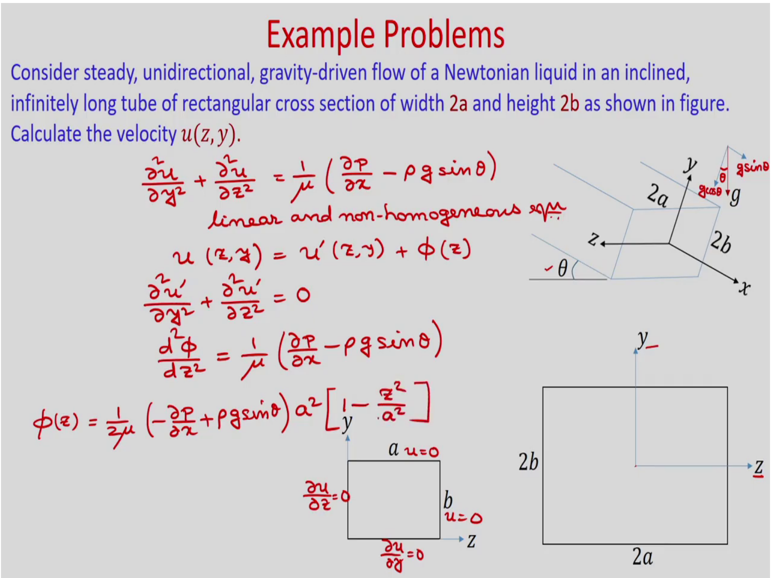
Viscous Fluid Flow

Prof. Amaresh Dalal
Department of Mechanical Engineering
Indian Institute of Technology, Guwahati

Module - 05
Steady, Two-dimensional Rectilinear Flows
Lecture - 04
Example Problems

Hello everyone. So, in today's class, we will solve some problems.

(Refer Slide Time: 00:38)



Let us consider the first problem. Consider steady, unidirectional, gravity-driven flow of a Newtonian liquid in an inclined, infinitely long tube of rectangular cross section of width 2a and height 2b as shown in the figure. Calculate the velocity u as function of z and y.

So, this is two-dimensional problem. So, you can see that u is function of z and y and this rectangular duct is inclined with horizontal with an angle theta. So, gravity is acting in the downward direction so, obviously, this is theta and these component of the gravity is g sin theta and this is g cos theta. So, obviously, you can see g sin theta is acting in the axial direction.

So, governing equation whatever we have derived so, you can see the governing equation we derived as del 2 u by del y square plus del 2 u by del z square is equal to 1 by mu del p by del x minus rho g sin theta ok. So, you can see it is a pressure and gravity driven flow.

So, if you see the governing equation, whatever we have derived the velocity distribution u as function of z, y while solving these rectangular duct, this is the same equation where we have additional term minus rho g sin theta ok. So, you can see that you can adapt the methodology whatever we have adopted while solving the fluid flow inside rectangular duct, you can divide this problem into two sub problems where one problem is just plane Poiseuille flow, one-dimensional fluid flow and the other one we can solve using separation of variables method.

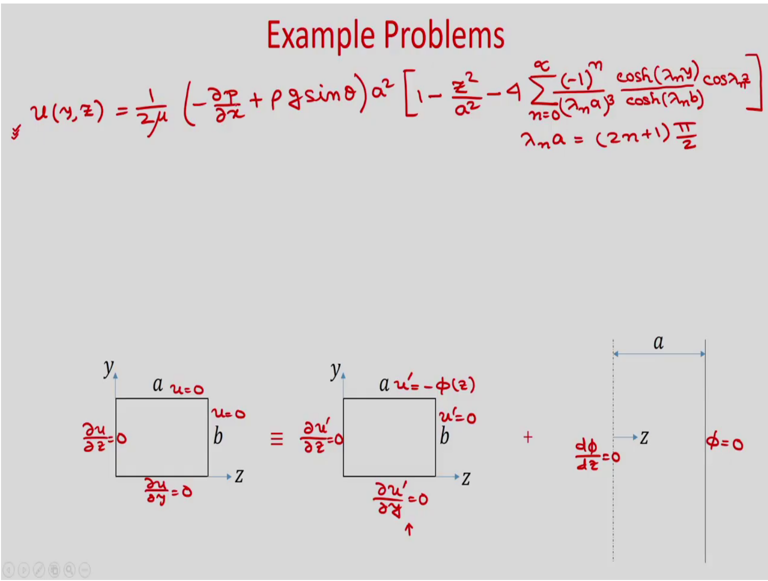
So, obviously, you can see this is linear and non-homogeneous equation. So, we cannot use the separation of variables method directly so, we will split into two problems where u which is function of z and y as solution of two sub problems; one is u prime z, y and one is phi which is function of z. So, this is the problem of one-dimensional plane Poiseuille flow and this is the solution from the two-dimensional problem.

So, you can see if you substitute it here, you will get two equations, one is del 2 u prime by del y square plus del 2 u prime by del z square is equal to 0 and another equation you will get d 2 phi by d z square is equal to 1 by mu del p by del x minus rho g sin theta ok. So, obviously, whatever solution method we adopted, you can see that one-quarter of the domain we will solve ok.

So, keeping the origin at the center so, this is your z direction, this is your y direction and obviously, you can see this is the domain for which we are solving and this is the boundary condition u is equal to 0, u is equal to 0 and these are the symmetry line. So, we have del u by del y is equal to 0 and this is del u by del z is equal to 0 ok.

