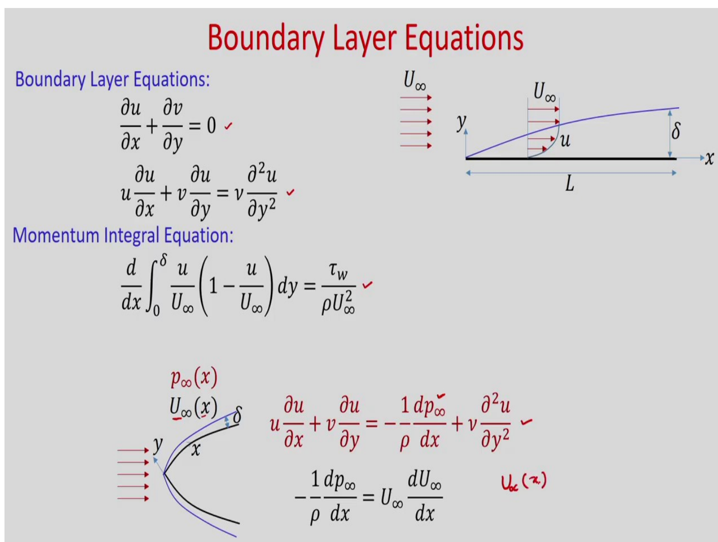
Viscous Fluid Flow
Prof. Amaresh Dalal
Department of Mechanical Engineering
Indian Institute of Technology, Guwahati

Module - 08
Laminar Boundary Layers - II
Lecture - 01
Karman-Pohlhausen Method for Non-zero Pressure Gradient Flows

Hello everyone. So, in today's class, we will first derive the momentum integral equation considering the general boundary layer momentum equation with non-zero pressure gradient and then, we will solve this momentum integral equation using Karman-Pohlhausen Method for flows with Non-Zero Pressure Gradient.

(Refer Slide Time: 00:59)



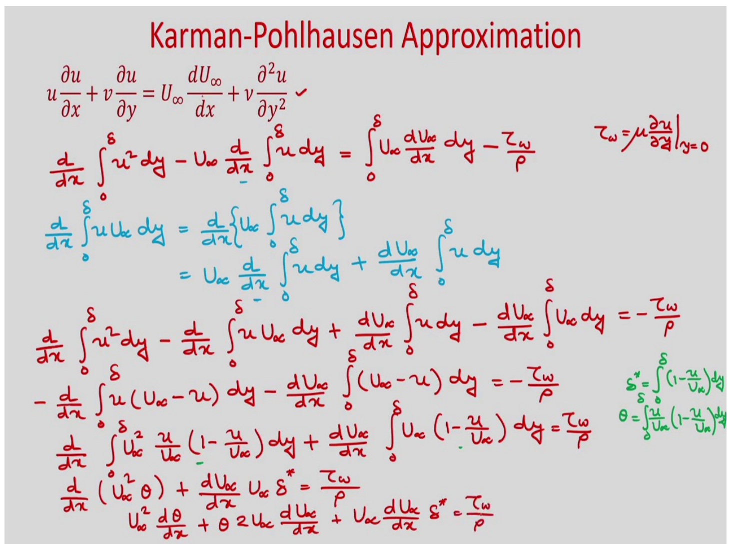
As you know that for flow over flat plate, already we have written this boundary layer equation. This is the continuity equation and this is the momentum equation and using approximate method, we have derived this momentum integral equation. Now, we will consider non-zero pressure gradient.

So, we consider where this free stream velocity U infinity is function of x; where, x is the direction along the surface and y is the normal to the surface and delta is the boundary layer thickness at any location x. So, for this, this is the general boundary layer equation, where we have non-zero pressure gradient.

And using Bernoulli's equation, we can write this pressure gradient term minus 1 by rho dp infinity by dx equal to U infinity dU infinity by dx; where, U infinity is function of x. Now, we will adopt the same procedure which we discussed in lecture 3 of module 7, we will consider this boundary layer equation.

We will integrate inside the boundary layer and we will derive the momentum integral equation. The difference with the earlier lecture is that we have one additional term which is pressure gradient term.

(Refer Slide Time: 02:40)



So, you recall the earlier lecture, where we derived this momentum integral equation. So, this is the boundary layer equation. Now, if you integrate it and follow the same procedure as we have derived in lecture 3 of module 7, first let us write the left-hand side.

So, if you remember the left-hand side, we have written as d of dx integral 0 to delta u square dy is equal to minus U infinity d of dx integral 0 to delta u dy; so, this is the left-hand side terms and in the right-hand side, we have now this term ok. So, we need to integrate 0 to delta U infinity dU infinity by dx dy and this term will become minus tau w by rho, where tau w is the wall shear stress; that means, tau w is equal to mu del u by del y at y is equal to 0.

So, you can see that U infinity is function of x; but this integral is we are integrating with respect to dy. So, obviously, this term we can take outside the integral because this U infinity and dU infinity by dx are function of x. Now, let us write this term d of dx integral 0 to delta u infinity dy ok.

So, now this we can write as U infinity is function of x, you can take it outside the integral. So, you can write d of dx U infinity integral 0 to delta u dy. So, this term, now we can write as U infinity d of dx integral 0 to delta u dy plus dU infinity by dx integral 0 to delta u dy.

