Viscous Fluid Flow
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Module - 01
Introduction
Lecture - 03
Derivation of incompressible Navier-Stokes equations

Hello everyone. So, today we will derive the Navier-Stokes equation starting from the Reynolds transport theorem. So, first, we will derive the integral form of linear momentum conservation from the Reynolds transport theorem then we will write this equation in terms of the differential equation for linear momentum conservation.

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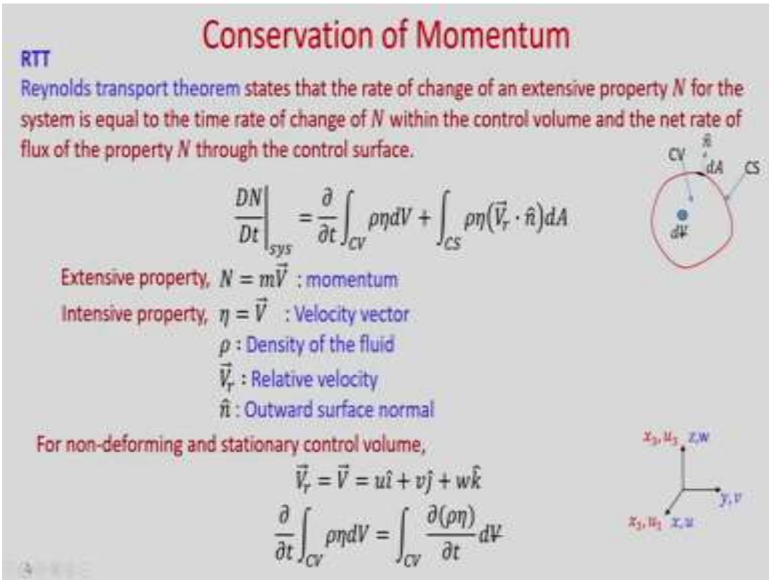

So, if you remember Reynolds transport theorem states that the rate of change of an extensive property N for the system is equal to the time rate of change of N within the control volume and the net rate of flux of the property N through the control surface. So, we can see this is the general form of RTT, where DN/Dt system is the rate of change of momentum in this particular case and where N is the extensive property that is equal to momentum.

Because we are conserving the momentum; so, N will be mV, where V is the velocity vector. And, in the right-hand side η is the intensive property and if you take N per unit mass then;

obviously, it will become a velocity vector. And, in this expression ρ is the density of the fluid and V_r is the relative velocity and n is the outward surface normal.

So, if you consider this arbitrary control volume, where red color is the control surface. So, if you take one elemental area dA. So, outward normal is n and if you consider one elemental volume that is your dV. So, if you consider non-deforming and stationary control volume, then this relative velocity V_r will be just V which is nothing but $u\hat{\imath} + v\hat{\jmath} + w\hat{k}$.

$\hat{\imath}$, $\hat{\jmath}$, $\hat{k}$ are the unit normal in x, y and z direction respectively and the velocities u, v, w are the velocities in the direction x, y, z respectively. So, in that particular case the first term in the RTT in the right hand side, you can see this you can write as

$$\left.\frac{DN}{Dt}\right|_{sys} = \int_{CV} \frac{\partial(\rho\vec{V})}{\partial t} d\forall + \int_{CS} \rho\vec{V}(\vec{V}.\hat{n}) dA$$

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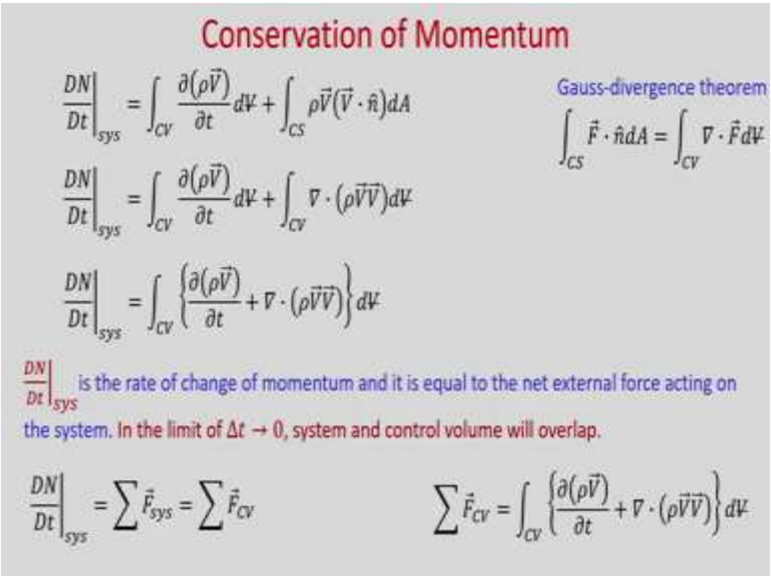

We have substituted η is equal to V in this case. So, now this surface integral we will convert to volume integral using Gauss divergence theorem. So, this is the general form

$$\int_{CS} \vec{F}.\hat{n} dA = \int_{CV} \nabla.\vec{F} d\forall$$

So, you can see this we can write as

$$\left.\frac{DN}{Dt}\right|_{sys} = \int_{CV} \frac{\partial(\rho\vec{V})}{\partial t} d\forall + \int_{CS} \nabla.(\rho\vec{V}\,\vec{V}.\hat{n}) d\forall$$

So, now if you take common then you can write

$$\left.\frac{DN}{Dt}\right|_{sys} = \int_{CV} \{\frac{\partial(\rho\vec{V})}{\partial t} + \int_{CS} \nabla.(\rho\vec{V}\,\vec{V}.\hat{n})\}\, d\forall$$

So, here V is the velocity vector and this $\forall$ is the volume ok. So, as we discussed that DN/Dt system is the rate of change of momentum and it is equal to the net external force acting on the system. So, we know that in the limit of Δt tends to 0 system and control volume will overlap. So, whatever net external force acting on the system, it will be acting on the control volume.

