Viscous Fluid Flow
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Module - 07
Laminar Boundary Layers - I
Lecture - 02
Blasius Flow Over A Flat Plate: Similarity Solution

Hello everyone. So, in last class we discuss about the Boundary layer theory and we derived the boundary layer equations. Today we will consider Flow Over A Flat Plate it is a simple problem that Blasius solved this problem using Similarity transformation.

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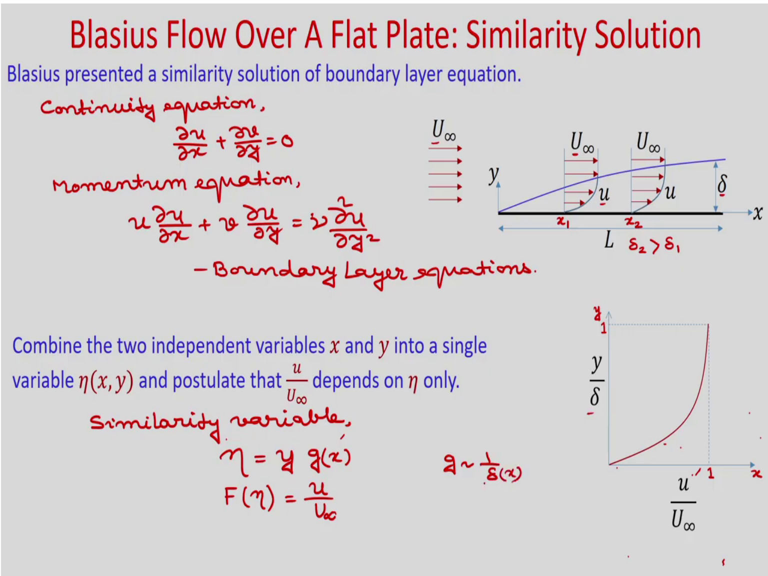

Consider this flat plate of length L, x is the axial direction and y is the normal to this plate.

Now, we have this free stream velocity U infinity and this flow is taking place over this flat plate. So, as you discussed that there will be formation of the boundary layer and delta is known as boundary layer thickness. Now you can see that the velocity u will vary inside the boundary layer and outside it will have the same velocity as free stream velocity U infinity.

Now if you consider two locations in axial direction this is let us say x 1 and this is the location x 2, then you can see obviously, the free stream velocity will remain same however, the boundary layer thickness is increasing. So, at x 2 location boundary layer thickness delta 2 will be greater than delta 1 where delta 1 is the boundary layer thickness at position x 1.

Now, Blasius observed that if we plots u by U infinity with the non dimensional normal distance, y by delta then all this velocity profile falls in same curve. So, you can see if you plot in x direction u by U infinity and in y direction if you plot y by delta, then at different locations all the velocity profile will fall in same curve.

So, here you can see obviously, u by U infinity will be 1 and y by delta; obviously, it will be 1 and after that there will be no change in the velocity profile it will remain 1. So, Blasius observed it and he used the similarity transformation technique to solve this flow over flat plate. First let us write down the governing equations for flow over flat plate obviously, in this particular case you know the axial pressure gradient is 0.

So, we have continuity equation, del u by del x plus del v by del y is equal to 0 and we have the momentum equation. This is the boundary layer equation u del u by del x plus v del u by del y equal to nu del 2 u by del y square. So, these equations are known as boundary layer equations. So, now, we are interested to find the velocity profile and we want to find what is the shear stress acting on the flat plate and the total drag force acting on the flat plate.

And at the same time we need to find that delta which is your boundary layer thickness as function of x. Blasius use this similarity transformation technique to solve this problem and he combined these two variables x and y into an independent variable eta where u by U infinity becomes function of eta only. So, he used the similarity variable, eta as y into some function g which is function of x ok.

So; obviously, g is having the scale of 1 by delta, then we have the F which is function of eta is the non dimensional velocity u by U infinity. So, you can see that we are actually defining one independent variable eta which is function of two independent variables x and y and F is the velocity profile which is function of eta which is u by U infinity and this g is function of x because you know that g the g will scale with the 1 by delta and delta is function of x.

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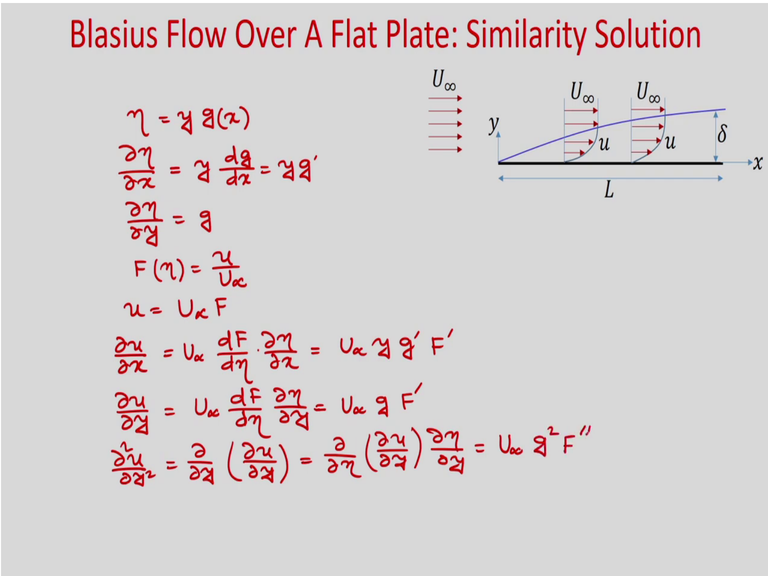

Now, we have eta as function of y and x. Now we can write del eta by del x as y now d g by d x g is function of x. So, you can write ordinary derivative is equal to let us say y g prime g prime is representation of d g by d x and similarly del eta by del y we can write as g ok and we have already written that F is equal to u by U infinity.