So, you know the solution of this problem easily because this is the solution phi z from plane Poiseuille flow so, this is equal to 1 by twice mu minus del p by del x plus rho g sin theta a square and it is 1 minus z square by a square ok.

(Refer Slide Time: 05:19)

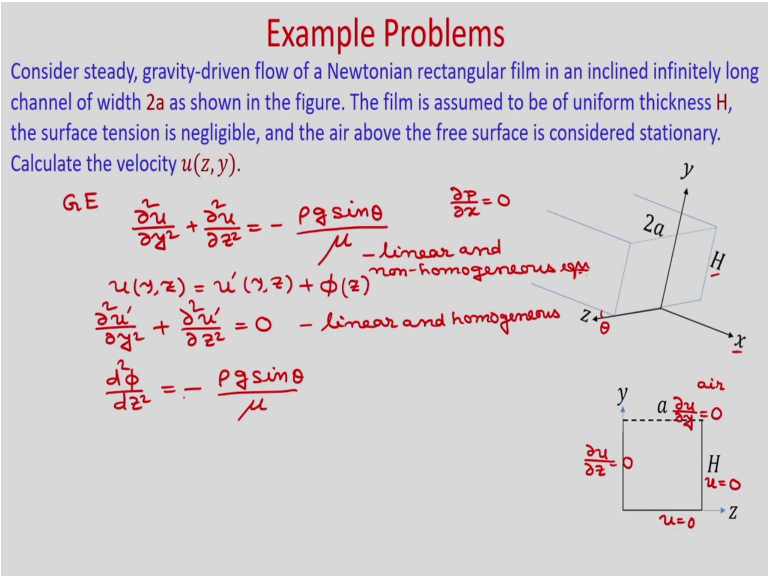


The other problem now, you can see that we can split into two problem so, this is the problem, original problem, u is equal to 0, u is equal to 0, del u by del y is equal to 0 and this is del u by del z is equal to 0. So, we are splitting into two sub problems. So, here boundary condition will be u prime is equal to minus phi z, this is u prime is equal to 0, here del u prime by del y is equal to 0 and here del u prime by del z is equal to 0 and this is the problem where d phi by dz is equal to 0 and phi is equal to 0 ok.

So, if you get or if you write the final solution, then you will get u which is function of y, z is equal to 1 by twice mu minus del p by del x plus rho g sin theta a square 1 minus z square by a square minus 4 summation of n is equal to 0 to infinity minus 1 to the power n divided by lambda n a whole cube cos hyperbolic lambda n y divided by cos hyperbolic lambda n b cos lambda n z. So, this is the final solution and where lambda n a is equal to twice n plus 1 pi by 2.

So, you can see for this problem, we used separation of variables method and you can see z is the homogeneous direction and finally, we got this as the velocity distribution. So, if you compare with the solution of rectangular duct so, you can see that minus del p by del x just we have replaced with minus del p by del x plus rho g sin theta ok. So, if say it is purely gravity-driven flow, then you can put del p by del x is 0 ok, del p by del x as 0.

(Refer Slide Time: 07:47)



Now, let us consider the second problem. Consider steady, gravity-driven flow of a Newtonian rectangular film in an inclined infinitely long channel of width 2a as shown in the figure. The film is assumed to be of uniform thickness H, the surface tension is negligible and the air above the free surface is considered stationary. Calculate the velocity u as function of z and y.

So, you can see this is the problem where x is the axial direction, film thickness is H and width is 2a and you can see that it is inclined with the horizontal as angle theta ok. So, obviously, we will have the same governing equation whatever we have written in last problem except it is purely gravity driven-flow right, there is no pressure gradient because it is free surface flow right. So, del p by del x will be 0 and in this case, the origin we are taking at the center of the bottom wall ok and x is the axial direction.

So, you can see that from this problem as the boundary conditions are symmetric and the geometry is symmetric, we can take half of the domain and if you consider half of the domain, then you can see this is the width a and height is H and velocity boundary conditions u is equal to 0, bottom u is equal to 0, left boundary symmetry plane right so, it will be del u by del z is equal to 0.

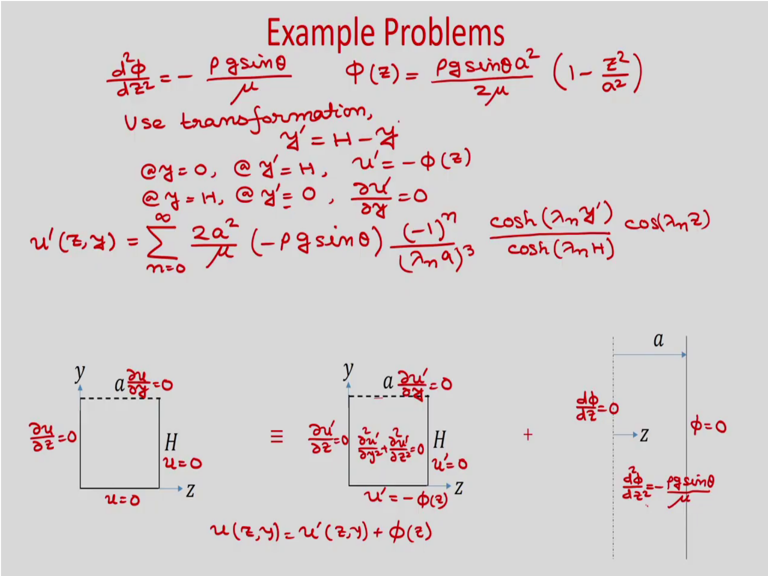
What will be the boundary condition of the top boundary? So, as it is mentioned that the air above the free surface is considered stationary. So, whatever outside air is there so, it is stationary so, obviously, shear stress acting on this free surface ok will be 0 so, that means, del u by del y will be 0 right because this is, there is a there will be shear stress continuity and as air is stationary, shear stress will be 0 from the air side and as to maintain the shear stress continuity at the interface so, shear stress from the liquid side it will be 0 hence, del u by del y will be 0 on the top boundary.