So, we can write

$$\left.\frac{DN}{Dt}\right|_{sys} = \sum \vec{F}_{sys} = \sum \vec{F}_{CV}$$

So, where this is the net external force acting on the control volume. So obviously if you put $\left.\frac{DN}{Dt}\right|_{sys}$ is equal to $\sum \vec{F}_{CV}$ here, then we can write this equation as

$$\sum \vec{F}_{CV} = \int_{CV} \left\{\frac{\partial\left(\rho\vec{V}\right)}{\partial t} + \nabla.\left(\rho\vec{V}\,\vec{V}\right)\right\} d\forall$$

So, now this equation you can see that we have written in the integral form, now we need to convert it into differential form. So, what is this net external force acting on the control volume, that we need to determine. So, if you consider the control volume. So, there will be two types of forces acting on this fluid element. One is on the surface which is known as surface forces and another force will be acting on the bulk fluid that is the body force.

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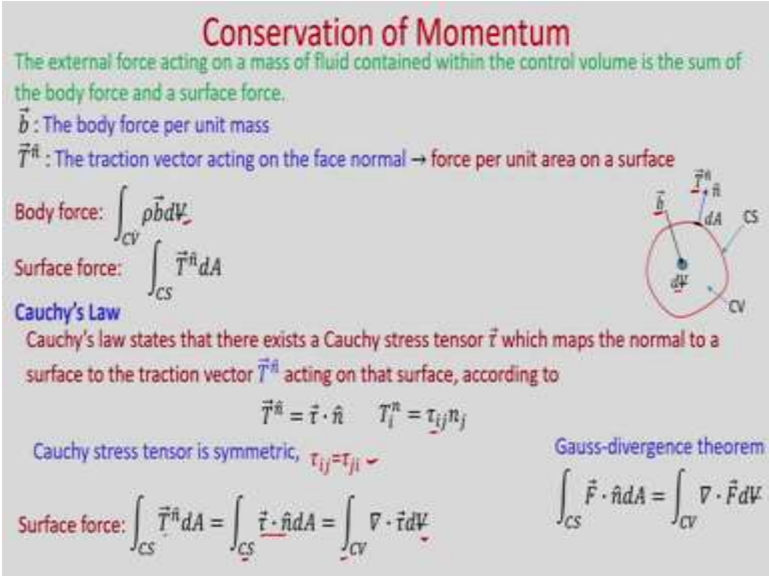

So, if we consider that $\vec{b}$ as the body force per unit mass then we can write body force is equal to v$\int_{CV} \rho \vec{b} d\mathcal{V}$, $\rho d\vec{\mathcal{V}}$ is the dm; that means, $\rho \vec{b} d\vec{\mathcal{V}}$. So, you can see this is the arbitrary control volume. This body force $\vec{b}$ is acting on this elemental control volume $d\vec{\mathcal{V}}$ and $\hat{n}$ is the outward surface normal. And, on the surface we have the traction vector $\vec{T}^{\hat{n}}$ acting on the face normal, which is force per unit area on a surface kind of thing.

So, you can see that this is a surface force and this is the body force. So, the external force acting on a mass of a fluid contained within the control volume is the sum of the body force and a surface force and we are telling that $\vec{b}$ is the body force per unit mass. So, that we can write $\int_{CV} \rho \vec{b} d\mathcal{V}$ and $\vec{T}^{\hat{n}}$ is the traction vector acting on the face normally. So, this is acting on the surface. So, we can write surface force as area integral $\int_{CS} \vec{T}^{\hat{n}} dA$.

So, now this surface force which is your area integral we have we need to convert it into volume integral. So, for that we will use Cauchy's law. What it states? Cauchy's law states that there exists a Cauchy stress tensor $\vec{\tau}$ which is second order tensor which maps the normal to a surface to the traction vector $\vec{T}^{\hat{n}}$ acting on that surface, according to $\vec{T}^{\hat{n}}$ is equal to $\vec{\tau}.\hat{n}$.

So, in tensorial form if you write, then you can write

$$T_i^n = \tau_{ij} n_j$$

So, you can see τ_{ij} is the second order tensor. So, Cauchy stress tensor is symmetric. So, we can write τ_{ij} is equal to τ_{ji}. So, now if we write the surface force; so, you can see

$$\int_{CS} \vec{T}^{\hat{n}} dA = \int_{CS} \vec{\tau}.\hat{n} dA$$

Now, we will use Gauss divergence theorem to convert these surface integral to volume integral. So, you can see we can write surface integral

$$\int_{CS} \vec{F}.\hat{n} dA = \int_{CV} \nabla.\vec{F} d\forall$$

So,

$$\int_{CS} \vec{T}^{\hat{n}} dA = \int_{CS} \vec{\tau}.\hat{n} dA = \int_{CV} \nabla.\vec{\tau}.dV$$

So now, both the forces we have written in terms of volume integral.

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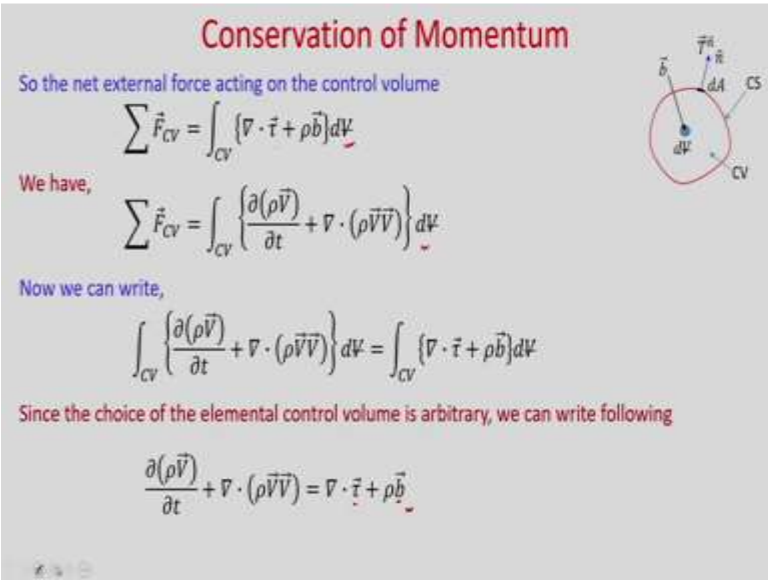

So, now we can write the net external force acting on the control volume. So,

$$\sum \vec{F}_{CV} = \int_{CV} \{\nabla.\vec{\tau} + \rho\vec{b}\} d\forall$$

Already this we have derived from the Reynolds transport theorem. So, now, if we equate this then we can write this term in the left-hand side and this term in the right-hand side.