So, we can write u is equal to U infinity F. So, we write del u by del x is equal to now you can see U infinity is constant d F by d eta and del eta by del x. So, del eta by del x already we have found y g prime. So, it will be U infinity y g prime and F prime. F prime is representation of d f by d eta.

Similarly, del u by del y we can write as U infinity d F by d eta and del eta by del y ok and del eta by del y is nothing, but g. So, you can write U infinity g F prime ok. So, and we have in the boundary layer equation del 2 u by del y square. So, this we can write as del of del y del u by del y is equal to. So, you can write del of del eta del u by del y and del eta by del y ok.

So, you can see del of del eta or del u by del y g is function of x. So, it will be just F double prime and del eta by del y is g. So, already 1 g is there. So, it will be U infinity g square F double prime. So, now, let us consider the momentum equation and find the value of velocity v.

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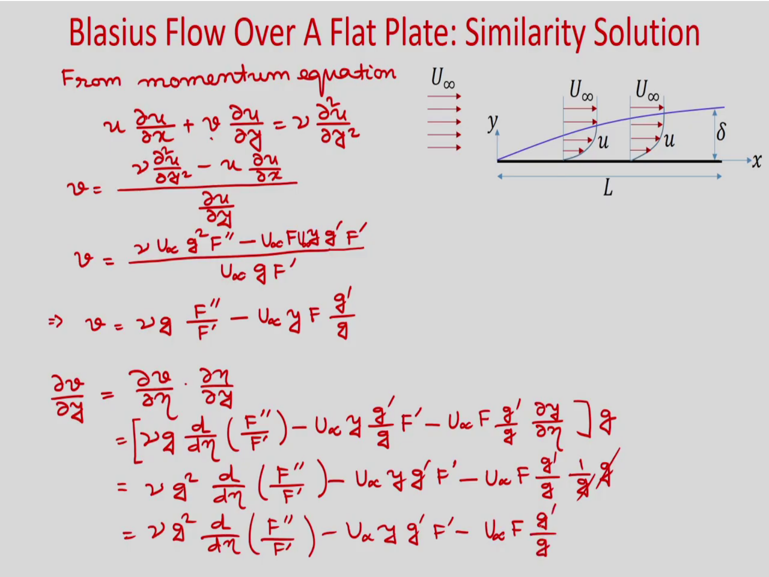

So, we have from momentum equation u del u by del x plus v del u by del y is equal to nu del 2 u by del y square. So, let us find the value of v. So, v we can write as nu del 2 u by del y square minus u del u by del x divided by del u by del y ok. Now we have found all these derivatives in terms of eta and F.

So, you can write it v is equal to. So, nu del 2 u by del y square we are found U infinity g square F double prime minus u, u can write U infinity into F and del u by del x we have found as y g prime F prime ok and del u by del y is U infinity g F prime. So, we can write v is equal to. So, this U infinity will get cancelled one g will be there. So, nu g F double prime by F prime minus u infinity.

So, y another U infinity will be there because u is equal to infinity F and del u by del x 1 U infinity will be there. So, one U infinity will get cancelled in one U infinity will be here y F and g prime by g ok this F prime f prime will get cancelled. So, now, we can find the gradient del v by del y is equal to del v by del eta del eta by del y ok.

So, now, you can see here. So, del v by del eta. So, these we can write nu g these are not function of eta because g is function of x only. So, you can write d of d eta F double prime by F prime minus U infinity y g prime by g and we have this F. So, this is F function of eta. So, del F by del eta.

So, it will be F prime and minus now we have y right. So, this we can write as U infinity F g prime by g del y by del eta ok. So, this into del eta by del y. So, del eta by del y is g ok. So, now, we can write if you simplify it. So, you can write nu. So, this g if you multiply g square it will be d of d eta F double prime divided by F prime minus u infinity.

So, these g if you multiply it will get cancelled you will get y g prime F prime and minus U infinity f ok g prime by g and what is del y by del eta? So, del eta by del y already we have found as g. So, it will be 1 by g right and you are multiplying with another g ok. So, these g g will get cancelled. So, you will finally, you will get nu g square d of d eta F double prime F prime minus U infinity y g prime F prime minus U infinity F g prime by g.