So, now, let us write down the governing equation. So, it will be the same as problem 1 except del p by del x is equal to 0. So, governing equation for this problem del 2 u by del y square plus del 2 u by del z square is equal to minus rho g sin theta by mu ok. So, del p by del x is 0 in this case ok because it is free surface flow.

So, now, similar way, we can split into two problems. Why we are splitting? Because these problems we cannot solve using separation of variables method because it is linear and non-homogeneous equation right ok. So, we are splitting into two problems. One solution is u prime y, z and one problem is phi which is function of z. So, if you put it here, you are going to get two problems; one is del 2 u prime by del y square plus del 2 u prime by del z square is equal to 0 ok.

So, this equation now is linear and homogeneous. So, we can use separation of variables method ok and another problem we will get d 2 phi by d z square is equal to minus rho g sin theta by mu ok subjected to these boundary conditions.

(Refer Slide Time: 12:22)



Now, for these two sub problems, let us write down the boundary conditions. We know that u which is function of z, y is equal to u prime which is function of z, y plus phi which is function of z only. So, now, this is the original problem where this is the free surface so, del u by del y is equal to 0, this is the wall so, u is equal to 0, bottom boundary is wall so, u is equal to 0 and this is symmetry line so, del u by del z is equal to 0.

So, here, the boundary condition we have written for the main problem and for del 2 u prime by del y square plus del 2 u prime by del z square is equal to 0 for these problem, what will be the boundary conditions? So, you can see from here and this is the problem where phi is function of z only. So, you can see this is will be just d 2 phi by d z square is equal to minus rho g sin theta by mu ok. So, these boundary condition will be phi is equal to 0, here it is symmetry line so, it will be d phi by dz is equal to 0.

So, now, you can see from here that if phi is equal to 0, here u is equal to 0 so, u prime will be 0 in the right boundary ok. Left boundary if you take the derivative with respect to z ok so, you will get del u prime by del z is equal to 0 ok. The top boundary obviously, you can see del u by del y is equal to 0 so, del u prime by del y will be 0 and the bottom boundary, you can see that u is equal to 0 so, u prime will be minus phi z so, u prime will be minus phi z ok.

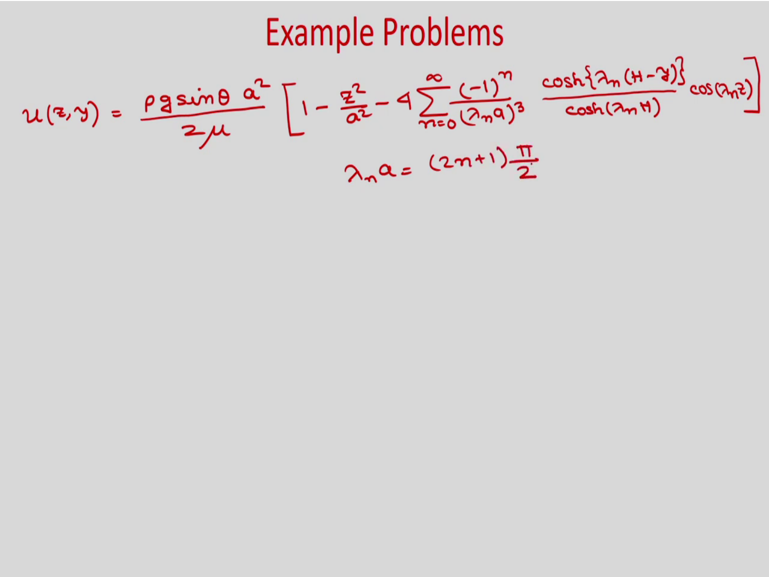
So, you can see now, the solution of this you know right. So, d 2 phi by dz square is equal to minus rho g sin theta by mu. So, this solution you know, phi which is function of z, it will be rho g sin theta a square by twice mu 1 minus z square by a square ok. Now, what about this problem? So, you can see obviously, z direction is homogeneous direction and the governing equation is linear and homogeneous so, obviously, you can use separation of variables method.

So, now, let us use the solution which way we solved the problem of flow inside rectangular duct, in that case, you can see bottom wall was del u prime by del y is equal to 0 and the top a boundary was u prime is equal to minus phi z. So, what we will do? We will use that solution, just we will use some transformation. So, we will use this transformation y prime is equal to H minus y. So, what will happen?

So, we can see that when at y is equal to 0 so, we can write at y prime is equal to H right so, u prime will be minus phi z and at y is equal to H so, that means, at y prime is equal to 0 right del u prime by del y is equal to 0. So, you can see with these boundary conditions, it resembles with the problem that we solved for the flow inside rectangular duct.