And, since the choice of the elemental control volume is arbitrary; so, we can write the following

$$\sum \vec{F}_{CV} = \int_{CV} \left\{ \frac{\partial(\rho\vec{V})}{\partial t} + \nabla.\left(\rho\vec{V}\,\vec{V}\right) \right\} d\forall$$

So, we can write

$$\int_{CV} \left\{ \frac{\partial(\rho\vec{V})}{\partial t} + \nabla.\left(\rho\vec{V}\,\vec{V}\right) \right\} d\forall = \int_{CV} \{\nabla.\vec{\tau} + \rho\vec{b}\}\, d\forall$$

So, you can see this equation we have written in differential form right. But, you can see here τ is unknown because it is a tensor and it is unknown; so obviously, this equation is not closed. So, you need to determine this τ in terms of the known parameter velocity or pressure. And b is obviously, the body force and it is a constant body force.

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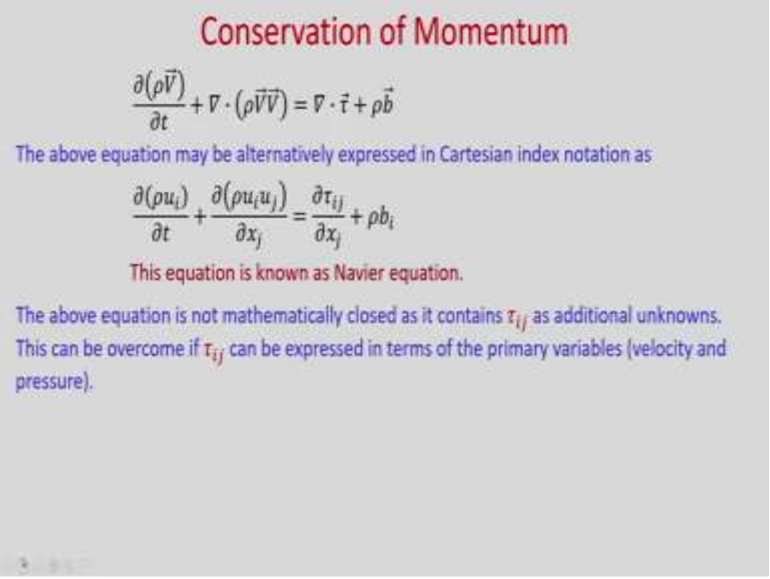

So, whatever equation now we have derived, this equation may be alternatively expressed in Cartesian index notation. So, we can write it as interstitial form. So, you can see

$$\frac{\partial(\rho u_i)}{\partial t} + \frac{\partial(\rho u_i u_j)}{\partial x_j} = \frac{\partial \tau_{ij}}{\partial x_j} + \rho b_i$$

where τ_{ij} is the second order tensor and these equation actually scientist Navier derived.

So, due to his name, it is this equation is known as Navier equation so; obviously, you can see the above equation is not mathematically close as it containsτ_{ij} as additional unknowns. So, this can be overcome if τ_{ij} can be expressed in terms of the primary variables, velocity and pressure. So, to find the shear stress tensor tau, we will use constitutive equation.

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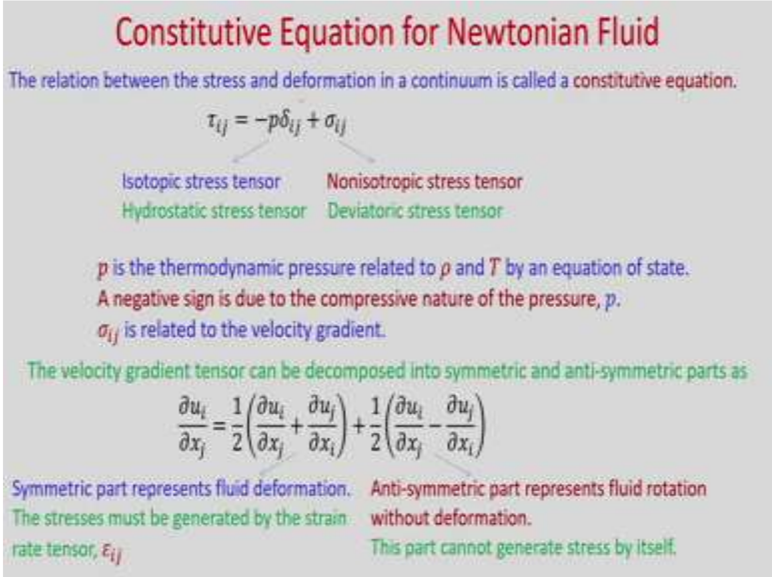

So, what is constitutive equation? The relation between the stress and deformation in a continuum is called a constitutive equation. So, now, whatever shear stress tensor is unknown τ_{ij} can be written in terms of two terms ok. One is isotropic stress plus one is non-isotropic stress tensor

$$\tau_{ij} = -p\delta_{ij} + \sigma_{ij}$$

So, $-p\delta_{ij}$ is known as hydrostatic stress tensor and σ_{ij} is known as deviatoric stress tensor. If you consider a static fluid then you know that only the normal stress will be acting on the surface and it is independent of the orientation of the surface. So obviously, it is the stress is isotropic in nature and if the fluid is static then; obviously, the pressure will be acting on a normal to the surface.