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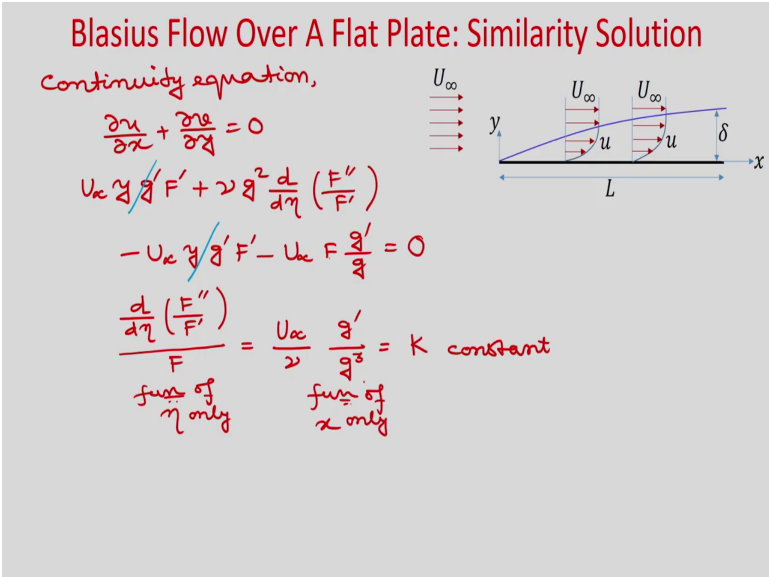

Now, let us consider the continuity equation and in the continuity equation del u by del x already we have found in terms of function of eta and also we have found del v by del y. So, we can write the continuity equation del u by del x plus del v by del y is equal to 0. So, del u by del x we have written as U infinity y g prime F prime and del v by del y we have written as nu g square d of d eta F double prime by F prime minus U infinity y g prime F prime minus U infinity F g prime by g is equal to 0.

So, you can see this term and this term will get cancel. So, we will get d of d eta F double prime by F prime divided by. So, here F is there. F is equal to U infinity by nu g prime by. So, g square if you bring in the denominator it will be g cube. So, here you can see in this equation we have separated the variables because left hand side it is function of eta only and right hand side it is function of x only.

So, we can write that these equal to some constant. So, equal to we can write K which is constant ok. So, now, you can see this is function of eta only and this is function of x only ok. So, we have separated the variables. So, it should be equal to some constant K.

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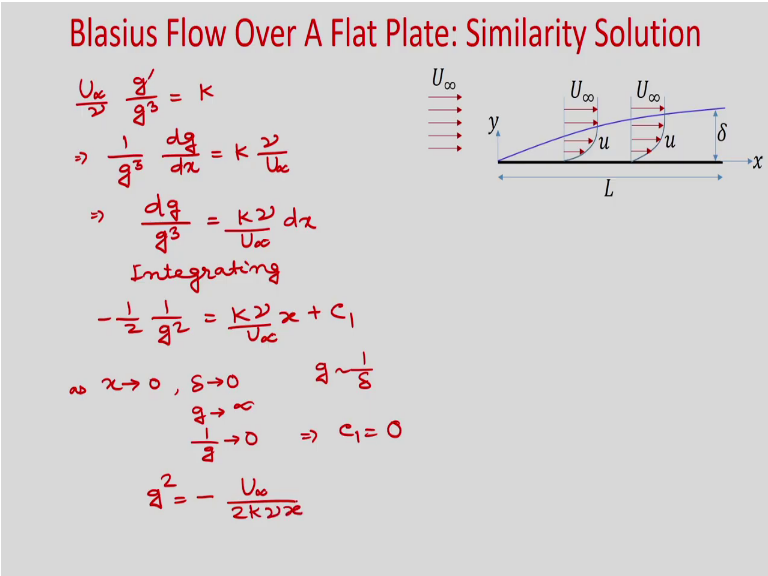

So, now, the we can write U infinity by nu g prime by g cube is equal to K. So, we can write 1 by g cube, d g by d x g prime we are writing as d g by d x is equal to K nu by U infinity ok. So, we can write d g by g cube is equal to K nu by U infinity d x ok. So, now, integrate this. So, integrating we can write. So, it g to the power minus 3. So, minus 3 plus 1. So, it will be minus 2. So, minus half 1 by g square is equal to K nu by U infinity x plus constant C 1.

So, integration constant C 1 now we can see that as x tends to 0; that means, near to the leading edge ok. So, x tends to 0 you can see delta tends to 0 right the boundary layer thickness tends to 0 that means, if x tends to 0 we know that g is the scale of 1 by delta. So, as x tends to 0. So, g tends to infinity; that means, 1 by g tends to 0 ok.

So, if 1 by g tends to 0 at x equal to 0 then you can see from here the integration constant will be C 1 is equal to 0. So, if C 1 is 0. So, you can write g square is equal to minus U infinity divided by twice K nu x.