So, if you see we have this term; so, this term is equivalent to this term. So, we will replace this term with this minus this. So, now we can write this d of dx integral 0 to delta u square dy, so if you take in the left-hand side; so, in the right-hand side, we can write minus d of dx integral 0 to delta u U infinity dy plus dU infinity by dx integral 0 to delta u dy and whatever right-hand side, this term we can take in the left-hand side and it will become minus.

So, dU infinity by dx we are writing outside the integral; integral 0 to delta U infinity dy is equal to minus tau w by rho. So, now these two if we write together, we can write minus d of dx integral 0 to delta we will take u outside. So, it will become U infinity minus u dy and minus dU infinity by dx, if we take outside, then we can write integral 0 to delta U infinity minus u dy is equal to minus tau w by rho ok.

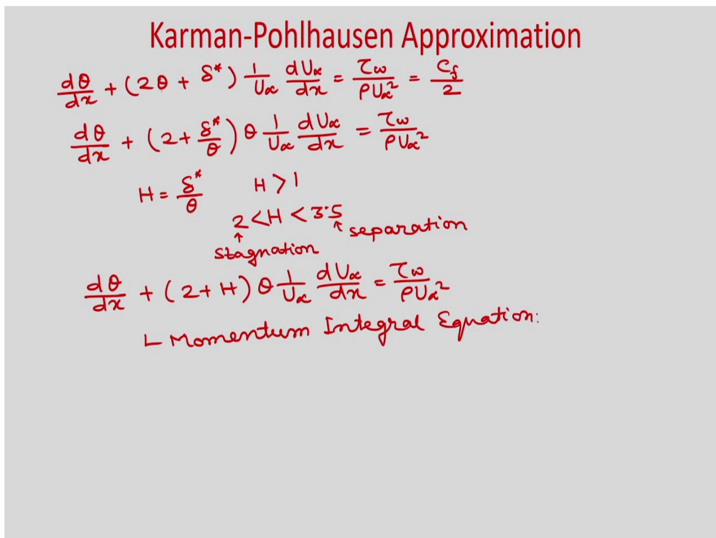
So, just multiply both side with negative sign. So, we will get d of dx integral 0 to delta. Now, we are writing U infinity square, we are multiplying and also dividing by infinity square. So, one will take at u by U infinity. So, it will become 1 minus u by U infinity dy minus it will become plus dU infinity by dx integral 0 to delta U infinity 1 minus u by U infinity dy is equal to tau w by rho.

So, now if you recall the displacement thickness. So, we have defined delta star as displacement thickness as integral 0 to delta 1 minus u by U infinity dy and momentum thickness theta we can write integral 0 to delta u by U infinity 1 minus u by U infinity dy.

So, now, this term, we can replace with momentum thickness and this term, we can replace with displacement thickness. As U infinity is not function of y, so you can take it outside the integral. So, you can write d of dx U infinity square theta plus dU infinity by dx. U infinity, you are taking outside the integral, it will become delta star is equal to tau w by rho.

Now, if you further simplify it, we can write as U infinity square d theta by dx plus theta 2 U infinity dU infinity by dx plus U infinity dU infinity by dx delta star is equal to tau w by rho. Now, let us divide both side with U infinity square; but remember U infinity is function of x.

(Refer Slide Time: 09:23)

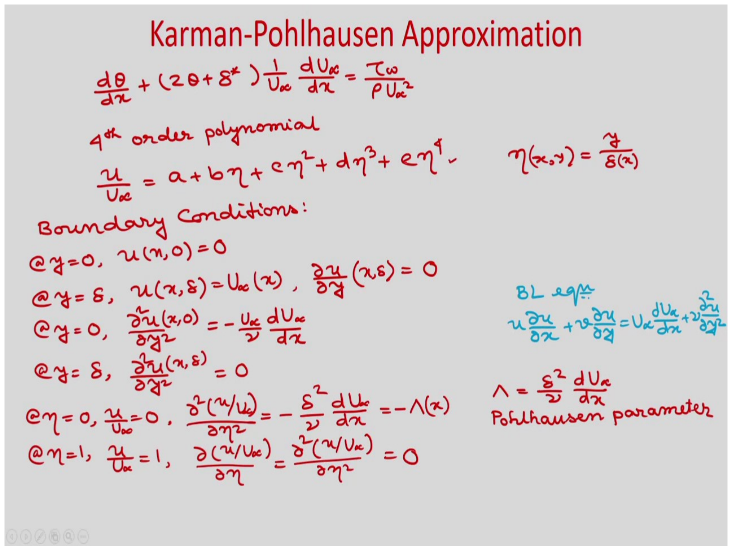


So, dividing both side by U infinity square, we can write d theta by dx plus 2 theta plus delta star 1 by U infinity dU infinity by dx is equal to tau w by rho U infinity square. So, this we can also write C f by 2. So, here now if we take theta outside, we can write d theta by dx into 2 plus delta star by theta theta 1 by U infinity dU infinity by dx is equal to tau w by rho U infinity square.

So, if you recall that we have defined shape factor H as delta star by theta right. So, we know that H must be greater than 1 and H value between 3.5 and 2 ok. So, in 2, you will get at stagnation point and this is the separation flow ok. So, you can write the momentum integral equation as d theta by dx into 2 plus H theta 1 by U infinity dU infinity by dx is equal to tau w by rho U infinity square.