And, this stress tensor is isotropic and that we can relate with the isotopic second-order tensor Kronecker delta ok. So, you can see the first term which is your hydrostatic part that we have written as $-p\delta_{ij}$, where δ_{ij} is the second-order isotropic tensor. And, p is the thermodynamic pressure acting on the normal to the surface and p is the thermodynamic pressure related to density and temperature by an equation of state.

And, here the negative sign is coming due to the compressive nature of the pressure. So, you can see that this term will be there in the moving fluid as well. So, that's why we have added and it is isotropic stress tensor. And, another term will be there due to the viscous effect and that is known as deviatoric stress tensor. And, this σ_{ij} we need to write in terms of the velocity gradient. So, σ_{ij} is related to the velocity gradient and the velocity gradient tensor can be decomposed into symmetric and anti-symmetric parts as you can see you can write

$$\frac{\partial u_i}{\partial x_j} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) + \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right)$$

So, essentially you can see this $\frac{1}{2}\frac{\partial u_j}{\partial x_i}$ and here $-\frac{1}{2}\frac{\partial u_j}{\partial x_i}$ will get canceled; only you will get $\frac{\partial u_i}{\partial x_j}$. But, we have written in terms of the symmetric and anti-symmetric parts. So, you can see the last term in the right-hand side, this is anti-symmetric part represents fluid rotation without deformation right.

It is coming due to the rotation of the fluid and this part cannot generate stress by itself. Whereas, the first term in the right hand side you can see that it is symmetric part and it represents fluid deformation and the stresses must be generated by the strain rate tensor ε_{ij}. So, you can see that whatever σ_{ij} we are talking about the deviatoric stress tensor. So, this will be just due to the presence of this strain rate tensor ε_{ij}.

So, when we consider a moving fluid; so, we can see that there will be two terms. One is minus $-p\delta_{ij}$ which is hydrostatic part, it will be present if the fluid were at rest. And, the other part will be the σ_{ij} which is deviatoric stress tensor and it is coming due to the viscous effect. And, you can see that we need to determine this σ_{ij} in terms of the velocity gradient.

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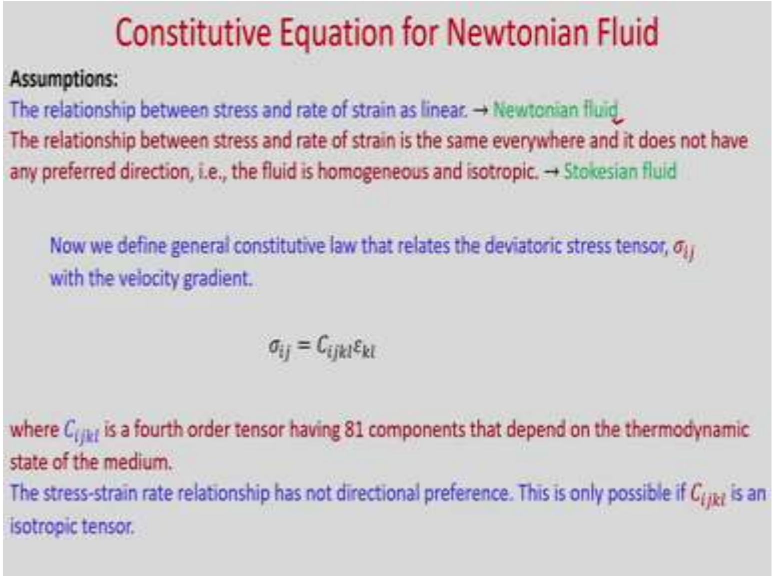

So, here we will make certain assumptions to find the deviatoric stress tensor. What are the assumptions? The relationship between stress and rate of strain is linear; so, we are considering the linear relationship between stress and strain rate ok. And, whatever fluids obey this relationship those are known as a Newtonian fluid.

So, you can see now we are making assumptions that it is a Newtonian fluid, where the stress and strain rate relationship is linear ok. The second assumption we are taking, the relationship between stress and rate of strain is the same everywhere and it does not have any preferred direction. What does it mean that same everywhere? That means, it is homogeneous and there is no proper direction; that means, it is isotropic so; that means, the fluid is homogeneous and isotropic ok.

And, whatever fluids actually having these homogeneous and isotropic media those are known as Stokesian fluid. So now, we define general constitutive law that relates the deviatoric stress tensor σ_{ij} with the velocity gradient as this. So,

$$\sigma_{ij} = C_{ijkl}\varepsilon_{kl}$$

C_{ijkl} is a fourth-order tensor and having 81 components that depend on the thermodynamic state of the medium.

The stress-strain rate relationship has not directional preference so; that means, it is isotropic. So, this is only possible; that means, σ_{ij} will be isotropic only if C_{ijkl} is an isotropic tensor. So, this has to be one isotropic tensor.

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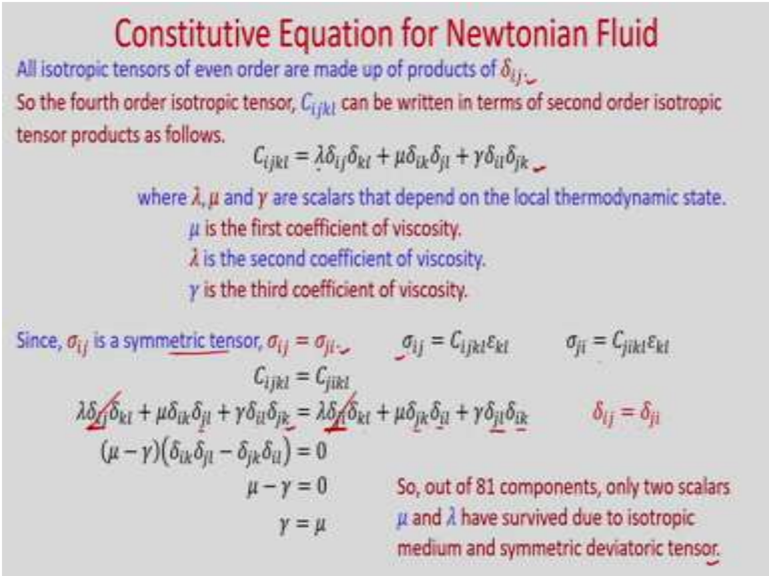

So, you can refer to some mathematics book of tensor, where you can actually this fourth-order tensor you can express in terms of second-order tensor, isotopic tensor, Kronecker delta. So, all isotropic tensors of even order are made up of products of δ_{ij}. So, the fourth-order isotopic tensor C_{ijkl} can be written in terms of second order isotopic tensor products as follows.

