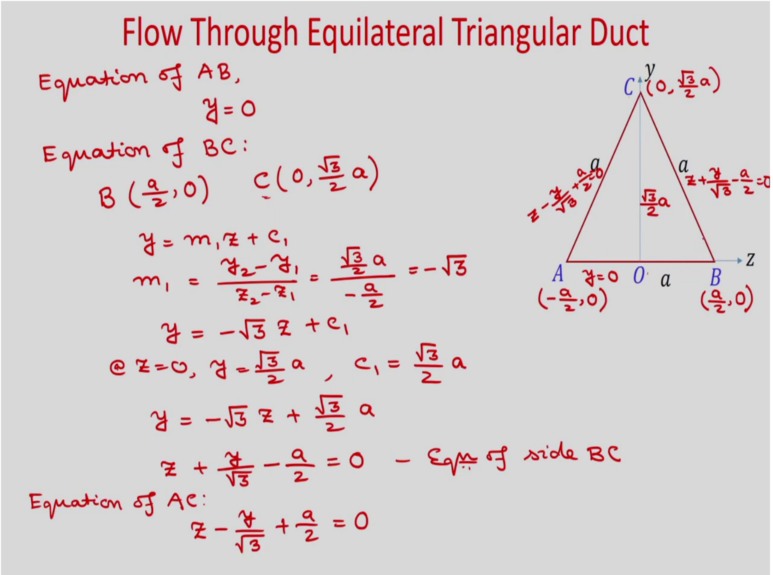

So, if you see that let us say that O is the origin ok and y is measured from this wall and this is A B and this is C. So, if you see obviously equation for this line A B will be y is equal to 0 right. So, equation of line A B ok, it will be y is equal to 0 right. Now let us find what will be the equation for line B C. So, if you consider equation of B C let us find ok.

So, you can see from this geometry you will be able to find what is the co ordinate of B and C you can see B coordinate will be z will be a by 2 because side length is a, so what obviously it will be a by 2. And you can see y is 0 right, so it will be 0. And if you considered C, so for C the coordinate will be z is equal to 0 ok and y will be the height of it right.

So, you know that for equilateral triangular duct what will be the height? The height will be root 3 by 2 times side ok. So, this will be root 3 by 2 a, so it will be root 3 by 2 a. So, now we have found the coordinate of B and C, so B is a by 2 0 and C is 0 root 3 by 2 a. So, now we know the coordinate, so we will be able to find the equation of this wall B C. So, let us assume that equation will be obviously, it is line.

So, we can assume y is equal to m 1 z plus c 1 where c 1 is constant and m 1 you can find as y 2 minus y 1 divided by z 2 minus z 1. So, from this coordinates you can see obviously, in this case y 2 if you see it will be root 3 by 2 a, and y 1 is 0 and z 2; z 2 is 0 and z 1 is a by 2, so it will be minus a by 2. So, it will be minus root 3 right.

So, from here if you can see the equation of this line will be y is equal to minus root 3 z plus c 1. Now if you write at z is equal to 0 y is equal to root 3 by 2 a, so you can find the value of C 1. So, C 1 will be root 3 by 2 a, so if you put the value here. So, you are going to get the equation of this line B C as y is equal to minus root 3 z plus root 3 by 2 a.

Now if you rearrange it you can write the equation as z plus y by root 3 minus a by 2 is equal to 0, so this is the equation of side B C ok. So, you can see this will be z plus y by root 3 minus a by 2 is equal to 0 and bottom is y is equal to 0. So, similarly now you can find the equation for the side A C, so you know the point A obviously, it will be minus a by 2 0 y is 0 and C is 0 root 3 by 2 a.

And in similar process if you find the equation you will get equation of side A C ok. In similar way if you write then it will become z minus y by root 3 plus a by 2 is equal to 0 ok, so this line equation will be z minus y by root 3 plus a by 2 is equal to 0. So, now you know the equations for these 3 side walls. So obviously, now whatever velocity we will assume inside this triangular duct it will satisfy the boundary condition at walls.