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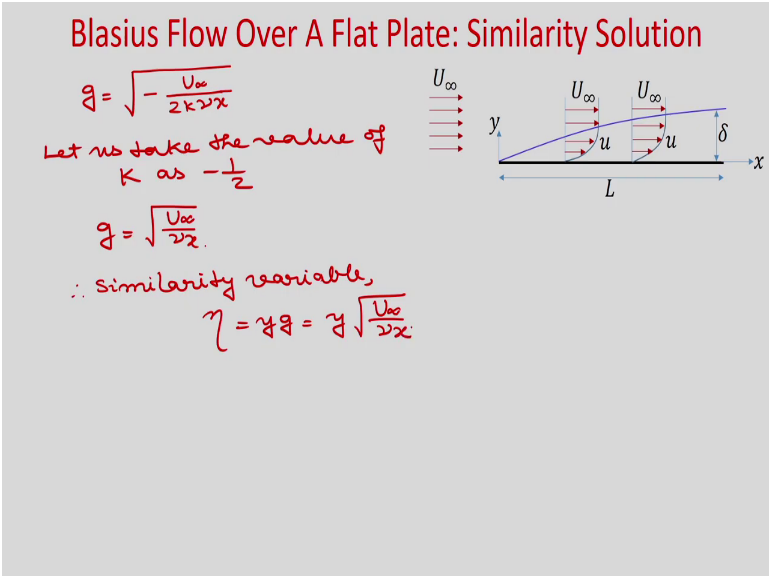

So, you can write g is equal to minus U infinity divided by twice K nu x whole square root ok. So, you can see that g is the scale of 1 by delta right. So; obviously, ah we cannot have the negative sign inside the root. So, what we can take that for convenience we can take the K value as minus half because K is a constant. So, any value you can take and for convenience we can take K is equal to minus half. So, let us take the value of K as minus half ok.

So, if K is minus half. So, now, you can write g. So, K is equal to minus half. So, it will become g is equal to root U infinity by nu x ok. So, we have the similarity variable eta is equal to y g; that means, y root U infinity by nu x. As we have found the value of K is equal to minus half. So, now, if we take the left hand side term that is which is function of eta then we can write d of d eta F double prime by F prime divided by F is equal to K and this is minus half ok.

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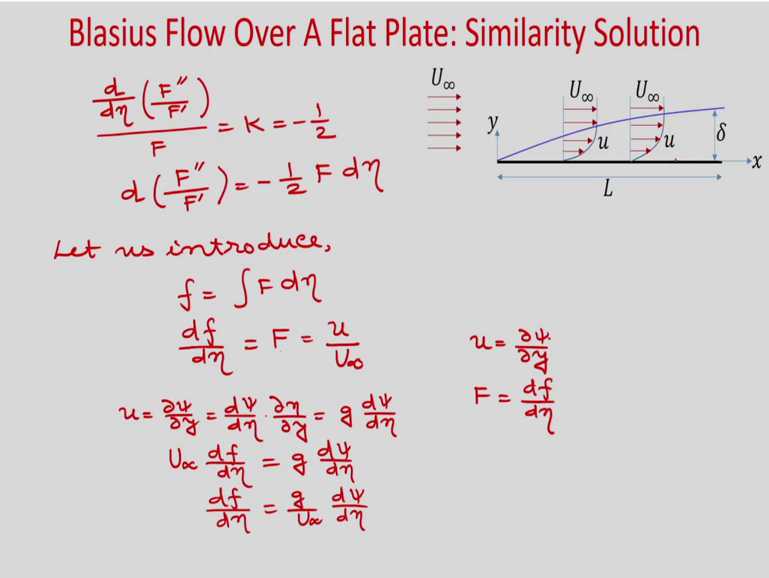

We can write d F double prime by F prime is equal to minus half F d eta. So, now, we will introduce one new variable small f ok where d f by d eta is equal to capital F. So, let us introduce that f is equal to integral F d eta; that means, we are introducing d f by d eta is equal to f and f is nothing, but u by U infinity right.

So, you can see that we have written this F, F is representation of the velocity right and this is d f by d eta. So, we are defining this F as integral F d eta. So, what is the physical significance of this F?

Like F capital F is the physical significance of that it is non dimensional velocity and what is the physical significance of this F? So, if you can see that whatever way we define the stream function the velocity in terms of the stream function gradient. So, we write u is equal to del psi by del y right.

So, you can see if you compare. So, you can see that we have written F is equal to d f by d eta so; obviously, you can see that this small f is equivalent to the stream function. So, the physical significance of this small f is the it is equivalent to the stream function ok. So, you can show that u is equal to del psi by del y ok. So, that is d psi by d eta into del eta by del y and del eta by del y is g. So, d psi by d eta.

And we have d f by d eta is equal to F is equal to u by u infinity. So, you can write u is equal to U infinity in to df by d eta is equal to g d psi by d eta; that means, df by d eta is equal to g by U infinity d psi by d eta. So, you can see that the physical significance of f is that it is a stream function ok and you know that the stream function will be constant at the wall ok.

So, along the wall we have one streamline and the it is a constant stream function. So, we can assume that psi is equal to 0 on the wall. So, now, let us simplify it.