So, we can write

$$C_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu\delta_{ik}\delta_{jl} + \gamma\delta_{il}\delta_{jk}$$

You please refer some book of tensor, you will find that we can represent this isotropic fourth order tensor in this expression. Here λ, μ and γ are scalars that depend on the local thermodynamic state, where μ is the first coefficient of viscosity.

So, that you know; so, in the fluid we are discussing about the dynamic viscosity; so, this is first coefficient of viscosity, λ is the second coefficient of viscosity and γ is the third coefficient of viscosity. So, you can see that where in general fourth order tensor are having 81 components, but assuming that it is isotropic and homogeneous, we reduced it to three unknowns; λ, μ and γ.

Now, this σ_{ij} is a symmetric tensor that, we have already seen. So, we can write σ_{ij} is equal to σ_{ji}. So, using this we can actually bring down these three unknowns to two unknowns. So, we have already expressed

$$\sigma_{ij} = C_{ijkl}\varepsilon_{kl}$$

sigma ij is equal to C ijkl epsilon kl and similarly you can write σ_{ji}. So, we will interchange this; so,

$$\sigma_{ji} = C_{jikl}\varepsilon_{kl}$$

So, using symmetric tensor this relation you can see that we can write C_{ijkl} is equal to C_{jikl}, from these two expression you can see. So, C_{ijkl} already we have expressed these. So, in the left hand side we have written this. Now, in right hand side you change the indices ok; so C_{jikl}. So, you can see. So, we can write

$$\lambda\delta_{ij}\delta_{kl} + \mu\delta_{ik}\delta_{jl} + \gamma\delta_{il}\delta_{jk} = \lambda\delta_{ji}\delta_{kl} + \mu\delta_{jk}\delta_{jl} + \gamma\delta_{jl}\delta_{ik}$$

Now, we know that Kronecker delta is symmetric tensor. So, we can write δ_{ij} is equal to δ_{ji}. So, you can see this δ_{ji} is equal to δ_{ij}. So, in the both side we have this term. So, it will be get cancelled. So, you can write

$$(\mu - \gamma)\left(\delta_{ik}\delta_{jl} - \delta_{jk}\delta_{il}\right) = 0$$

But, the second term in the bracket cannot be 0 right so; obviously, the first term in the bracket will be 0. So,

$$\mu - \gamma = 0$$

that means,

$$\gamma = \mu$$

That means, the third coefficient of viscosity is equal to the first coefficient of viscosity as we assumed σ_{ij} as symmetric tensor. So, you can see that out of 81 components only two scalars μ and λ have survived due to isotropic medium and symmetric deviatoric tensor. So now, if you put it in this expression γ is equal to μ and let us evaluate this deviatoric stress tensor.

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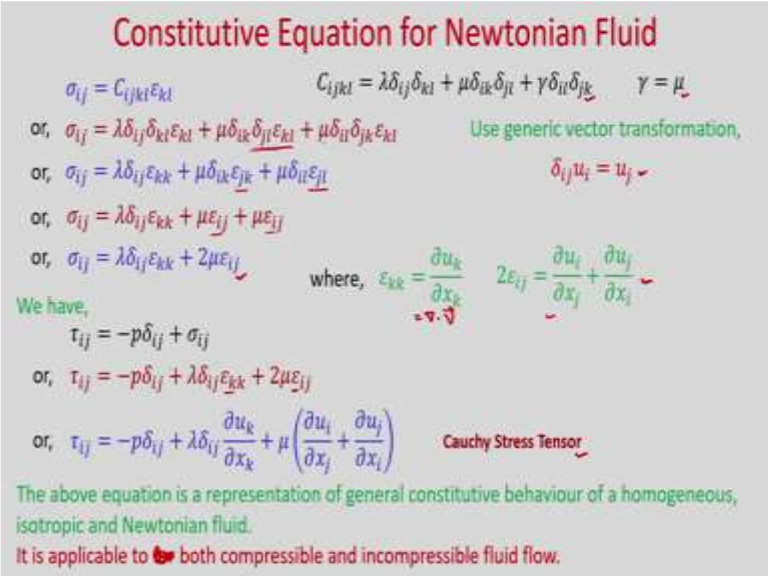

So, this is the expression of this fourth order isotropic stress tensor and also we have shown that γ is equal to μ. And, we have already represented this deviatoric stress tensor is equal to $C_{ijkl}\varepsilon_{kl}$, where ε_{kl} is the strain rate. So, σ_{ij} now if you put this expression and if you write ε_{kl} then we can write

$$\sigma_{ij} = \lambda\delta_{ij}\delta_{kl}\varepsilon_{kl} + \mu\delta_{ik}\delta_{jl}\varepsilon_{kl} + \gamma\delta_{il}\delta_{jk}\varepsilon_{kl}$$

So, now, we will use this generic vector transformation. So, if you write $\delta_{ij}u_i$, this will become u_j ok. So, this is the transformation you can seen in a tensor book. So, if you use this transformation; so, you can see $\delta_{kl}\varepsilon_{kl}$, if we put it here; so, we can write ε_{kk} so,$\delta_{jl}\varepsilon_{kl}$. So, it will be ε_{jk}. So, you can see ε_{jk} and $\delta_{jk}\varepsilon_{kl}$,; so, it will be ε_{jl},; so, ε_{jl}.