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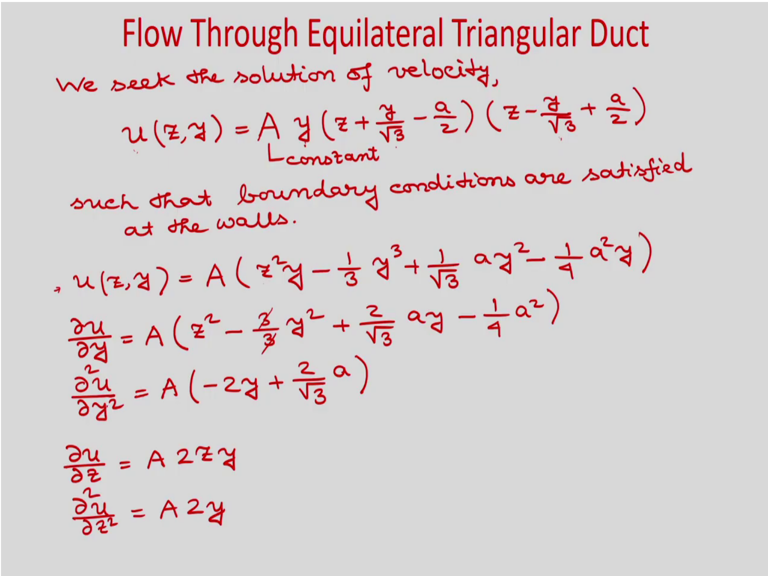

So, now we will seek the solution of the velocity as we seek the solution of velocity ok. So, let us say that we will assume u which is function of z and y will be some constant A into the equations of all these 3 sides, z plus y by root 3 minus a by 2 into z minus y by root 3 plus a by 2.

So, you can see this is the equation for the bottom wall A B and this is the equation for right side wall; that means, B C and this is the equation for left side wall; that means, A C. So obviously, we are seeking the solution of velocity as u z y is equal to some constant A, so A is constant and it is to be determine satisfying the governing equation.

So now, we can see that at the wall if you put bottom wall, so y is equal to 0 you know that is the equation. So obviously, y is equal to 0, so it will velocity will become 0. Similarly, right side wall you can see that this equal to 0. So, obviously u will become 0 and in the left side wall this will be 0 and u will be 0.

So, that means, these velocity profile whatever we have assume, so that is actually satisfying the boundary condition at the wall ok. So, now we need to find the constant A. So, we are seeking the solution of the velocity profile such that boundary conditions boundary conditions are satisfied ok at the wall. So, now we have assume the velocity profile we know the governing equation, so will put those differentials and will find the constant A.

So, now you can see that if you multiply this you will get u z y as A and we will get after multiplication you will get z square y minus 1 by 3 y cube plus 1 by root 3 a y square minus 1 by 4 a square y ok. So now, take the derivative with respect to y, so you will get del u by del y; so from here you can see. So, it will be just z square it will become minus 3 by 3 y square this 3 3 will get cancel plus 2 by root 3 a y minus 1 by 4 a square.

And if you take the derivative of this with respect to y then you write del 2 u by del y square. So, now, from here you can see you will get A, so this will become 0 then the second term will become minus 2 y then it will become plus 2 by root 3 a ok. Similarly now you take the derivative with respect to z. So, del u by del z if you find then you can see from this equation what you can write?

The first term will become A 2 z y and all the, these terms are not function of z. So, will become 0 and if you write del 2 u by del z square then it will become to A 2 y ok. So, now we know the governing equation and we know the derivatives you substitute these and find the value of constant A.