So, this is the Momentum Integral Equation. So, now, we will use Karman-Pohlhausen method to solve this momentum integral equation. So, for that, we will assume the velocity profile as 4th degree polynomial.

(Refer Slide Time: 11:32)



So, we have the momentum integral equation as d theta by dx plus 2 theta plus delta star 1 by U infinity dU infinity by dx is equal to tau w by rho U infinity square. Now, we will assume the velocity profile as 4th order polynomial. So, we will write u by U infinity as a plus b eta plus c eta square plus d eta cube plus e eta to the power 4; where, eta is function of x, y right.

And it is y by delta, where delta is function of x. So, now, you can see that we have 5 coefficients, these coefficients are function of x. So, we need 5 boundary conditions. So, let us write down the boundary conditions; 3 boundary condition, we will get in straightforward way and 2 boundary conditions, we will derive from the governing equation.

So, boundary conditions, we can write as at y is equal to 0 ok, u is equal to 0 ok; at y is equal to delta, u will be the free stream velocity U infinity which is function of x and also, we have the del u by del y at x delta will be 0. Because U infinity x does not vary along the y. So, del u by del y will be 0. Now, let us write down the boundary layer equation and satisfy it at the wall as well as at the edge of the bound layer so that we get two derived boundary conditions.

So, what is your boundary layer equation? So, boundary layer equation, we have written as u del u by del x plus v del u by del y is equal to U infinity dU infinity by dx plus nu del 2 u by del y square ok. So, now if you satisfy it at the wall; that means, at y is equal to 0, u is equal to 0, v equal to 0.

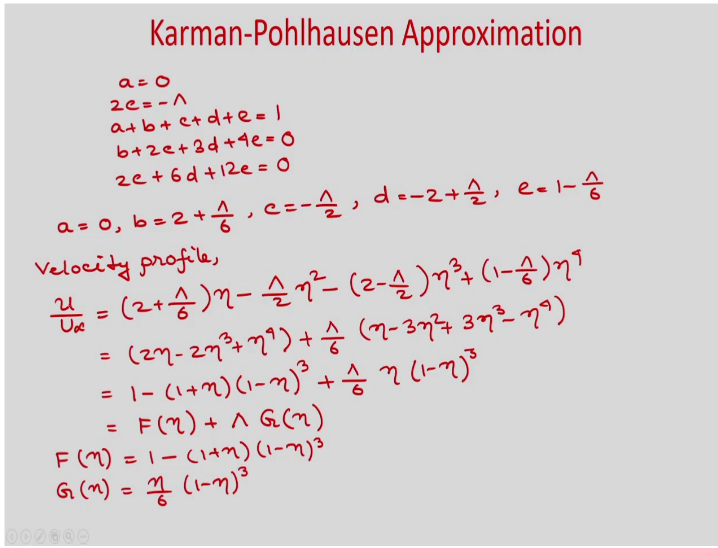
So, that means, inertia terms are 0. So, you will get the boundary condition at y is equal to 0, del 2 u at y is equal to 0 is equal to minus U infinity by nu dU infinity by dx ok and at y is equal to delta, what will happen? So, at y is equal to delta you can see that del u by del y will become 0 ok. And at the edge of the boundary layer, obviously, you can see that we have only U infinity which is function of x and it does not vary with u infinity.

So, del u by del y is equal to 0 everywhere. So, you will get del 2 u by del y square also will be 0 everywhere. So, it will be 0. So, now, we can write these boundary conditions in terms of u by U infinity. So, we can write at eta is equal to 0, u by u is equal to 0, del 2 u by u del eta square is equal to minus delta square by nu dU by dx; sorry, it is U infinity. So, dU infinity by dx.

So, you can see that now this parameter we will just write as lambda. So, we will write lambda as delta square by nu dU infinity by dx and this is known as Pohlhausen parameter; Pohlhausen parameter ok. So, this will be just minus lambda and you can see that delta is function of x, dU infinity by dx is function of x, it will be function of x.

And similarly, at eta is equal to 1, u by U infinity will be 1 and del u by U infinity divided by del eta and del 2 u by U infinity by del eta square will also be 0 ok. So, from these boundary conditions, we have written in terms of u by U infinity. Now, you find satisfying these boundary conditions from here, you find the values of these five coefficients.

(Refer Slide Time: 16:59)



So, if you satisfy the boundary conditions, you will get a is equal to 0; 2c is equal to minus lambda; a plus b plus c plus d plus e is equal to 1; b plus 2c plus 3d plus 4e is equal to 0 and 2c plus 6d plus 12e is equal to 0. So, from this 5 equations, you find the values of a, b, c, d and e.