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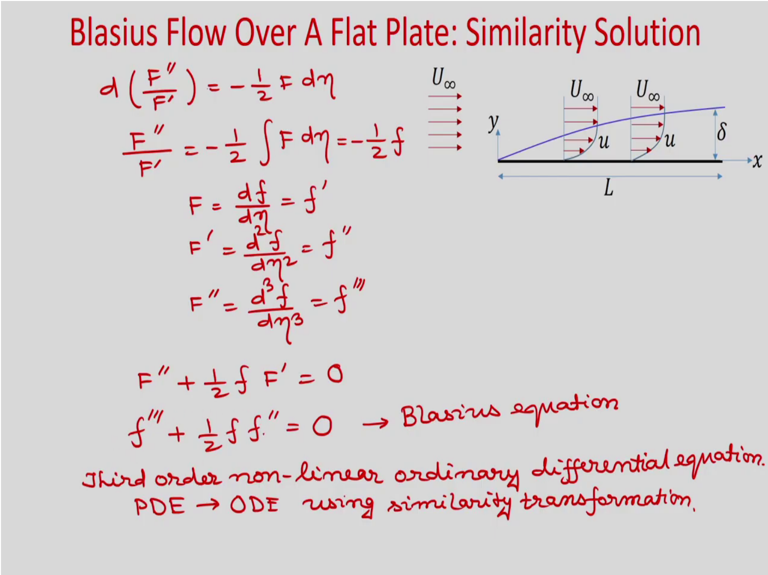

So, we can write d F double prime by F prime now we have written in terms of minus F d eta this we have written and now if you integrate then you can write F double prime by F prime is equal to minus half integral F d eta and integral F d eta we have defined as small f minus half f.

So, you can write that. So, we can see that F we have written as d f by d eta right which you can represent that F prime so, obviously, F prime will be d 2 f by d eta square. So, it will be F double prime and F double prime will be just d cube f by d eta cube is equal to f triple prime. So, now, if you put it here. So, this equation you can see you can write F double prime plus half f F prime is equal to 0.

So, F double prime is f triple prime plus half f and F prime is F double prime is equal to 0. So, you can see that this is ordinary differential equation. So, we used the similarity variable approach to convert the partial differential equation to ordinary differential equation. So, that we can integrate this equation and we can have the solution of the velocity profile.

Now, you can see this is a third order ordinary differential equation and it is non-linear because we have this term f, F double prime. So, this is non-linear equation. So, this is third ordered non-linear ordinary differential equation ok. So, this equation you can see that using similarity variable approach, we convert it the partial differential equation to ordinary differential equation ok using similarity variable approach or similarity transformation.

So, the equation whatever we have derived this equation is known as Blasius equation because Blasius first times solve this equation using similarity transformation and these equation is known as Blasius equation. Now let us discuss about the boundary conditions because to solve these ordinary differential equations you need three boundary conditions right because it is a third order ordinary differential equation so, you need three boundary conditions.

So, we know the boundary conditions in terms of velocity. So, at the wall you know that velocity is 0 mostly boundary condition at y tends to infinity; obviously, we have the u is equal to U infinity and at the same time there is no change in the velocity gradient outside the boundary layer right. So, at y tends to delta or y tends to infinity, we can write that del u by del y is equal to 0.

So, we have written this ordinary differential equation as a function of eta. So, let us convert these boundary conditions in terms of f.

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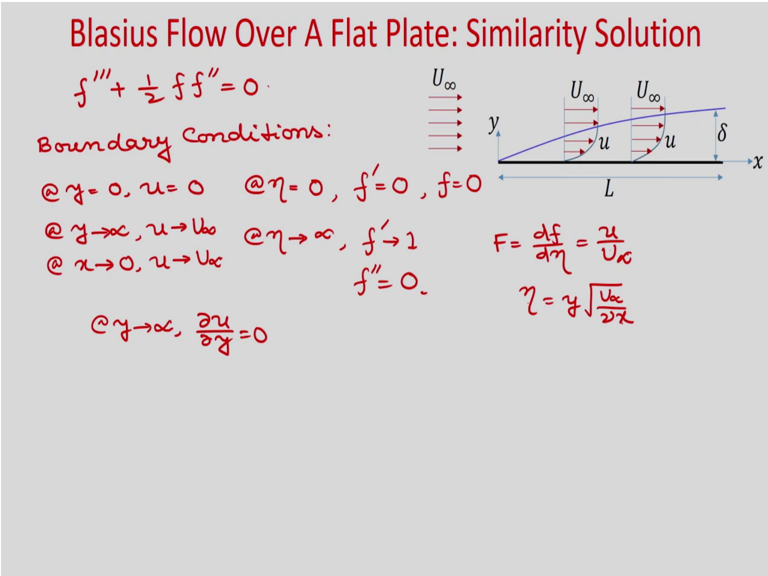

So, we have the equation f triple prime plus half f f double prime is equal to 0. So, we have you can see that highest order derivative is third order. So, you need three boundary conditions. So, you can write boundary conditions. So, at y is equal to 0 we have u is equal to 0 right.

So, you can write at eta is equal to 0 ok. So, you can see that if u is equal to 0. So, we have F is equal to d f by d eta and this we have written as u by U infinity. So, if u is equal to 0 you can see here u is equal to 0 then obviously, f will be 0; that means, F prime will be 0. So, eta is equal to 0 F prime is equal to 0 ok and also we are assuming that this is a streamline so, obviously, the stream function along the flat plate will be constant.

So, that constant we are taking let us say 0. So, we can write f is equal to 0 because f is the representation of the stream function and we are assuming that f is equal to 0 on the flat plate and at y tends to infinity we have u tends to U infinity ok and also you have at x tends to 0 we have u tends to U infinity. So, you can see at x tends to 0 we have the free stream velocity u infinity. So, these two boundary conditions we can combine as eta tends to infinity because you know that eta is equal to y root U infinity by nu x ok.