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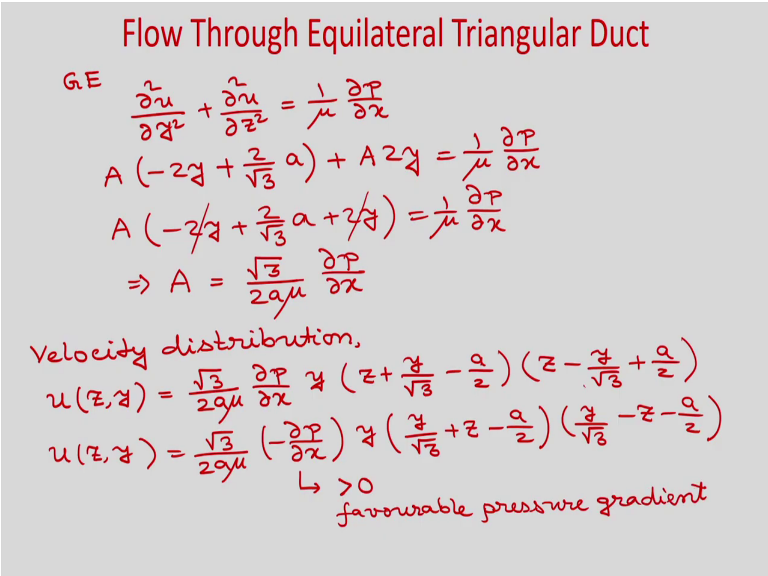

So, governing equation del 2 u by del y square plus del 2 u by del z square is equal to 1 by mu del p by del x by ok. So, del 2 y by del y square we have found as A minus 2 y plus 2 by root 3 a and del 2 u by del z square we have found as plus A 2 y and right hand side you have 1 by mu del p del x ok.

So, you can see that if you take A common then it will become minus 2 y plus 2 by root 3 a plus 2 y is equal to 1 by mu del p by del x and this will get cancel, then you can find A as root 3 by twice a mu del p by del x. So, you can see del p by del x is constant mu is constant for a liquid, so obviously A is constant.

Now, we have found the constant A now you will be able to find the velocity profile which satisfy the boundary conditions. So, velocity distribution now we can write as u which is function of z and y. So, a we can write root 3 by twice a mu del p by del x and other terms y into z plus y by root 3 minus a by 2 z minus y by root 3 plus a by 2, so you can see this velocity profile.

Obviously, satisfy the boundary condition at the wall because these are the equations of each side walls. Now we will write this pressure gradient as minus del p by del x, so that it will become favourable pressure gradient right for which you will get the velocity in the axial direction.

So, we will get u z y is equal to, so we will take from here just we will take minus outside and we will put in the minus del p by del x; so we will write root 3 by twice a mu minus del p by del x ok and we will get y this will just write as y by root 3 plus z minus a by 2. And if you take minus outside, so you will get y by root 3 minus z minus a by 2.

So, this is the velocity profile where minus del p del x obviously, is greater than 0 right. It is greater than 0; that means, it is favorable pressure gradient. So now, let us find where the maximum velocity will occur ok. So, it is equilateral triangular duct and obviously, the velocity will be maximum where del u by del y will be 0 and del u by del z will be 0 ok; so let us satisfy that.

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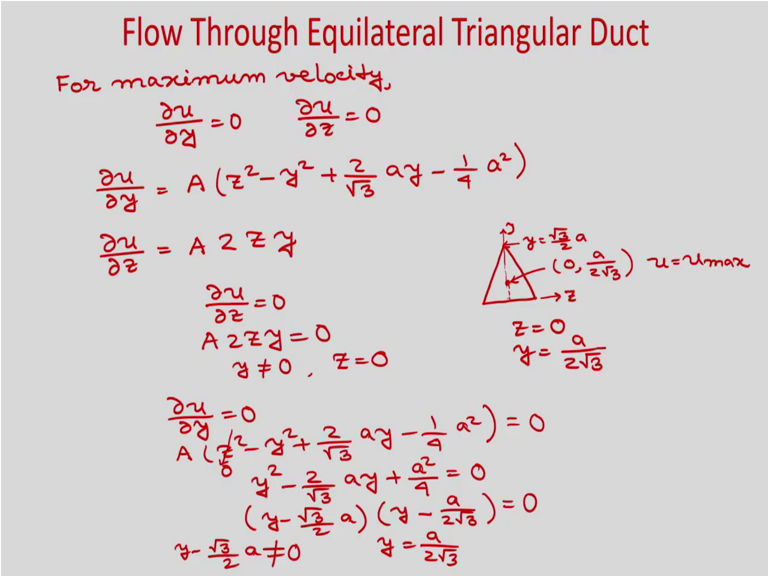

So we have already found the value of del u by del y, so for maximum velocity e ok. Del u by del y will be 0 as well as del u by del z will be 0. So, del u by del y already we have found as A z square minus y square plus 2 by root 3 a y minus 1 by 4 a square. And del u by del z we have found as A 2 z y ok.