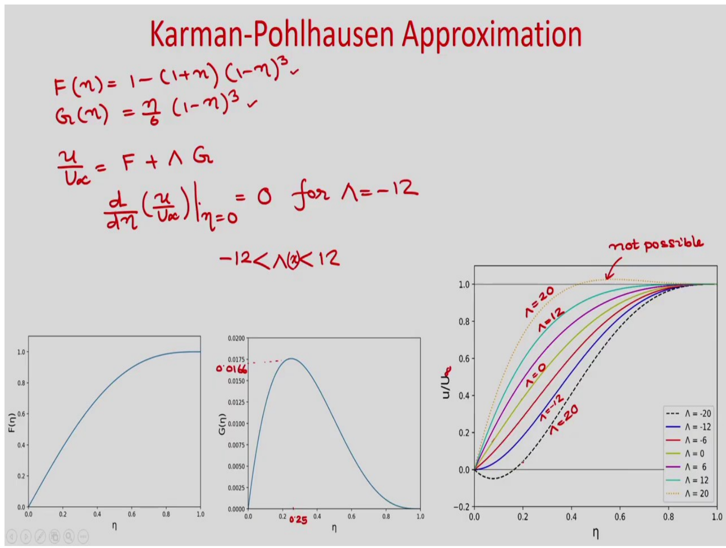
So, we can write the final value a is equal to 0; b is equal to 2 plus lambda by 6; c is equal to minus lambda by 2; d is equal to minus 2 plus lambda by 2 and e is equal to 1 minus lambda by 6 ok. So, now all these coefficients, you substitute in the polynomial of these velocity profile. So, you will get the velocity profile u by U infinity as 2 plus lambda by 6 eta minus lambda by 2 eta square minus 2 minus lambda by 2 eta cube plus 1 minus lambda by 6 eta to the power 4 ok.

So, if you rearrange you will get, 2 eta minus 2 eta cube plus eta to the power 4 plus lambda by 6 eta minus 3 eta square plus 3 eta cube minus eta to the power 4. So, further if you simplify you will get, 1 minus 1 plus eta 1 minus eta whole cube plus gamma by 6 eta 1 minus eta whole cube ok.

So, now, you can see that this term, we can represent as F eta and this term 1 by 6 eta 1 minus eta cube, we can represent with another function G. So, we can write gamma G eta. So, we can see that F eta is a function of eta as 1 minus 1 plus eta 1 minus eta cube and G is the function eta by 6, 1 minus eta cube ok.

Now, what we will do? We will just plot u by U infinity versus eta for different values of this lambda ok. So, lambda is Pohlhausen parameter. So, for different values of lambda, we will plot the u by U infinity.

(Refer Slide Time: 20:11)



So, you can see that if you plot this F versus eta ok. So, this function F monotonically increases with eta ok. And if you see the G. So, if you write down the function F as 1 minus 1 plus eta 1 minus eta whole cube; G as eta by 6 1 minus eta cube and the velocity profile u by U infinity as F plus lambda G ok.

So, this G if u plot with eta, you can see here that G first increases from 0, at eta is equal to 0 to a maximum value. So, this maximum value will be 0.0166 at eta is equal to 0.25 ok. So, somewhere here, it will be 0.25. Then, it drops up to 0 at eta is equal to 1. So, these are the variation of these functions F and G.

And then, if you plot the velocity profile u by U infinity versus eta for different values of lambda ok; lambda you can see that this is a Pohlhausen parameter which is a pressure parameter because we have the velocity gradient dU infinity by dx. So, if lambda is equal to 0; that means, you do not have any pressure gradient that means, flow over a flat plate.

If you see for lambda is equal to 0, so this is the plot lambda is equal to 0. So, this is for flow over flat plate and at eta tends to 1, you can see that u become U infinity ok. That means, at y is eta we have defined as y by delta. So, that means, at the edge of the boundary layer, it will become 1 and u will become U infinity; that means, u by U infinity is becoming 1.

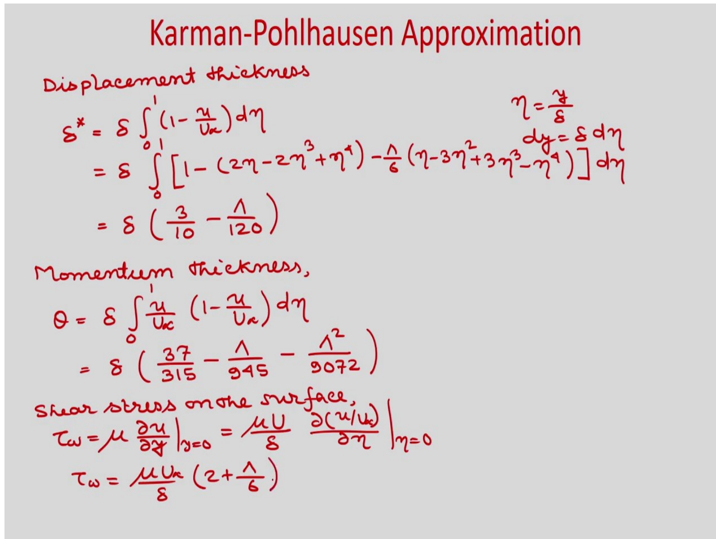
So, this is the lambda is equal to 12 and if you further increase the value of lambda, you can see that u is becoming higher than the free stream velocity U infinity, which is not physically correct right. So, this is for lambda is equal to 20. So, this is not physically correct. So, it is not possible right. This is not possible.

If you decrease the value of lambda; so, you can see this is lambda is equal to minus 6, this is lambda is equal to minus 12. So, if you see here lambda is equal to minus 12, actually you will get the del u by U infinity at eta is equal to 0; that means, d of d eta at u by U infinity at eta is equal to 0 will be 0 for lambda is equal to minus 12. So, that means, at that point, the separation will takes place ok.