So, you know the solution for u prime here so, you can write u prime z, y is equal to summation of n is equal to 0 to infinity twice a square by mu minus rho g sin theta minus 1 to the power n by lambda n a whole cube cos hyperbolic lambda n y prime so, we will not write y because we are using similar boundary condition in terms of y prime so, lambda n y prime divided by cos hyperbolic lambda n H and cos lambda n z ok. So, this is the solution of u prime.

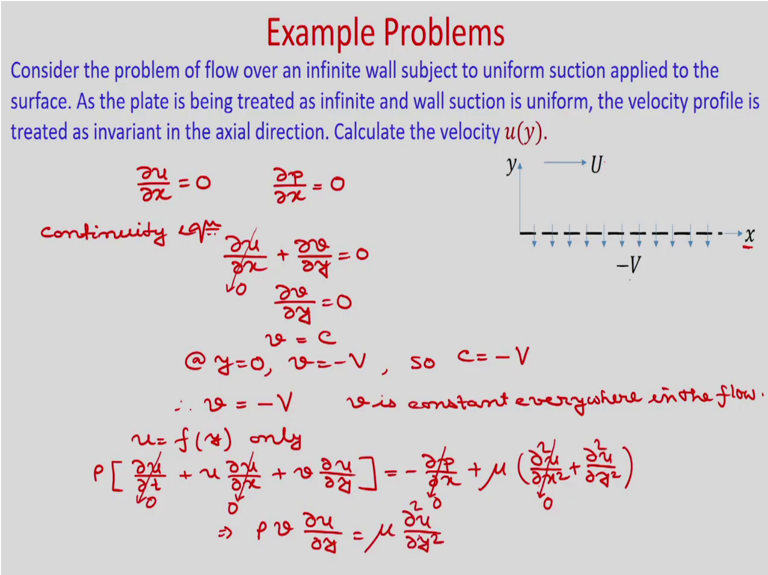
(Refer Slide Time: 17:38)



So, now you know the solution of phi z, you know the solution of u prime z y, and this y prime you know as H minus y so, you put all those things and write the final velocity profile. So, you can write u z, y which is your final velocity profile as rho g sin theta a square divided by twice mu 1 minus z square by a square minus 4 summation of n is equal to 0 to infinity minus 1 to the power n divided by lambda n a whole cube cos hyperbolic ok.

In place of y prime, we will write H minus y, H minus y divided by cos hyperbolic lambda n H cos lambda n z where lambda n a is equal to twice n plus 1 pi by 2.

(Refer Slide Time: 18:56)



Now, we will solve another problem not two-dimensional steady state problem it is only one-dimensional problem flow over a porous flat plate. Consider the problem of flow over an infinite wall subject to uniform suction applied to the surface. As the plate is being treated as infinite and wall suction is uniform, the velocity profile is treated as invariant in the axial direction. Calculate the velocity u y.

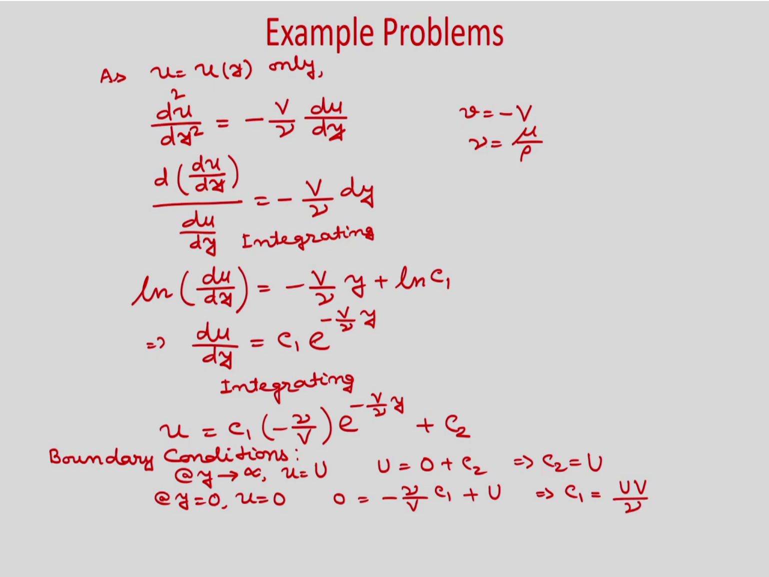
So, you can see this is the infinitely long flat plate which is porous and uniform suction is applied ok. So, as the velocity is applied in the negative y direction so, it is minus V ok, v is equal to minus V and this is the axial direction and we have uniform axial velocity as y ok, this is the free stream velocity ok.

So, you can see it is invariant in the axial direction that means, del u by del x is 0 ok and obviously, as it is flow over a flat plate so, del p by del x is also 0 ok. So, from continuity equation what we can write del u by del x plus del v by del y is equal to 0. So, this is 0 because the velocity profile is treated as invariant in the axial direction right. So, del v by del y is equal to 0 and if you integrate it so, v will be just constant.

What is the boundary condition? At y is equal to 0, v is equal to minus V ok. So, c is equal to minus V. So, your v velocity profile is constant, v is equal to minus V everywhere ok. So, v is constant ok of magnitude minus V everywhere in the flow ok and as del u by del x is 0 so, u is function of y only right and if you write down the x momentum equation so, you can see rho del u by del t plus u del u by del x plus v del u by del y two-dimensional x momentum equation we are writing minus del p by del x plus mu del 2 u by del x square plus del 2 u by del y square.