So, you can see if we put del u by del z is equal to 0. So, del u by del z is equal to 0 then; obviously, A 2 z y should be 0 ok. So, we can see A is constant. So, it cannot be 0 and y cannot be 0 because if you see we have taken for this equilateral triangular duct this is z and y is measured from the bottom ok. So, at y is equal to 0 velocity is 0 right which is not maximum velocity right, so it is a wall. So, at y is equal to 0 you will get velocity as 0, so y cannot be 0; so y cannot be 0.

So; that means, z must be 0, so at z is equal to 0 somewhere we will get the maximum velocity. So now, satisfy del u by del y is equal to 0, so del u by del y is equal to 0. So, from here you can see A z square minus y square plus 2 by root 3 ay minus 1 by 4 a square is equal to 0.

So, we know that at z is equal to 0 somewhere this velocity will become maximum, so let us this is equal to 0 and A cannot be 0 right because A is constant. So, you can write this as y square minus 2 by root 3 a y plus a square by 4 is equal to 0 ok. So, now if you can write it as y minus root 3 by 2 a, into y minus a by 2 root 3 is equal to 0.

So, you can see that y minus root 3 by 2 a, cannot be 0 because at y is equal to root 3 by 2 a; that means, this is the point ok; where y is equal to root 3 by 2 a. So, this is obviously velocity will be 0 ok it will not become maximum here, so velocity is 0 we know at the wall; so it cannot be 0.

So, y minus root 3 by 2 a, cannot be 0 so obviously, y is equal to a by 2 root 3. So, we have found the location where the velocity will become maximum that is your z is equal to 0 and y is equal to a by 2 root 3 ok so; that means, somewhere here ok. So, which is your centroid of these triangular cross section.

So, at this point ok, we have z is equal to 0 y is equal to a by 2 root 3 and u will become u max. Now we will found the location at which the velocity will become maximum right. So, now, let us put that coordinate and find the maximum velocity ok.

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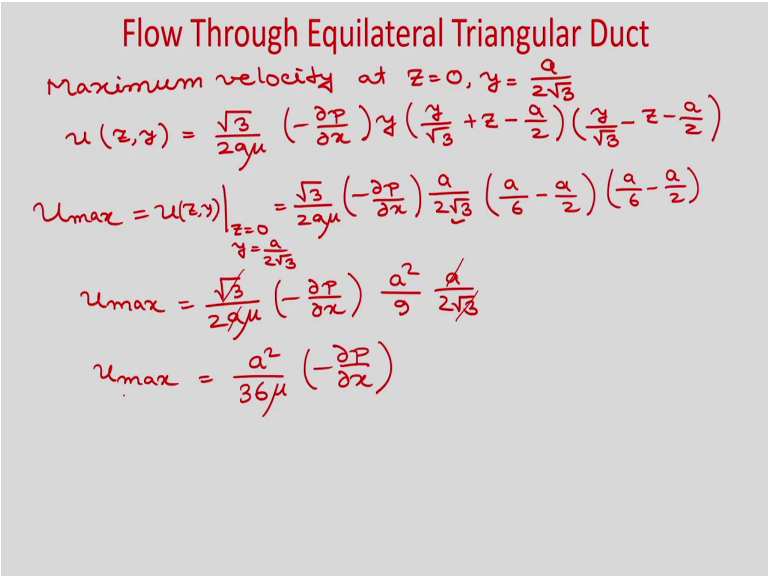

$$u(z,y) = \frac{\sqrt{3}}{2a\mu}\left(-\frac{\partial p}{\partial x}\right) y \left(\frac{y}{\sqrt{3}} + z - \frac{a}{2}\right)\left(\frac{y}{\sqrt{3}} - z - \frac{a}{2}\right)$$