If you further increase the value of lambda, let us say lambda is equal to 20. So, you can see that we have a negative velocity; that means, flow is reversed right and boundary layer theory is not valid in these flow separations right. So, that means, lambda value we have to be limited in the range of less than 12 and greater than minus 12 ok.

So, this lambda value which is function of x should be in between minus 12 and 12. Because if it goes above lambdas is equal to 12, then velocity will become greater than 1; u by U infinity will become greater than 1 which is not possible and if you go below lambda value of minus 12, then back flow will occur, where this boundary layer theory is not valid. So, now let us find the displacement thickness and the momentum thickness and then, we will find the shear stress.

(Refer Slide Time: 24:48)



So, now if we find the displacement thickness, so delta star we can write as delta 0 to 1 1 minus u by U infinity d eta ok because we have written eta is equal to y by delta. So, dy you can write delta d eta.

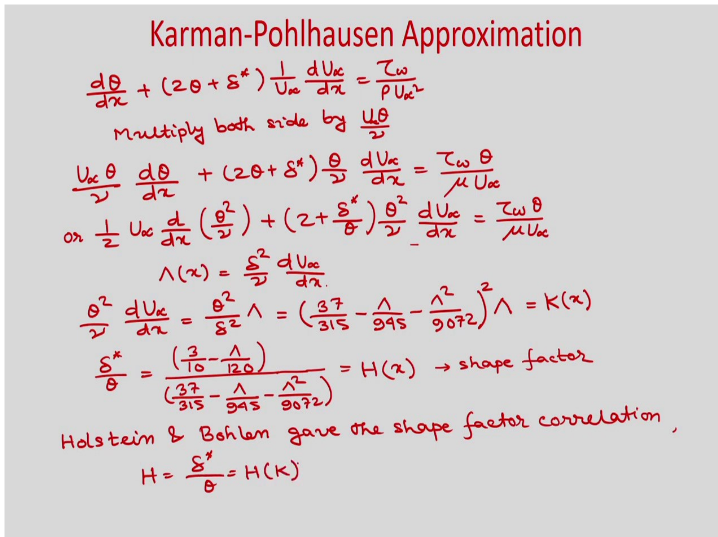
So, we have changed the limit from 0 to 1. So, if you put delta 0 to 1 1 minus u by U infinity, we have as 2 eta minus 2 eta cube plus eta to the power 4 minus lambda by 6 eta minus 3 eta square plus 3 eta cube minus eta to the power 4 d eta. So, if you perform this integration, you will get delta 3 by 10 minus lambda by 120.

Similarly, if you evaluate the momentum thickness. So, theta we can write as delta integral 0 to 1 u by U infinity 1 minus u by U infinity d eta ok. So, if you put the value of u by U infinity value here and evaluate the integral, we will get theta is equal to delta 37 by 315 minus lambda by 945 minus lambda square 9072.

You can see that now we have represented these displacement thickness and momentum thickness in terms of unknown parameter delta, which is boundary layer thickness. Now, let us find what is the wall shear stress. So, you can see that shear stress on the surface tau w will be just mu del u by del y at y is equal to 0. So, it will be mu U by delta del u by U del eta at eta is equal to 0.

So, now you put and evaluate this derivative; then finally, you will get tau w as mu U infinity by delta 2 plus lambda by 6. So, you can see now all the terms; this displacement thickness, momentum thickness and the wall shear stress, we have represented in terms of unknown parameter delta. So, now, we need to evaluate the delta and for that, we will use the momentum integral equation.

(Refer Slide Time: 28:15)



So, we have this momentum integral equation d theta by dx, we have derived today 2 theta plus delta star 1 by U infinity dU infinity by dx is equal to tau w by rho U infinity square. Now, you can see that we have evaluated theta, delta star, tau w in terms of delta. So, now, you multiply both side by U infinity theta by nu ok.

So, if you do that you will get, U infinity theta by nu d theta by dx plus 2 theta plus delta star theta by nu dU infinity by dx is equal to tau w theta by mu U infinity ok. So, further you can take it inside. So, you can write half U infinity d of dx theta square by nu plus 2 plus delta star by theta.

So, if you take theta outside, so it will become theta square by nu dU infinity by dx is equal to tau w theta by mu U infinity ok. So, now what we will do? We will just express all these terms in terms of lambda, which we have already represented as delta square by nu dU infinity by dx.

So, you can see that we can represent this term, theta square by nu dU infinity by dx as theta square by delta square lambda because this is the lambda value and lambda value you know. So, theta already we have represented in terms of lambda.

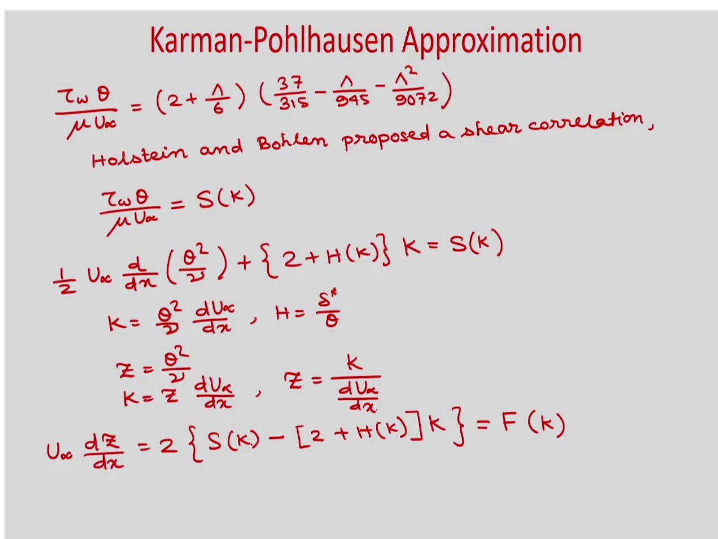
So, we can write 37 by 315 minus lambda by 945 minus lambda square by 9072 whole square lambda. You can see that let us represent that this as K which is function of x because you can see that lambda is function of x right.