So, you can see it is a steady state problem. So, del u by del t is equal to 0. So, this is fully developed. So, del u by del x is 0, flow over a flat plate del p by del x is 0 and as del u by del x is 0 everywhere so, del 2 u by del x square also will be 0. So, finally, we are getting rho v del u by del y is equal to mu del 2 u by del y square ok.

(Refer Slide Time: 22:11)



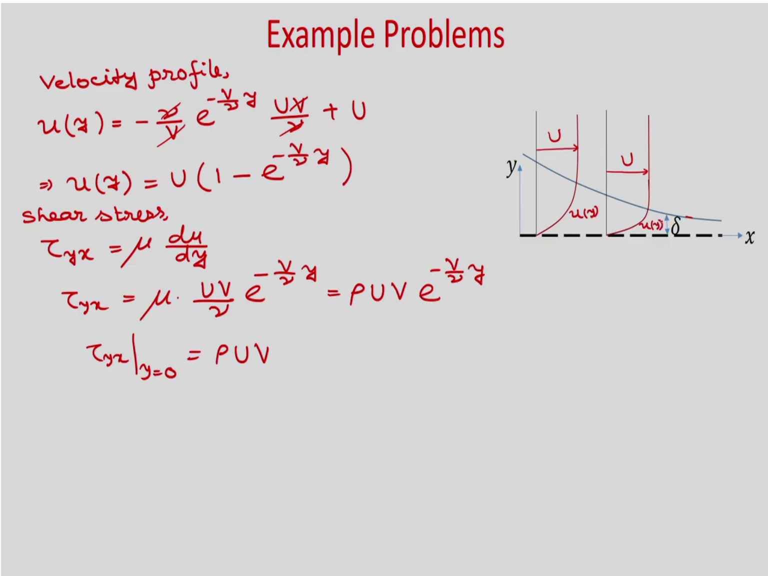
So, as u is function of y only so, we can write d 2 u by dy square is equal to minus V by nu du by dy. So, v is equal to minus V we have put and nu is equal to mu by rho ok. So, this is the kinematic viscosity. So, this is mu by rho. So, now, what we can write? We can write d of du by dy by du by dy is equal to minus V by nu dy ok.

So, now, if you integrate it so, you will get ln du by dy ok integrating, minus V by nu y plus constant ln c 1. So, this we can write du by dy is equal to c 1 e to the power minus V by nu y ok. So, now again you integrate so, integrating you will get u is equal to c 1 so, if you integrate it, you will get minus nu by V e to the power minus V by nu y plus c 2.

So, now, what are the boundary condition for velocity u? So, obviously, at y is equal to 0 that means, over the flat plate, it is no slip condition so, u will be 0 and away from the plate, y tends to infinity so, it will have the free stream velocity u ok. So, these are the boundary conditions.

So, boundary conditions ok at y tends to infinity so, it will have the velocity same as free stream velocity U. So, if you see from here so, you can see it will be U is equal to this will become 0 plus c 2 so, c 2 is equal to U and at y is equal to 0, u is equal to 0. So, you will get 0 and this you will get as minus nu by V c 1 and c 2 is U. So, you will get c 1 is equal to UV by nu.

(Refer Slide Time: 25:11)

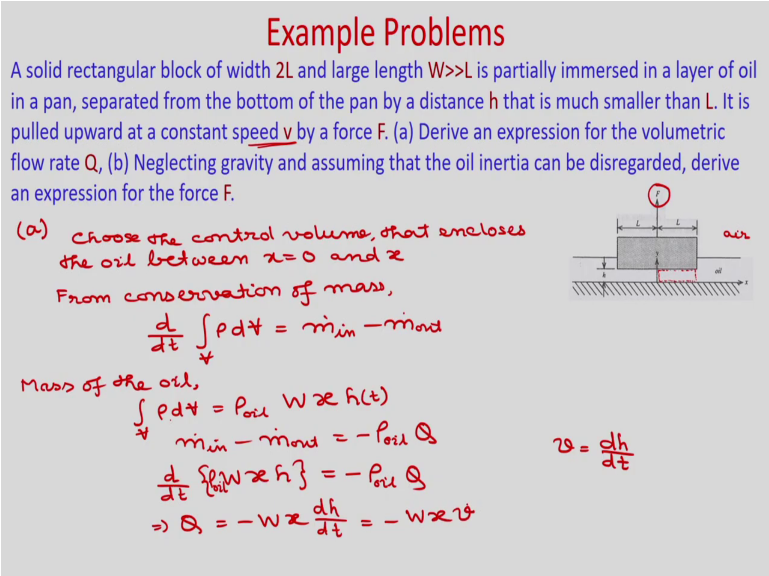


So, now, put the values of c 1 and c 2 in this equation and get the velocity profile. So, velocity profile you will get now, u which is function of y as minus nu by V e to the power minus V by nu y c 1, c 1 is UV by nu plus U. So, here V, V will get cancel, nu, nu will get cancel so, you will get u which is function of y, if you take U outside so, we will the bracket so, we will get 1 minus e to the power minus V by nu y.

And if you want to calculate the shear stress distribution inside the fluid domain so, it will be just tau yx equal to mu du by dy ok. So, tau yx will be just mu UV by nu e to the power minus V by nu y. So, this you will get as rho UV because mu by rho so, mu, mu will get cancelled. So, rho UV e to the power minus v by nu y and on the bottom plate tau u will be tau will be y is equal to 0 if you put so, it will be rho UV ok.