So, again you just use this vector transformation. So, it will be $\delta_{ik}\varepsilon_{jk}$. So, you can see it will become ε_{ij} and $\delta_{il}\varepsilon_{jl}$, it will be ε_{ij}. So, you can see now σ_{ij} which is your deviatoric stress we can write

$$\sigma_{ij} = \lambda\delta_{ij}\varepsilon_{kk} + 2\mu\varepsilon_{ij}$$

So, λ is the second coefficient of viscosity and μ is the first coefficient of viscosity, where ε_{kk} we can write $\frac{\partial u_{kk}}{\partial x_k}$ which is nothing, but divergence of $\vec{V}$. And, this $2\mu\varepsilon_{ij}$ you can write in terms of velocity gradient $\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}$.

So, now these ε_{ij} let us put in the stress tensor τ_{ij}. So,

$$\tau_{ij} = -p\delta_{ij} + \lambda\delta_{ij}\frac{\partial u_k}{\partial x_k} + \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

So, this is known as Cauchy stress tensor. So, you can see the above equation is a representation of general constitutive behavior of a homogeneous, isotropic and Newtonian fluid. And, it is applicable to both compressible and incompressible fluid flow because, we have not done any assumption till now based on this compressible or incompressible.

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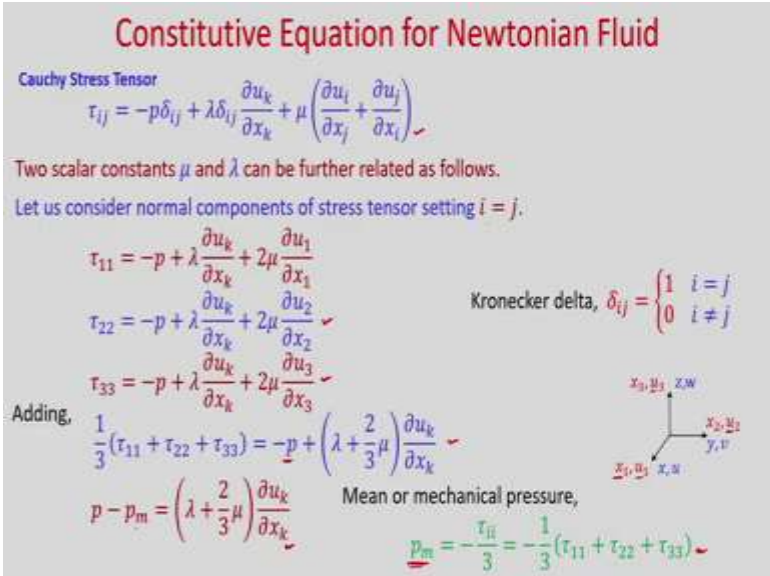

So, now you can see that we have derived this now here μ and λ are unknown, but we can relate it using the normal components of the stress tensor. So, let us write down the normal components of the stress tensor setting i is equal to j. So, you can see τ_{11} we can write

$$\tau_{11} = -p + \lambda\frac{\partial u_k}{\partial x_k} + 2\mu\frac{\partial u_1}{\partial x_1}$$

Similarly, u_2 is the velocity in the x_2 direction and u_3 is the velocity in the x_3 direction. So, in Cartesian coordinate we have just represented these normal components of the stress tensor. So, similarly if you put i is equal to j is equal to 2, then you will get this expression

$$\tau_{22} = -p + \lambda\frac{\partial u_k}{\partial x_k} + 2\mu\frac{\partial u_2}{\partial x_2}$$

and if i is equal to j is equal to 3; so, you will get this expression.

$$\tau_{33} = -p + \lambda \frac{\partial u_k}{\partial x_k} + 2\mu \frac{\partial u_3}{\partial x_3}$$

So, if you add these three you will get

$$\frac{1}{3}(\tau_{11} + \tau_{22} + \tau_{33}) = -p + \left(\lambda + \frac{2}{3}\mu\right)\frac{\partial u_k}{\partial x_k}$$

So, after adding and dividing by 3 just we have written this expression and this expression in the left hand side, this actually will relate with the mean or mechanical pressures.

So, this mean or mechanical pressure we can write

$$p_m = -\frac{\tau_{ii}}{3} = -\frac{1}{3}(\tau_{11} + \tau_{22} + \tau_{33})$$

So, you can see this mean or mechanical pressure is nothing but the arithmetic average of the normal components of the stresses. So, this is known as mean or mechanical pressure p_m and here you can see p is there, p already we have discussed that it is a thermodynamic pressure.

So, the difference between these thermodynamic pressure and the mechanical pressure we can write it as

$$p - p_m = \left(\lambda + \frac{2}{3}\mu\right)\frac{\partial u_k}{\partial x_k}$$

So, you can see this is the expression to find the difference between this thermodynamic pressure and the mechanical pressure. And, these expression you can see it is; obviously, in general we have written, but we can actually relate this λ and μ for Stokesian fluid ok.

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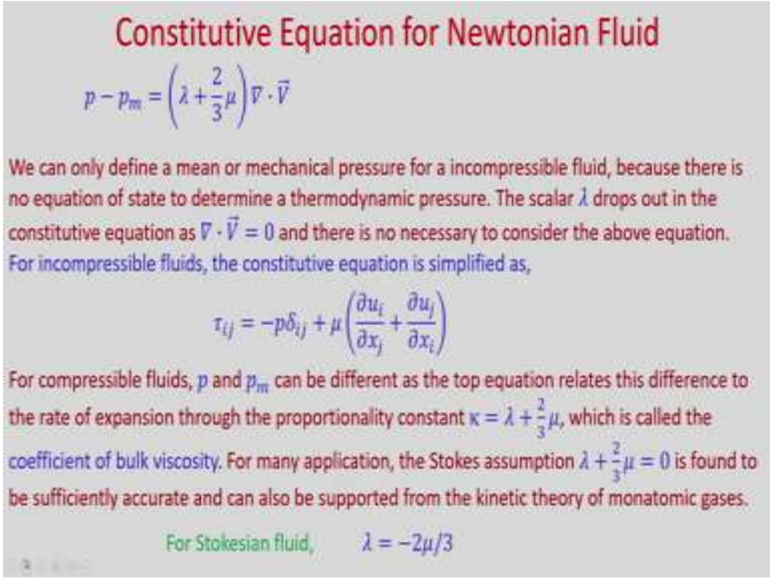

So, now let us discuss; obviously, $\frac{\partial u_k}{\partial x_k}$, you can write $\nabla.\vec{V}$. And, from here you can see for an incompressible fluid $\nabla.\vec{V}$ is equal to 0; that means, we can only define a mean or mechanical pressure for an incompressible fluid because there is no equation of state to determine a thermodynamic pressure.