So, you can see that as y tends to infinity. So, eta tends to infinity and as x tends to 0 y tends to infinity. So, these two boundary conditions will club into one boundary condition that is at eta tends to infinity. So, we will have. So, u tends to infinity; that means, you will have that f prime as 1 ok. So, f prime tends to 1 because u tends to infinity. So, d f by d eta will be 1 that means, f prime tends to 1 and also we have the velocity gradient will be 0 right.

So, del u by del y will be 0; that means, d 2 f by d eta square will be 0; that means, f double prime will be 0. So, you can see that at y tends to infinity ok. So, we have the velocity gradient as 0 del u by del y is equal to 0 because we have the free stream velocity only right. So, outside the boundary layer we have the uniform free stream velocity infinity hence del u by del y is equal to 0 and if del u by del y is equal to 0; that means, d 2 f by d eta square is equal to 0 right.

So, that we are writing f double prime is equal to 0. So, now, we need to solve this ordinary differential equation along with these boundary conditions and you can use some suitable numerical techniques like Runge Kutta method and you can get the velocity profile from the solution of this ordinary differential equation. So, now after solutions we are plotting this velocity profile.

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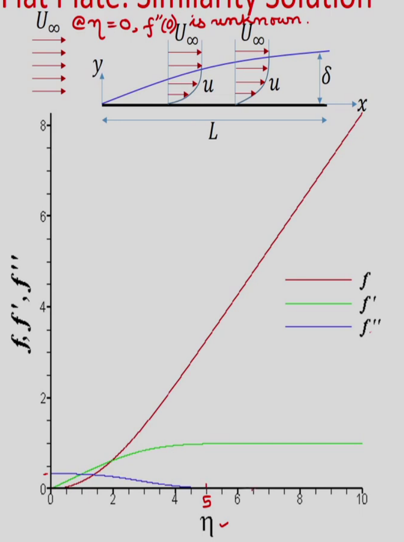
Blasius Flow Over A Flat Plate: Similarity Solution

$\eta = y\sqrt{\frac{U_\infty}{\nu x}}$	f	$\frac{df}{d\eta} = \frac{u}{U_\infty}$	$\frac{d^2 f}{d\eta^2}$
0.0	0.0	0.0	0.33206
0.4	0.02656	0.13277	0.33147
0.8	0.10611	0.26471	0.32739
1.2	0.23795	0.39378	0.31659
1.6	0.42032	0.51676	0.29667
2.0	0.65003	0.62977	0.26675
2.4	0.92230	0.72899	0.22809
2.8	1.23099	0.81152	0.18401
3.2	1.56911	0.87609	0.13913
3.6	1.92954	0.92333	0.09809
4.0	2.30576	0.95552	0.06424
4.4	2.69238	0.97587	0.03897
4.8	3.08534	0.98779	0.02187
5.0	3.28329	0.99155	0.01591
5.2	3.48189	0.99425	0.01134
5.4	3.68094	0.99616	0.00793
5.6	3.88031	0.99748	0.00543
6.0	4.27964	0.99898	0.00240
7.0	5.27926	0.99992	0.00022
8.0	6.27923	1.00000	0.00001



So, you can see that eta is y root u unitary by nu x, this is the f which is the representation of the stream function df by d eta which is representation of the velocity which is u by U infinity and this is the d 2 f by d eta square; that means, del u by del y; that means, this is the representation of shear stress.

Because the velocity gradient; obviously, at the wall will give the representation of the shear stress. So, along the boundary layer if you want to find the velocity gradient that is d 2 f by d eta square in this particular case you can see that at eta is equal to 0 f double prime 0 is unknown ok.

So, you have to assume some value of f double prime 0 to start with this solution and ah. So, if you assume that value of f double prime 0 and you will get some solution and as at eta tends to infinity d f by d eta should be 1 because that is the boundary condition we have written right at eta tends to infinity d f by d eta should be 1.

So, you assume that d 2 f by d eta square and you see at eta tends to infinity what is the value df by d eta? So, if it converges that as 1 and it does not change after that then obviously, that is the correct assume value at eta is equal to 0 of this f double prime 0 and f double prime 0 that means, you are assuming at eta is equal to 0.

That means at the wall what is the velocity gradient right f double prime 0 ok. So, you can see as you do not know the value of f double prime at eta is equal to 0. So, you need to assume some value. So, that is known as shooting technique right. So, because you are starting with some gauge value and see at eta tends to infinity if d f by d eta is approaching to 1.

So, you can trial an error method you just assume it, but you can also use some shooting technique and some numerical technical like Newton Raphson method you can use to correctly choose the value of F double prime at eta is equal to 0. If you give at eta is equal to 0 F double prime 0 as 0.33206 then you will see that at eta tends to infinity d f by d eta will become 1 and obviously, outside that d 2 f by d eta square as you have free stream velocity it will approach to 0.

So, at different value of eta you can see we have shown the f value, d f by d eta value and d 2 f by d eta square. So, you can see when it is approaching to eta is equal to 5, you can see that d f by d eta is almost approaching to one right it is approaching to 1 so; that means, you have that u is equal to 0.99 of U infinity right.