$$u_{max} = u(z,y)\Big|_{z=0,\, y=\frac{a}{2\sqrt{3}}} = \frac{\sqrt{3}}{2a\mu}\left(-\frac{\partial p}{\partial x}\right)\frac{a}{2\sqrt{3}}\left(\frac{a}{6} - \frac{a}{2}\right)\left(\frac{a}{6} - \frac{a}{2}\right)$$

$$u_{max} = \frac{\sqrt{3}}{2a\mu}\left(-\frac{\partial p}{\partial x}\right)\frac{a^2}{9}\,\frac{a}{2\sqrt{3}}$$

$$u_{max} = \frac{a^2}{36\mu}\left(-\frac{\partial p}{\partial x}\right)$$

So, maximum velocity ok at z is equal to 0 and y is equal to a by 2 root 3 ok. So, we know the velocity profile u z y is equal to root 3 by 2 a mu minus del p by del x y y by root 3 plus z minus a by 2 into y by root 3 minus z minus a by 2 ok. So, u maximum will be u at z is equal to 0 y is equal to a by 2 root 3.

So, if you put it here you will get root 3 by twice a mu minus del p by del x ok, y will become a by 2 root 3 ok. Here z will become 0 so; obviously, you will get a by 6 right minus a by 2 and here also if you put z is equal to 0 this will also get a by 6 minus a by 2 ok. So, now if you find the maximum velocity you will get as root 3 by twice a mu minus del p by del x ok.

So, you can see this will become minus a by 3 right a by 6 minus a by 2 it will become minus a by 3, so it will be a square by 9. So, u max you will get as and another a by 2 root 3 is there ok this one ok. So, you will get now this a a will get cancel and this root 3 root 3 will get cancel, so you will get a square by 36 mu minus del p by del x.

So, this is the maximum velocity u max is a square by 36 mu minus del p by del x and you can see that obviously, minus del p by del x is positive ok; so u max will be obviously positive.

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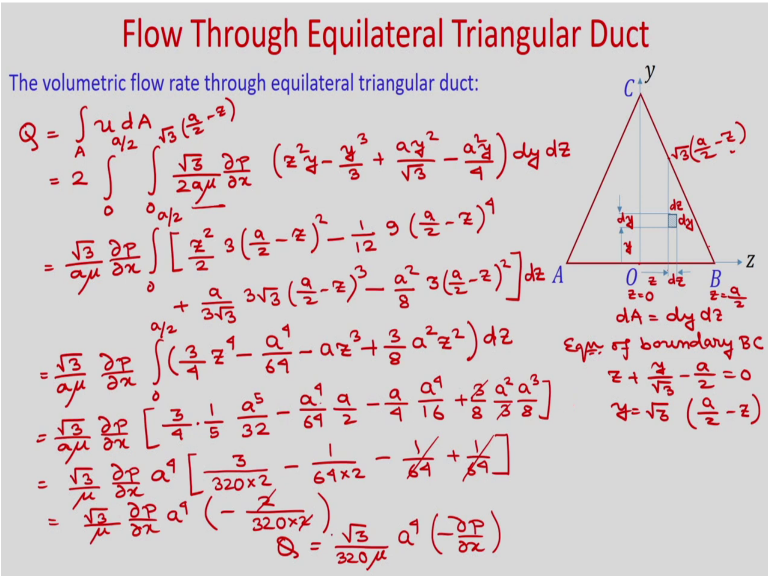

So, now we want to find the volumetric flow rate through equilateral triangular duct. So, what will do we will consider half of the domain here. So, origin is here and this is the z direction this is the y direction, if you consider half of the domain. Then if you can find the volumetric flow rate then we will just multiply with 2. So, you consider 1 elemental area of dy dz. So, this area dA equal to dy dz, so we are taking from the origin at a z distance as elemental distance dz and at y distance elemental distance dy.

So, this is the area and if you can see that; obviously z you can vary from z is equal to 0 to z is equal to a by 2 right. But here you can see that as you vary z is equal to 0 to a by 2 y also varies ok. So, that you can get from the equation from this side wall. So, what is the equation of boundary? B C ok.