So, if you see the velocity profile so, say if you plot the velocity profile at different location so, obviously, there will be boundary layer and this is the edge of the boundary layer and boundary layer thickness is delta and you can see outside this boundary layer, you will have free stream velocity U ok; free stream velocity U and inside you will have function of y, u is function of y ok.

(Refer Slide Time: 27:16)



Now, let us consider another problem which is a tangent problem. A solid rectangular block of width 2L and large length W much much greater than L is partially immersed in a layer of oil in a pan, separated from the bottom of the pan by a distance h that is much smaller than L. It is pulled upward at a constant speed v by a force F. Derive an expression for the volumetric flow rate Q, b neglecting gravity and assuming that the oil inertia can be disregarded, derive an expression for the force F.

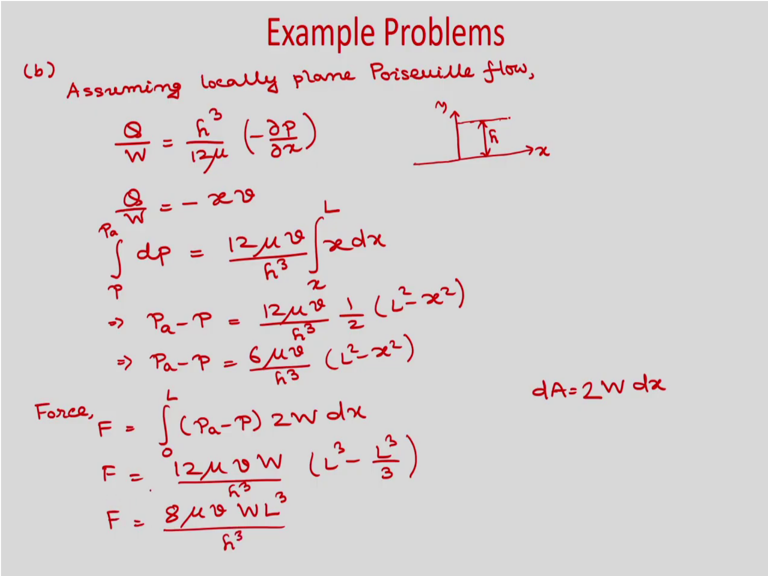
So, you can see ok. So, this is the oil outside we have air and this is the rectangular block of width 2L and the width W is perpendicular to this and that is much much larger than the length L. This is the distance h ok from the block to the bottom and this block is pulled outside with a velocity v by a force F ok. Now, we need to calculate the volumetric flow rate Q and the force F.

So, let us choose a control volume here. So, if you choose a control volume like this and apply the conservation of mass, then you can write choose the control volume that encloses the oil between x equal to 0 and x ok. So, from conservation of mass, what you can write? d dt of integral volume integral rho dv is equal to m dot in minus m dot out ok.

So, what will the mass of oil? So, rho dv is the mass of oil ok, integral v rho dv is the mass of oil. So, let us calculate the mass of the oil. So, mass of the oil so, integral rho dv so, obviously, rho is rho oil ok and dv is the volume so, volume is W x and h, h is function of t and m dot in minus m dot out, it will be just minus rho oil Q where Q is the volumetric flow rate.

So, if you put it here, you are going to get d of dt W x h is equal to minus rho oil Q ok. So, this rho oil is constant so, this will get cancelled and you will get Q is equal to minus W x dh by dt and you can see dh by dt so, it is nothing, but the velocity v ok. So, v is nothing, but dh by dt so, it will be minus W x v.

(Refer Slide Time: 31:08)



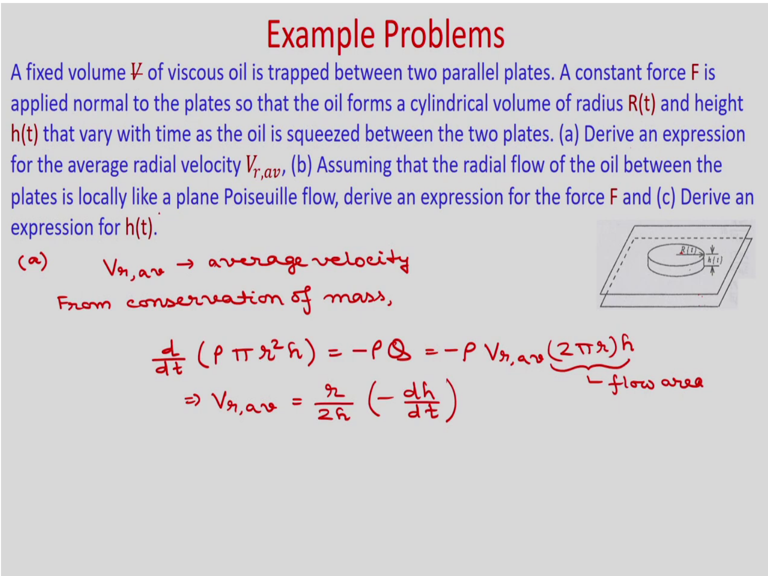
So, next we need to calculate the force F. So, first let us use or assume that locally, it is a plane Poiseuille flow, locally plane Poiseuille flow ok. So, you know what is Q by W for this particular case? It will be just h cube by 12 mu minus del p by del x ok. So, this we have taken the coordinate at the bottom ok because in this particular case, ok this is oil and this is the block surface so, this is the h so, for this, you can see Q by W will be h cube by 12 mu minus del p by del x.