So, you can see that here we are plotting in ah this x coordinate we have eta, in the y coordinate we have different f f prime at f double prime. So, you can see that f at wall it will be 0. So, this red color. So, you can see this is the variation of f with eta and it is representation of the stream function f prime this is representation of the velocity. So, at the wall it should be 0. So, the green color plot is the plot of f prime so; that means, at wall the velocity is 0 and it is gradually increasing and it is approaching to 1 ok.

So, as eta is increasing it is approaching to one and at eta is equal to 5 you can see that it is almost becoming 1 and after that there is no variation of this velocity. And f double prime you can see it is representation of the shear stress. So, maximum value of f double prime will be at the wall.

So, you can see here we are getting ok. So, it is 0.332. So, after that as you go inside the boundary layer, it is decreasing and if you go outside the boundary layer thickness. So, or outside the boundary layer then velocity gradient will be 0 and it is becoming 0. So, you can see at eta is equal to 5 it is almost becoming 0. So, these ordinary differential equation along with this boundary condition and using this shooting technique, you can solve numerically and plot this f f prime and f double prime.

So, you can use fourth order Runge Kutta method to solve this ordinary differential equation. Now we are interested to find what is the boundary layer thickness. So, here you can see that we can define the boundary layer thickness where the velocity u almost becomes 0.99 of the free stream velocity U infinity.

So, we have seen from this plot that at eta is equal to 5, it becomes almost 0.99. So, we can write that at eta is equal to 5, we have we can define the boundary layer thickness.

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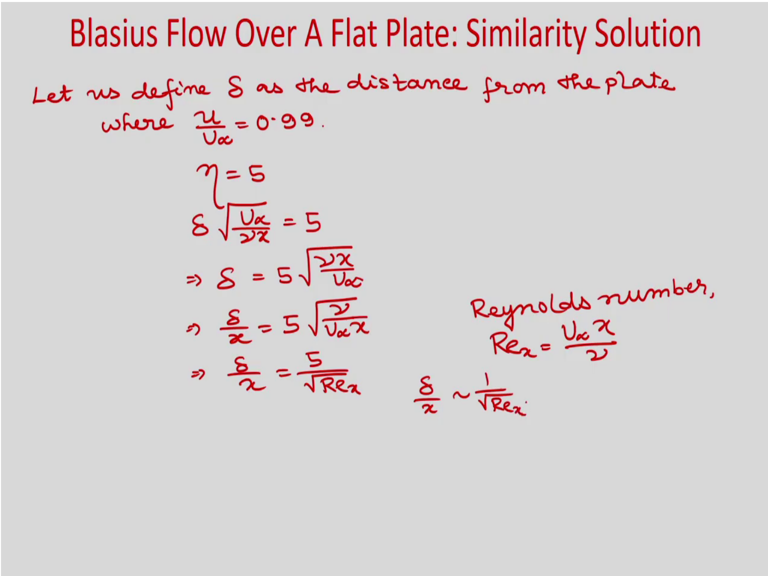

So, let us define delta which is your boundary layer thickness as the distance from the plate where u by U infinity will be 0.99 ok. So, eta equal to 5 it becomes 0.99. So, you can write delta U infinity by nu x because that is the y which is becoming the boundary layer thickness delta.

So, this is equal to 5. So, you can see that delta we can write as 5 root nu x by U infinite ok we can write delta by x is equal to 5 root nu by U infinity x. So, you can see that U infinity x by nu we can define the Reynolds number right. Reynolds number based on x. So, you can write Reynolds number as U infinity x by nu. So, you can write delta by x is equal to 5 by root R e x.

So, you can see from here that delta by x is order of 1 by root R e x. Now let us find what is the shear stress ok on the plate. So, first let us find the tau w.

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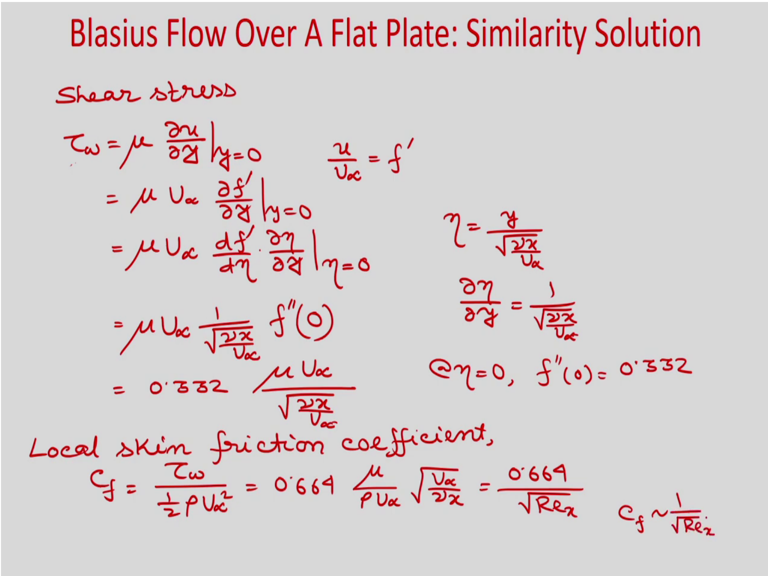

So, we can find the shear stress tau w as mu del u by del y at y is equal to 0 ok ah. So, we can write mu. So, you know u by U infinity as d f by d eta; that means, f prime. So, you can write del u by del y. So, you can write U infinity del f prime by del y at y is equal to 0 and we can write mu infinity df prime by d eta into del eta by del y ok.