So, that you know as z plus y by root 3 minus a by 2 is equal to 0, so y will be root 3 a by 2 minus z. So, you can see at a distance z y will be root 3 a by 2 minus z. So, as z varies y also will vary with this. So, now, if you want to find the volumetric flow rate q. So, we need to integrate over this area u d A ok.

So, now you can see we are considering half of the domain, so we will multiply with 2 and we are integrating from z is equal to 0 to a by 2. But here y will vary from y is equal to 0 to root 3 a by 2 minus z ok. So, it will be root 3 a by 2 minus z and what is the u? So, u expression if you remember, so it will be root 3 divided by twice a mu del p by del x and we have expression z square y minus y cube by 3 plus a y square by root 3 minus a square y by 4 then we have dy dz ok.

So, now you can see these terms root 3 by twice a mu del p by del x is constant you can take it outside. So, you can write it as root 3 by a mu del p by del x integral 0 to a by 2. Now you integrate and put the limit, so you can see if you integrate what you will get. So, you will get z square ok then you will get y square by 2 right, so y square by 2. So now, if you put y square, so y upper limit you put because in the lower limit it is 0.

So, upper limit if you put y square, so you will get 3 a by 2 minus z square. Then here we will get y to the power 4 by 4 right. So, you will get 1 by 4 into 3 12 and y to the power 4. So, you can see this is the upper limits, so you will get 9 a by 2 minus z to the power 4 then you will get a y cube by 3 root 3 right.

You will get a by 3 root 3 and y cube, so y cube you will get 3 root 3 a by 2 minus z cube. And then minus, so you will get a square by; so y square by 2. So, you will get a square by 8 and y square. So, y square you will get 3 a by 2 minus z square ok dz ok. So, this you simplify it first ok this term.

So, just you do the algebra then finally, you will get inside this integral you will get 3 by 4 z to the power 4 minus a to the power 4 by 64 minus a z cube plus 3 by 8 a square z square ok d z ok, so after simplifying it you will get final this expression. Now again you integrate it ok. So what you will get? Root 3 by a mu del p by del x ok.

So, here you can see it will be 3 by 4 and z to the power 5 by 5. So, it will be 1 by 5 and z to the power 5. That means, a to the power 5 by 32 ok minus, so it will be a to the power 4 by 64 z; so it will be a by 2 minus now you will get a z to the power 4 by 4. So, a by 4 and z to the power 4, so you will get a 4 by 16 plus 3 by 8 ok.

Now, a square z cube by 3, so a square by 3 z cube; so it will be a cube by 8. So, now, if you simplify it you can see every where you will have a to the power 5 right. So, a to the power 5 you take outside and here denominator a is there. So you can write root 3 by mu del p by del x a to the power 4 ok a to the power 5 by a, so a to the power 4. So, here you can see this you can write 3 by 320 into 2 minus 1 by 64 into 2 minus 1 by 64 and these 3 3 will get cancel, so it will be plus 1 by 64.

So, this will get cancel ok and this if you evaluate you will get root 3 by mu del p by del x a to the power 4. So, if you evaluate this you will get minus 2 by 320 into 2 ok. So, these 2 2 will get cancel and finally, you are going to get q as you can see here you are going to get root 3 by 320 mu a to the power 4 minus del p by del x.

So, you can see this is the favorable pressure gradient minus del p by del x, so Q is a positive quantity. Once you know q now you will be able to calculate what is the average velocity.