So, K will be function of x. And delta star by theta we can represent as; so, delta star we have found as 3 by 10 minus lambda by 120 in to delta and denominator also you will get delta into the 37 by 315 minus lambda by 945 minus lambda square by 9072 ok.

So, these we have already represented delta star by theta as shape factor H right. So, this is your H. So, you can see that lambda is function of x, so H is also function of x. So, this is your shape factor because it depends on the profile of the velocity. So, now, you can see that H is function of x, K is function of x.

So, and H is function of lambda and K is functional lambda. So, obviously, we can write. So, actually Holstein and Bohlen, they actually gave the shape factor correlation; gave the shape factor correlation ok as H is equal to delta star by theta which is function of also K ok. This is the K.

(Refer Slide Time: 32:56)



So, now evaluate this right-hand side term. So, we have right-hand side term as tau w theta by mu U infinity. So, tau w we have expressed in terms of lambda, theta also we have expressed in terms of lambda. If you put it here, if you simplify, you will get 2 plus lambda by 6 into 37 by 315 minus lambda by 945 minus lambda square by 9072 ok and this is also function of x ok.

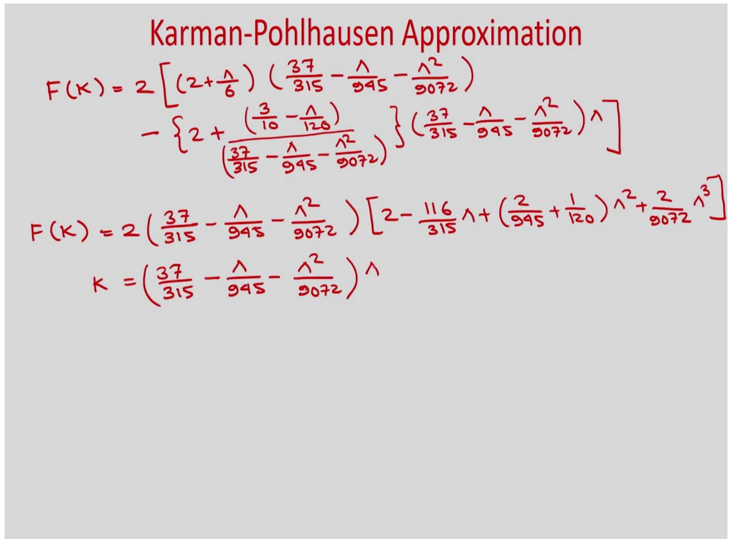
So, here also Holstein and Bohlen proposed a shear correlation. So, Holstein and Bohlen proposed a shear correlation ok. So, tau w theta by mu U infinity is equal to S which is your shear correlation which is function of K because K is function of x right. So, this also we can write in function of x.

So, now let us write down the momentum integral equation as half U infinity d of dx theta square by nu plus 2 plus H which is function of K, then this is K is equal to S which is function of K ok; where, we have seen that K is equal to theta square by nu dU by dx, H is delta star by theta ok.

Now, if you write that Z is equal to theta square by nu ok. So, we can write. So, if Z is equal to theta square by nu. So, from here, you can see that we can write K is equal to theta square by nu is Z dU infinity by dx ok. So, from here Z, we can write as K by dU infinity by dx ok.

So, now, if you put it here, you can see that we can write as U infinity dZ by dx because Z, we have written as theta square by nu is equal to 2 S minus 2 plus H which is function of K then K ok. So, you can see that U infinity dZ by dx, we have represented in terms of these correlation and this, now we can write as F which is function of K. So, we know what is S K; S K is nothing but this and H K also we can represent in terms of lambda; then, we can write this function F K ok. So, F K you can write in terms of lambda.

(Refer Slide Time: 36:30)

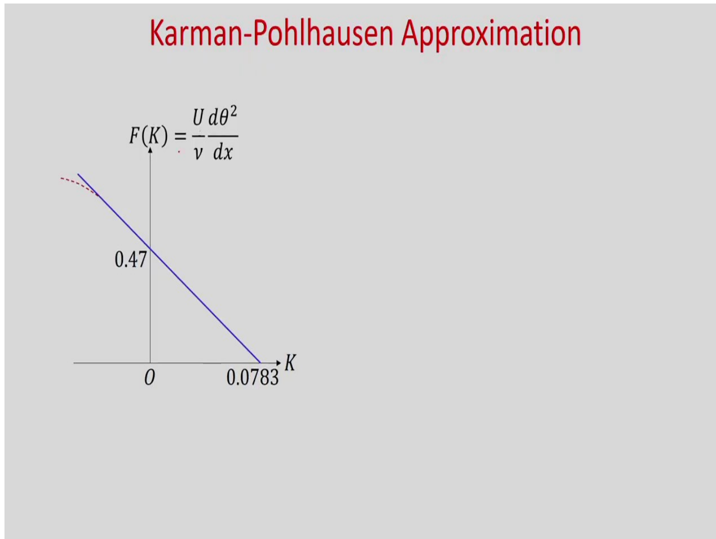


So, if you write F K in terms of lambda, then we will get 2, 2 plus lambda by 6, 37 by 315 minus lambda by 945 minus lambda square by 9072 minus 2 plus 3 by 10 minus lambda by 120 divided by 37 by 315 minus lambda by 945 minus lambda square by 9072.