So, we know from earlier expression Q by W is equal to minus xv. So, now if you equate it so, you will get dp is equal to 12 mu v h cube x dx. So, if you integrate it from p to p a where p is the atmospheric pressure from x equal to x to L ok. So, if you integrate it, you will get p a minus p is equal to 12 mu v by h cube and this x square by 2 so, half L square minus x square ok. So, you will get p a minus p is equal to 6 mu v by h cube L square minus x square.

Now, what about the force? So, force F ok so, it will be just integral p a minus p into area. So, what is the area? So, area you can see for this problem so, elemental area dA so, here you can see so, the elemental area dA on the block surface it will be dA is equal to W into dx and both side it will be 2 ok so, 2 W dx because we have taken from this side also you have to take so, it will be just 2 W into dx so, 2 W into dx and you have to integrate from 0 to L.

So, p a minus p, if you this if you put it here so, you will get F is equal to 12 mu v W by h cube L cube minus L cube by 3. So, it will be 2 by 3 and finally, F you will get 8 mu v W L cube divided by h cube. So, this is the force required to pull the block.

(Refer Slide Time: 34:24)



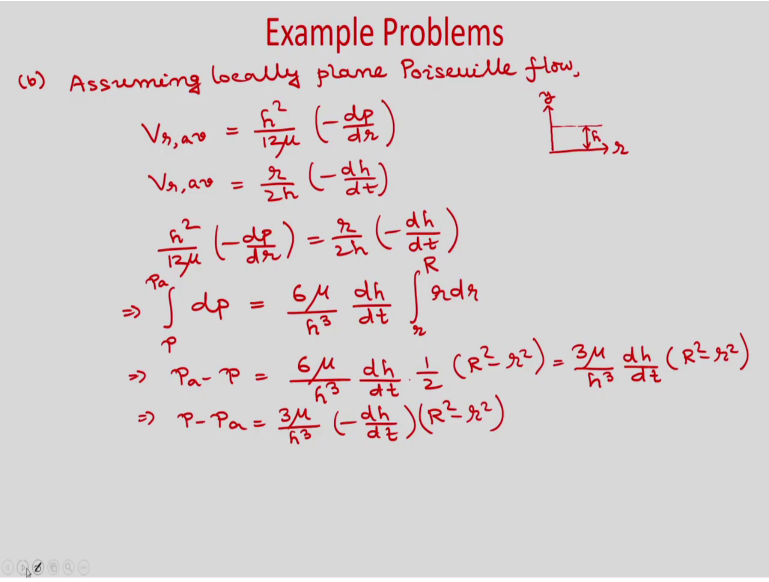
Now, let us consider the last problem, it is similar to the earlier problem. Let me read out the problem. A fixed volume V of viscous oil is trapped between two parallel plates. A constant force F is applied normal to the plates so that the oil forms a cylindrical volume of radius R which is function of t and height h that vary with time as the oil is squeezed between the two plates.

A derive an expression for the average radial velocity V r, average, b assuming that the radial flow of the oil between the plates is locally like a plane Poiseuille flow, derive an expression for the force F and c, derive an expression for h. So, we can see these are the two parallel plates ok. So, these two plates are squeezed and obviously, you consider this cylindrical volume ok where R is function of t and obviously, h is function of t. So, we need to calculate the average velocity V r, average and we need to calculate the force F and the expression of h t ok.

Again, from conservation of mass, what you can write? So, you can see if V r, average, this is the average velocity, then from conservation of mass ok so, d of dt the volume so, rho volume is pi r square h is equal to minus rho Q and Q if you know the V r, average, if you know the area so, you can write minus rho V r, average so, Q is V into a and area is so, twice pi r into h right. So, this is the volume and this is the area twice pi r ok into h. So, this is the flow area ok; flow area ok.

So, at a distance r so, it will be twice pi r into h so, that will give you the flow area. So, from here, rho is same. So, you can calculate V r, average is equal to r by twice h minus dh by dt.

(Refer Slide Time: 37:26)

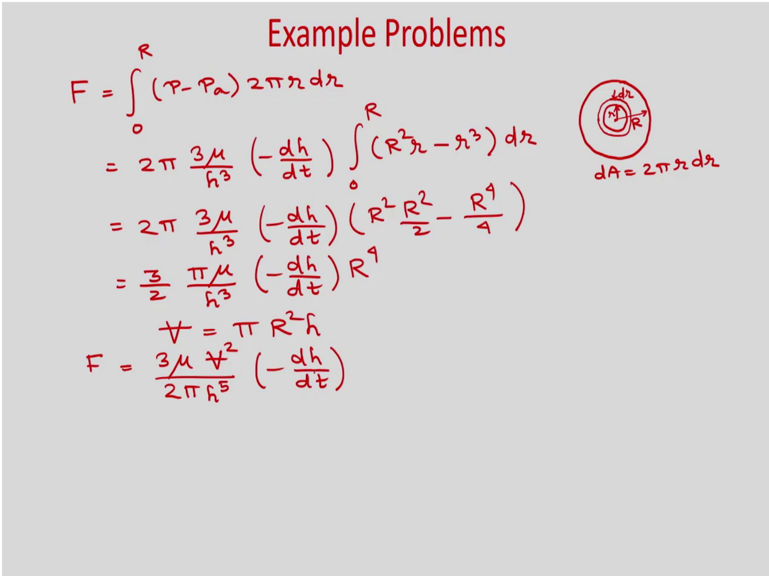


So, next we need to calculate the force F assuming locally plane Poiseuille flow. So, assuming locally plane Poiseuille flow ok, you know that V r, average so, this you have already calculated so, this is the direction r, this is the direction y and this is the height h ok. So, for this you know V r, average as h square by 12 mu minus dp by dr ok in the radial direction, the pressure gradient is dp by dr.