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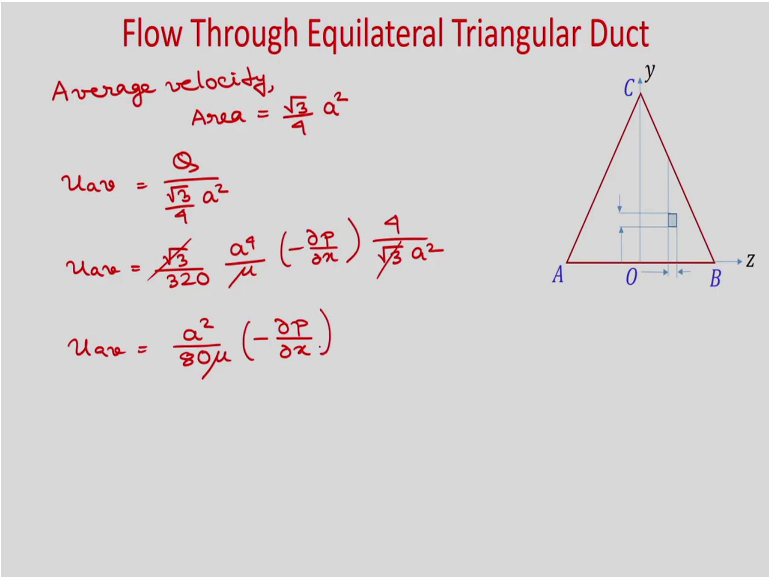

So, for this particular case, if you calculate the average velocity average velocity. So, area for this case you know root 3 by 4 a square right for equilateral triangular duct of side a area will be root 3 by 4 a square. So, u average will be just Q divided by root 3 by 4 a square ok. So, u average will be just root 3 by 320 a to the power 4 by mu minus del p by del x and you will get 4 by root 3 a square.

Root 3 root 3 will get cancel, so you will get u average as a square by 80 mu minus del p by del x. So, you can see that we have found the velocity distribution then we have found the location for the maximum velocity.

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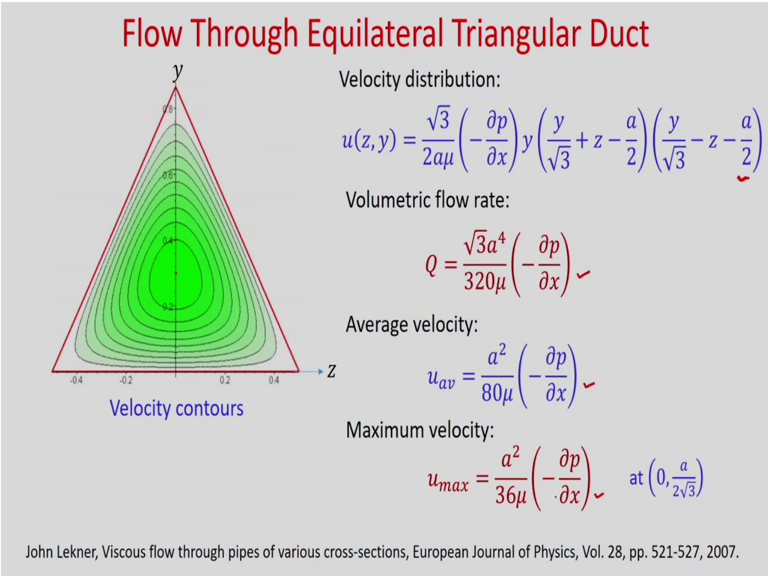

And if you can see from here that u z y we have expressed as function of y and z ok and volumetric flow rate as these and average velocity like this and maximum velocity a square by 36 mu minus del p by del x at z is equal to 0 and y is equal to a by 2 root 3. So, you can see the velocity contours. So, this is the lines you can see of equal velocity magnitude.

So, obviously you can see that it is 0 velocity at the walls ok. So, 0 velocity at the walls and the velocity is are actually following the boundary ok. So, velocity profile is following the boundary. So, these are the velocity contours and obviously, you can see your maximum velocity will become in this position right.

So, maximum velocity will occur at z is equal to 0 and y is equal to a by 2 root 3 and this is the value. So, in today's class we considered equilateral triangular duct and we found the velocity distribution for laminar steady incompressible fluid flow. In this case we have taken the x as the axial direction and we have assumed that pressure gradient is constant and also we have neglected the gravity.

To find the velocity distribution we assumed the velocity profile as a constant into the equations of this side lines and we have found satisfying the governing equation the value of A.

Once we get the velocity profile then we have found the location at which the maximum velocity will occur and for that we have put del u by del y is equal to 0 and del u by del z is equal to 0. And we found the location of maximum velocity as z is equal to 0 and y is equal to ay 2 root 3.

Thank you.