Now, K if you write in terms of lambda, then we will get 37 by 315 minus lambda by 945 minus lambda square by 9072 into lambda ok. So, this is the function K and if you simplify it, you will get F K as 2 into 37 by 315 minus lambda by 945 minus lambda square by 9072; 2 minus 116 by 315 lambda plus 2 by 945 plus 1 by 120 lambda square plus 2 by 9072 lambda cube.

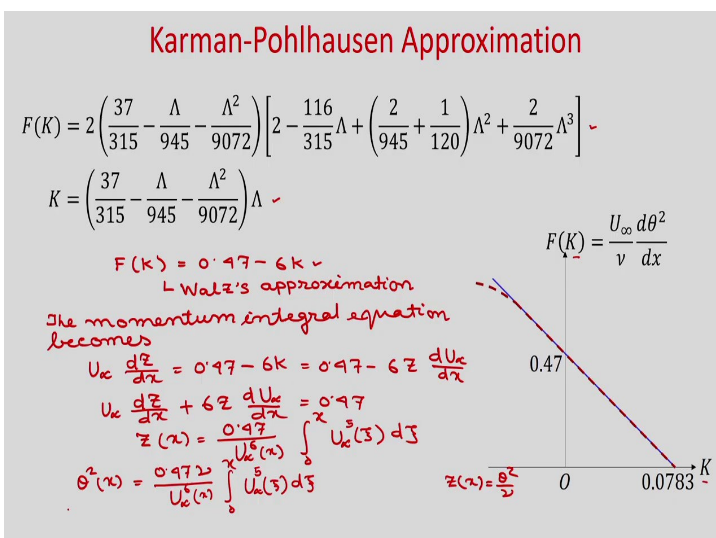
And K, we have represented already as 37 by 315 minus lambda by 945 minus lambda square by 9072 into lambda. So, now, you can see that lambda is the pressure parameter and which is function of x. So, for different values of lambda, if you find the K value and the value of this function F which is function of K and if you plot it, then we will get.

(Refer Slide Time: 39:00)



So, you can see that F K; F K is nothing but U by nu d theta square by dx. So, this if we this is U infinity. So, this in y axis and in the x axis if you plot for K, so for different values of lambda if you plot it, so you will get this curve ok and you can see that you can approximate this curve as a straight line right. Because you can see that this is a very complicated function right and it is very difficult to evaluate the value of F Ks.

(Refer Slide Time: 39:40)



So, from momentum integral equation, we found this F K, where K is also given by in terms of lambda like this. So, now, if you plot these functions F K versus K; so, for different values of lambda, then you will get this line red color dotted line. So, this is the plot of these functions F K versus K and for different values of lambda, you will see that all will fall in the same curve and this is the plot of F K versus K.

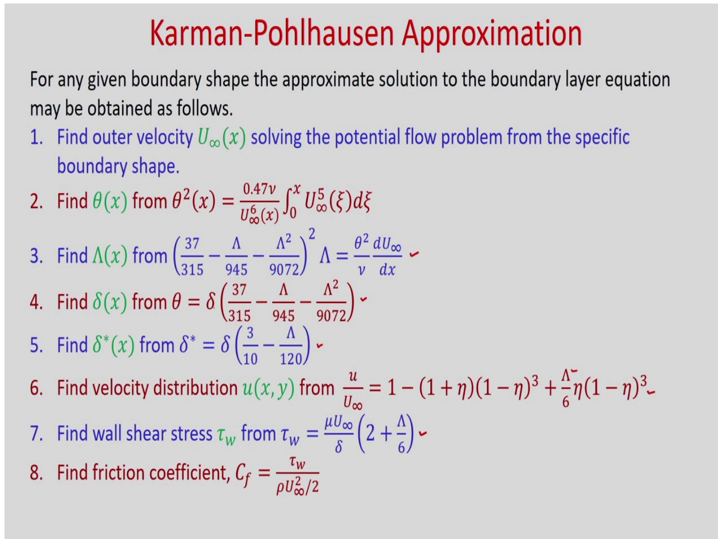
Now, you can see that it can be approximated by a straight line and we can write the equation of this straight line as F K is equal to 0.47 minus 6K. So, first Walz approximate it. So, due to his name, it is known as Walz's approximation. So, now, you can see that these two functions are very complex and very difficult to evaluate; but if we use this simple function linear function, then it will be easy to calculate the momentum thickness, displacement thickness and the shear stress.

So, if you use this function, then obviously, the momentum integral equation will become; the momentum integral equation becomes so you can see from here U infinity dZ by dx is equal to 0.47 minus 6K. So, now, if you write the value of K, then you can write 0.47 minus 6Z dU infinity by dx. So, if you rearrange, you can write as U infinity dZ by dx plus 6Z dU infinity by dx is equal to 0.47 ok.