At y is equal to 0; that means, eta is equal to 0. So, we know eta is equal to y by root nu x by U infinity. So, del eta by del y is 1 by root nu x by U infinity. So, if you put it here. So, we will get mu U infinity. So, del eta by del y is 1 by nu x by U infinity and this will become f double prime. So, f double prime at eta is equal to 0 ok.

So, you can see that at eta is equal to 0 f double prime 0 is 0.332 right 332 0 6. So, you can write 0.332. So, you can write as 0.332 mu U infinity divided by root nu x by U infinity. So, now, if you want to find what is the a skin friction coefficient local skin friction coefficient. So, you can write local skin friction coefficient. So, which is the representation of non-dimensional shear stress ok.

Which is the representation of non-dimensional shear stress. So, we can write C f is equal to tau w by half rho U infinity square ok. So, you can see from here it will become 0.664 mu by rho U infinity root U infinity by nu x ok. So, if you rearrange you can see you will get this as 0.664 by root R e x ok. So, in this case also you can see that local skin friction coefficient C f will be order of root R e x.

Now, if you want to find the total skin friction coefficient then you have to integrate it over the length right. Now let us find the drag force acting on the flat plate. So, let us assume that the plate width is b.

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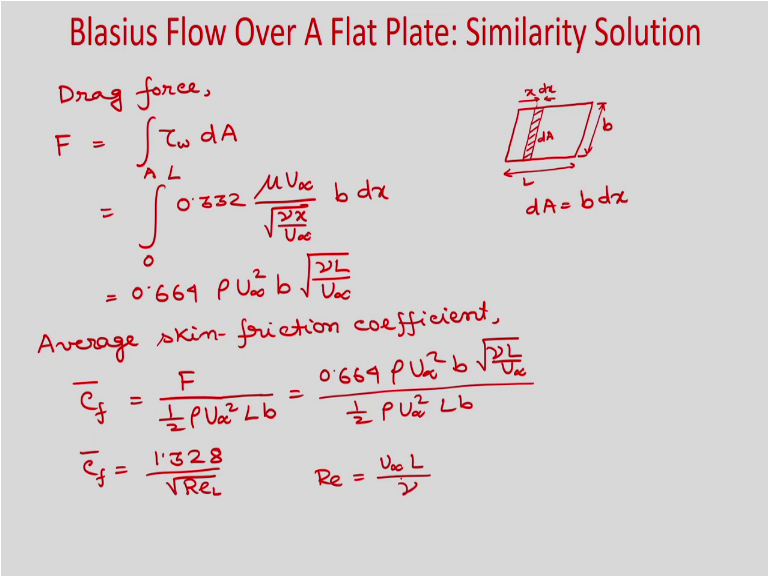

So, if we have this plate ok of length L and width is b and flow is taking place over this flat plate. So, at a distance x if you take one small elemental strip of length d x. So, you can see. So, this is the small area d a where d a in this case it is b into d x ok.

So, this shear stress is acting on this elemental area that is tau w and now we integrate over this surface then we will get the drag force. So, drag force we can calculate as F is equal to the area integral of this tau w into d A.

So, d A is equal to b dx. So, you can see tau w we have written as point 0.332 mu U infinity divided by root nu x by U infinity and d A is b dx and now x varies from 0 to L 0 to L ok. So, if you integrate it you will get 0.664 rho U infinity square b root nu L by U infinity ok. Now let us find the average skin friction coefficient ok.

So, C f bar we can write the non dimensional form of the drag force; that means, F by half rho U infinity square into the area. So, area is L into b. So, L into b ok. So, this is the f. So, 0.664 rho U infinity square b root nu l by U infinity divided by half rho U infinity square L into b. So, if you simplify it you are going to get this average skin friction coefficient as 1.328 divided by root R e l.

So, these Reynolds number we are defining based on the length of the plate; that means, U infinity which is free stream velocity L divided by the kinematic viscosity nu. So, in today's class we considered flow over a flat plate and we use the similarity transformation to convert these partial differential equations to ordinary differential equation.

Here we defined the similarity variable eta as function of two independent variables x and y, y into g x where g is the scale of 1 by delta because delta is the boundary layer thickness and delta varies with x. Using this similarity transformation we found the

Blasius equation which is third order non-linear ordinary differential equation and we discussed about the boundary conditions then you can use some numerical technique to solve the ordinary differential equation along with the boundary conditions.

Then we have shown the variation of f, f prime and f double prime with eta f is the representation of stream function, f prime which is representation of the velocity and f double prime is the representation of the shear stress. Then we have calculate the skin friction coefficient, the drag force and the average skin friction coefficient and we have shown that the boundary layer thickness delta and the skin friction coefficient C f is order of 1 by root Re x where Re x is the Reynolds number based on the length x.

Thank you.