So, now, we know V r, average from earlier expression r by twice h minus dh by dt so, now, you equate it. So, you will get h square by 12 mu minus dp by dr is equal to r by 2h minus dh by dt ok. So, you can write as dp is equal to 6 mu by h cube dh by dt r dr ok. So, now, we will integrate it from p to p a and any radius r to R.

So, we can write p a minus p is equal to 6 mu by h cube dh by dt ok. So, it will be r square by 2 so, it will be half R square minus r square ok. So, we will get 3 mu by h cube dh by dt R square minus r square ok or we can write p minus p a is equal to 3 mu by h cube minus dh by dt R square minus r square ok.

(Refer Slide Time: 39:53)



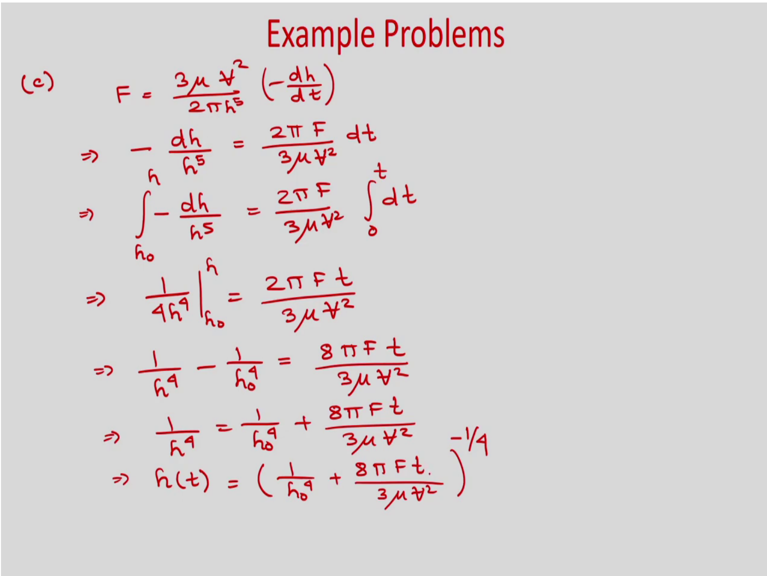
So, now, we need to calculate the pressure so, p into dA we can integrate over 0 to R. So, it is like you have this cylindrical volume so, it will be just R. So, at any radius R, you take one elemental distance dr so, the dA will be just twice pi r into dr. So, now, the force F ok on this plate you can see it will be just integral 0 to R p minus p a and dA is twice pi r dr ok so, on the plate.

So, now, you can see twice pi if you put the expression of p minus p a here so, you will get 3 mu by h cube minus dh by dt integral 0 to R R square r minus r cube dr. So, it will be twice pi 3 mu by h cube minus dh by dt so, it will be R square so, it will be R square by 2 so, if you put the upper limit R so, it will be R square by 2 minus R to the power by 4 by 4 so, it will be

R to the power 4 by 4 ok. So, finally, it will be just R to the power 4 by 4 so, it will be 3 by 2 pi mu h cube minus dh by dt R to the power 4.

Now, the volume V, we can write pi R square h so, this expression of this force, you can write F is equal to 3 mu V square which is your volume twice pi h to the power 5 minus dh by dt ok. So, this is the expression for the force F.

(Refer Slide Time: 42:07)



Now, we need to calculate the h as function of t ok. So, from this expression, whatever we have derived now so, F is equal to 3 mu V square by twice pi h to the power 5 minus dh by dt. You can write as minus dh by h to the power 5 is equal to twice pi F by 3 mu V square dt. So, now if you integrate it, h naught to h twice pi F 3 mu V square 0 to t dt ok.

So, it will be 1 by 4 h to the power 4 because h to the power minus 5 so, it will be minus 1 by 4 and minus is there so, it will be plus h naught to h twice pi F t by 3 mu V square ok. So, if you put it you will get 1 by h to the power 4 minus 1 by h naught to the power 4 is equal to so, it will be 4 so, it will be 8 pi F t by 3 mu v square.

So, 1 by h to the power 4 it will be 1 by h naught to the power 4 plus 8 pi F t by 3 mu V square. So, this is the expression to find the h. So, you can write that h which is function of t, you can see that it will be 1 by h naught to the power 4 plus 8 pi F t by 3 mu V square to the power minus 1 by 4. So, this is the height which is function of t.

So, in today's class, we considered few example problems. So, first two problems we considered based on the steady state two-dimensional problems where velocity is function of z and y, next problem we considered as flow over porous media so, flow over a porous flat plate and uniform suction is taking place with a velocity minus V and we found an expression for velocity U.

Next, two problems we considered where the block or the plates are moving and it is kind of tangent problem and using the conservation of mass ok, we found the volumetric flow rate or the average velocity and then, we calculated the force F acting on the block or the plate.

Thank you.