So, now, if you do some simplification, then we can write this Z x is equal to 0.47 divided by U infinity to the power 6 integral 0 to x U infinity to the power 5 zeta d zeta ok. So, this after integration, you can write Z x equal to this and now, you can find the Z x is equal to theta square by nu right. Z x equal to theta square by nu.

So, theta square you can write as 0.47 nu divided by U infinity to the power 6 integral 0 to x U infinity zeta d zeta U infinity to the power 5. So, you can see once you know theta, then the other parameters will be able to calculate.

(Refer Slide Time: 43:38)



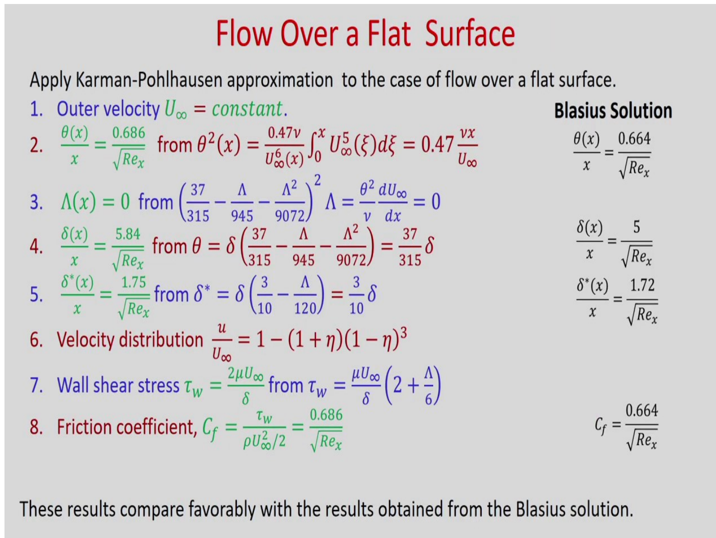
So, you can see for any given boundary shape, the approximate solution to the boundary layer equation may be obtained as follows. First you find the outer velocity U infinity ok; solving the potential flow problem from the specific boundary shape. So, once you know U infinity, now you can evaluate theta x ok.

So, from this expression from theta square x is equal to 0.47 nu by U infinity to the power 6 x integral 0 to x U infinity to the power 5 zeta d zeta. Now, once you know theta, then you find the lambda ok, from this expression ok. So, you can see that theta is known and U infinity is known. So, dU infinity by dx is known.

Then, you find delta x which is your boundary layer thickness from this expression ok. So, here we have already found the lambda. So, from here theta is known. So, delta you can find from this expression. Then, you find the displacement thickness delta star. So, delta star expression is this. So, lambda and delta are known. So, you can find delta star and also, you can find the velocity distribution from this expression, where lambda is known for a.

Now, find wall shear stress from this expression ok, because lambda and delta are known. So, obviously, and U infinity is also known. So, from this expression, you will be able to calculate the wall shear stress. So, once you know the wall shear stress, then you can calculate the friction coefficient because our ultimate goal is to find the wall shear stress and the displacement thickness and other parameters like momentum thickness and boundary layered thickness.

(Refer Slide Time: 45:25)



So, if you apply these to flow over flat plate, then you can see that for flow over flat plate; obviously, U infinity is constant ok. So, if U infinity is constant, then in this expression you can see that U infinity is constant, you can take it outside, you will get theta square is equal to 0.47 nu x by U infinity.

So, theta x by x you can write in terms of Reynolds number 0.686 divided by root Re x ok. Then, you can see from this expression, theta is known and U infinity is constant. So, dU infinity by dx is 0, so this will be 0. So, that means, lambda value will be 0 for flow over flat plate because pressure gradient is 0 you know. Then, you find from this expression this boundary layer thickness delta.

So, delta by x from this expression, you can find as 5.84 divided by root Re x. Then, from this expression as lambda is equal to 0, so delta star will be 3 by 10 delta and you can find delta star by x as 1.75 by root Re x and velocity distribution as lambda is 0, so you can find from this expression. Wall shear stress this lambda is 0, so you will get twice mu U infinity by delta and from here, you can find the fiction coefficient as 0.686 by root Re x.

So, you can see that this Karman-Pohlhausen approximation we have applied for flow over flat surface. Now, if you compare these results with the Blasius solution ok. So, you can see that theta x by x is 0.664 by root Re x; delta x by x as 5 by root Re x; delta start by x as 1.72 by root Re x and C f is 0.664 by root Re x.

So, you can see that these results compare favorably with the results obtained from the Blasius solution. So, in today's class, first we derived the momentum integral equation for the boundary layer momentum equation with non-zero pressure gradient. Then, we used Karman-Pohlhausen method to solve this momentum integral equation.

So, we have seen that if you represent u by U infinity as function of pressure parameter lambda, then this lambda value should be in between minus 12 and 12. Then, we have represented the displacement thickness, momentum thickness and the wall shear stress in terms of lambda. Then, we have written different correlations proposed by Holstein and Bohlen.

And we have finally, represented the function F in terms of K, and K we have represented in terms of lambda and that function is very complicated. So, after plotting, we have seen that we have expressed in terms of a linear function and from there, it is easy to find the momentum thickness and displacement thickness and the wall shear stress.

Thank you.