The scalar λ drops out in the constitutive equation as $\nabla.\vec{V}$ is equal to 0. So, this will become 0 so; obviously, you will not have any term related to λ and there is no necessary to consider this above equation. So, for the incompressible fluid you can see the constitutive equation we can write in a simplified form

$$\tau_{ij} = -p\delta_{ij} + \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

But, if you consider compressible fluid then you can see p and p_m; that means, thermodynamic pressure and the mechanical pressure can be different as part this equation. And, it relates this difference to rate of expansion through the proportionality constant

$$\kappa = \lambda + \frac{2}{3}\mu$$

And, this is called as coefficient of bulk viscosity.

So, this is known as coefficient of bulk viscosity and for many application the Stoke assumptions $\lambda + \frac{2}{3}\mu$ is equal to 0 is found to be sufficiently accurate and can also be supported from the kinetic theory of monatomic gases. So, now, for Stokesian fluid, we can write $\lambda + \frac{2}{3}\mu$ is equal to 0; that means,

$$\lambda = -\frac{2}{3}\mu$$

So, you can see here that we have written this expression λ is equal to $-\frac{2}{3}\mu$ based on the Stokes assumption ok.

And, this expression we can write for Stokesian fluid, we have related this second coefficient of viscosity in terms of the first coefficient of viscosity μ. So, now, if you consider Newtonian fluid and we consider Stokesian fluid assuming the Stokes hypothesis or invoking the Stokes hypothesis, then we can write the expression for this stress tensor as.

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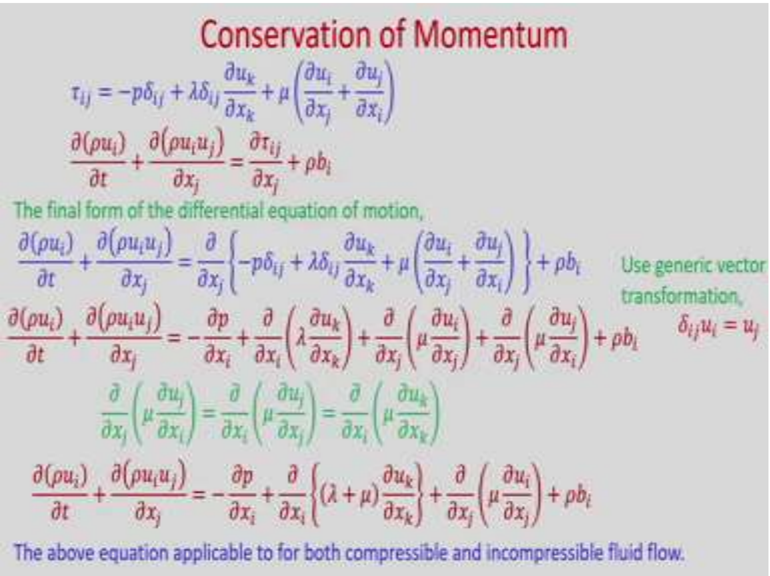

So, you can see that this is our expression of τ_{ij} and this is our expression which we derived from the RTT.

$$\tau_{ij} = -p\delta_{ij} + \lambda\delta_{ij}\frac{\partial u_k}{\partial x_k} + \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

So, invoking this τ_{ij}, here we can write in general this differential equation of motion as

$$\frac{\partial(\rho u_i)}{\partial t} + \frac{\partial(\rho u_i u_j)}{\partial x_j} = \frac{\partial}{\partial x_j}\left\{-p\delta_{ij} + \lambda\delta_{ij}\frac{\partial u_k}{\partial x_k} + \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)\right\} + \rho b_i$$

So, here we are not invoking any assumptions ok.

Now, we can simplify these terms. So, you can see we have

$$\frac{\partial(\rho u_i)}{\partial t} + \frac{\partial(\rho u_i u_j)}{\partial x_j} = \frac{\partial p}{\partial x_j} + \frac{\partial}{\partial x_i}\left(\lambda\frac{\partial u_k}{\partial x_k}\right) + \frac{\partial}{\partial x_j}\left(\mu\frac{\partial u_i}{\partial x_j}\right) + \frac{\partial}{\partial x_j}\left(\mu\frac{\partial u_j}{\partial x_i}\right) + \rho b_i$$

So, now this term will now write it as. So, this term we will write. So, now, we are changing this index. So, we can write here

$$\frac{\partial}{\partial x_j}\left(\mu\frac{\partial u_j}{\partial x_i}\right) = \frac{\partial}{\partial x_i}\left(\mu\frac{\partial u_j}{\partial x_j}\right) = \frac{\partial}{\partial x_i}\left(\mu\frac{\partial u_k}{\partial x_k}\right)$$

So, you can see we can write this term as this

$$\frac{\partial(\rho u_i)}{\partial t} + \frac{\partial(\rho u_i u_j)}{\partial x_j} = -\frac{\partial p}{\partial x_j} + \frac{\partial}{\partial x_i}\left\{(\lambda + \mu)\frac{\partial u_k}{\partial x_k}\right\} + \frac{\partial}{\partial x_j}\left(\mu\frac{\partial u_j}{\partial x_i}\right) + \rho b_i$$

So, you can see this is the equation which is applicable to both compressible and incompressible fluid flow. So, now in that general equation let us invoke the Stokes hypothesis. And, we can write the second coefficient of viscosity λ is equal to $-\frac{2}{3}\mu$.

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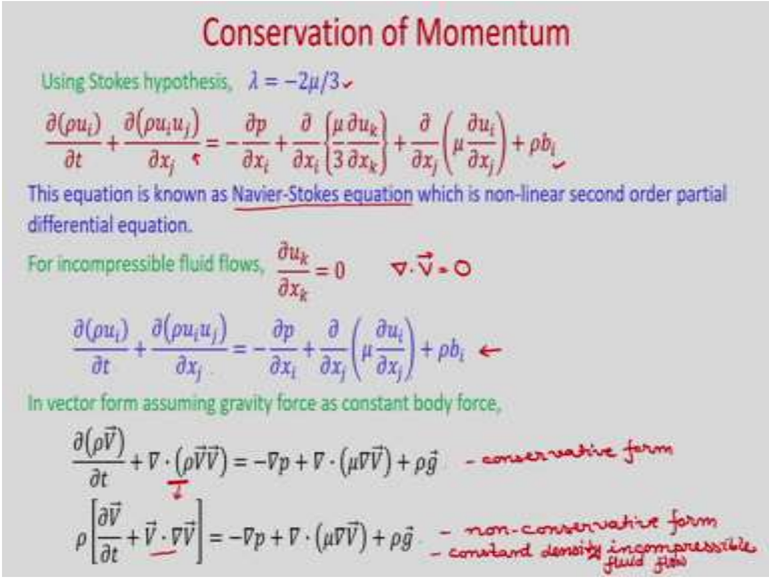

So, if you put it in that expression, then you can see

$$\frac{\partial(\rho u_i)}{\partial t} + \frac{\partial(\rho u_i u_j)}{\partial x_j} = -\frac{\partial p}{\partial x_j} + \frac{\partial}{\partial x_i}\left\{\frac{\mu}{3}\frac{\partial u_k}{\partial x_k}\right\} + \frac{\partial}{\partial x_j}\left(\mu\frac{\partial u_j}{\partial x_i}\right) + \rho b_i$$

So, this is the expression. Now, this is known as Navier-Stokes equation which is you can see it is non-linear due to this convective term ok. And, it is second order partial differential equation.

Now, we will make the assumptions, if we assume that it is incompressible fluid flows then; obviously, you can see $\frac{\partial u_k}{\partial x_k}$ is equal to 0. So, if you make it 0, then you can write the Navier-Stoke equation for incompressible fluid flow as this.

$$\frac{\partial(\rho u_i)}{\partial t} + \frac{\partial(\rho u_i u_j)}{\partial x_j} = -\frac{\partial p}{\partial x_j} + \frac{\partial}{\partial x_j}\left(\mu\frac{\partial u_j}{\partial x_i}\right) + \rho b_i$$

So, you can see this is the temporal term, this is the convective term, this is the pressure gradient term and this is the viscous term. So, now it is simplified because we have only this dynamic viscosity μ right plus any body force term. So, in general we can have gravity as body force. So, in vector form assuming a gravity force as constant body force ok. So, we can write this

$$\frac{\partial(\rho\vec{V})}{\partial t} + \nabla.\left(\rho\vec{V}\vec{V}\right) = -\nabla p + \nabla.\left(\mu\nabla\vec{V}\right) + \rho\vec{g}$$

So, this is in the vector form we have written and it is in conservative form ok; in conservative form. And, this term if you extend then you will get

$$\rho\left[\frac{\partial(\vec{V})}{\partial t}+\vec{V}.\nabla\vec{V}\right]==-\nabla p+\nabla.\left(\mu\nabla\vec{V}\right)+\rho\vec{g}$$

So, from this conservative form to non-conservative form, you need to elaborate this term and invoke the continuity equation then you will get this equation.

And, you can see it is a constant density incompressible fluid flow if you assume, then you can take ρ in the outside so, for constant density incompressible flow. So, you can take ρ outside and this is the expression in non-conservative form. So, in today's class, we started with the Reynolds transport theorem. And, we conserved the momentum where the extensive property N, we have considered as momentum mV and the intensive property η is equal to velocity vector V.

And, DN/Dt system is the change of rate of momentum and it is equal to the net external force acting on the system. And, we have seen there are two types of forces acting on the fluid element. One is body force which is your volumetric force. So, that is we have considered b as the body force per unit mass and we have the traction vector acting on the surface normal to the surface.

So, that is acting on the surface and this is surface force. So, these two now we have written in terms of volume integral using the Gauss divergence theorem. And finally, we have written the integral form of the conservation equation to the differential form of the conservation equation. Now, when we wrote that; so, we used Cauchy's law and we expressed the traction factor in terms of the shear stress tensor.

And, this shear stress tensor is composed of two components; one is hydrostatic shear stress tensor and other one is a deviatoric stress sensor which is coming due to the movement of the fluid and due to the presence of viscosity. So, you can see the hydrostatic part anyway it will be there if the fluid is static and that we have related with the second-order isotopic tensor δ_{ij}.

So, we have written $-p\delta_{ij}$ and minus sign is coming as pressure is compressive in nature. Then we have expressed this deviatoric stress in terms of the velocity gradient and that we have used the fourth order tensor. And, we assume that σ_{ij} as it is a symmetric tensor to make it symmetric and isotropic we should have this C_{ijkl} should be isotropic in nature.

And, it can be expressed in terms of three unknown parameters and the Kronecker delta. After rigorous derivation we have found the deviatoric stress, where we have written in terms of first coefficient of viscosity, second coefficient of viscosity and third coefficient of viscosity. And, as the shear stress is symmetric σ_{ij} is equal to σ_{ji}, we have written the third coefficient of viscosity is equal to first coefficient of viscosity.

Then, we have written the Navier-Stoke equation in general after putting the expression of this stress tensor. And, then we have invoked the Stokes hypothesis ok, where we have used that λ is equal to $-\frac{2}{3}\mu$ and invoking that we have written finally the Navier-Stokes equation.

And, further, we simplified it for the constant density incompressible flow, where we have used that $\nabla\vec{V}$ is equal to 0. And, in both forms, we have written, in tensorial form as well as in vector form. And, in vector form, we have written this Navier-Stoke equation in conservative and non-conservative form.

Thank you.