Viscous Fluid Flow
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Module - 10
Stability Theory
Lecture - 03
Inviscid Analysis

Hello everyone. So, in last class we discussed about the Stability Analysis or Viscous Flow. Today, we will consider Inviscid Flow. So, you can see that when we have very high Reynold's number flow or Reynold's number tends to infinity or we have very low viscosity fluid flow. Then, we can neglect the viscous term.

And if you drop the viscous term, then you can see that will get a simplified form of the Orr-Sommerfeld equation which is known as Rayleigh's equation. And you can see that it will be very easier to analyze this equation.

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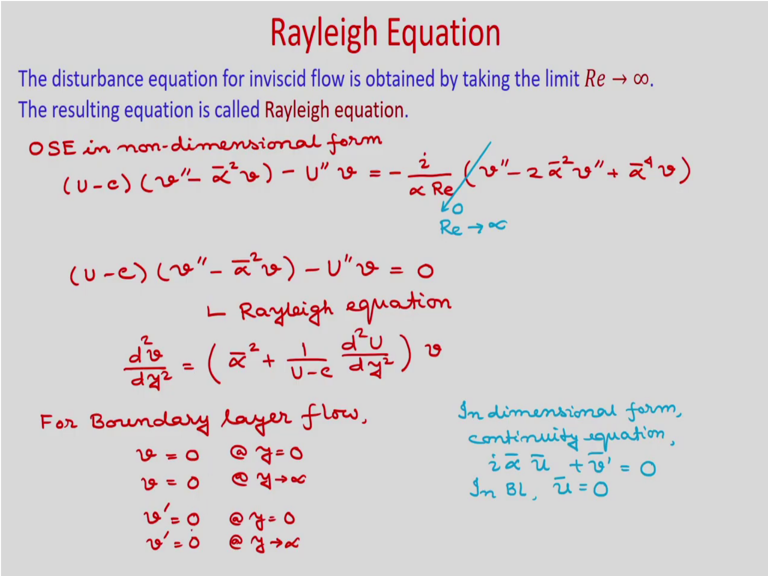

So, now, we can write the Orr-Sommerfeld equation. So, Orr-Sommerfeld equation you can see that we have U minus C v double prime minus alpha square v minus U double prime v is equal to minus i alpha R e v double prime minus 2 alpha bar square v double prime plus alpha to the power 4 v ok.

So, this is the equation we have written in non-dimensional form. So, this is Orr-Sommerfeld equation in non-dimensional form. So, you can see the disturbance equation for inviscid flow is obtained by taking the limit Reynold's number tends to infinity. So, you can see if you take Reynold's number tends to infinity then; obviously, this right-hand side term which is your viscous term, you can drop ok and the resulting equation is called Rayleigh equation.

So, you can drop this term when Reynold's number tends to infinity. Then we will get the equation as U minus C v double prime minus alpha square v minus U double prime v is equal to 0. So, this is known as Rayleigh equation ok. So, you can see that it is somewhat a simplified form and easier to solve than the Orr-Sommerfeld equation.

So, this now you can write. So, v double prime means it is the second derivative. So, d 2 v by d y square is equal to alpha square plus 1 divided by U minus c d 2 U by d y square. So, U double prime is d 2 U by d y square v. So, now, let us consider boundary layer flow.

So, if you consider boundary layer flow; obviously, at the flat plate at y is equal to 0. We have these disturbances as 0 as well as y tends to infinity these disturbances v will be 0. So, we can write for boundary layer flow, the disturbances will be 0 right v is equal to 0 at y is equal to 0 and v also will be 0 at y tends to infinity ok. So, you can see that since, this coefficient of this Rayleigh's equation are real. So, any complex eigenvalue will appear in conjugate pairs.

So, let us write down the continuity equation in dimensional form. So, in dimensional form, continuity equation is i alpha bar U bar plus v prime bar is equal to 0 ok. So, you can see that that disturbance is must vanish at infinity and at the walls ok. So, that we have already written here.

So, now, for boundary layer flow you can see in boundary layer. So, this U bar is equal to 0 right. So, if these disturbances are 0 at the boundary then, and also at y tends to infinity. Then, you can see v prime also will be 0.

So, we can write down the another boundary condition that v prime is 0 in non-dimensional form now we are writing. At y is equal to 0 and v prime is equal to 0 at y tends to infinity. So, this we are writing from these equation, because this is the continuity equation and this the U disturbances and; obviously, this U disturbances are 0 at y is equal to 0 and y tends to infinity. So, v prime bar is equal to 0.

So, in dimensional formula we are writing. So, in non-dimensional form if you represent then v prime will be 0 at y equal to 0; that means, at the wall and v prime is will be 0 at y tends to infinity. So, you can see for boundary layer flow, these boundary condition we can use. So, now, we will discuss about the Rayleigh's necessary condition for a inviscid stability analysis.

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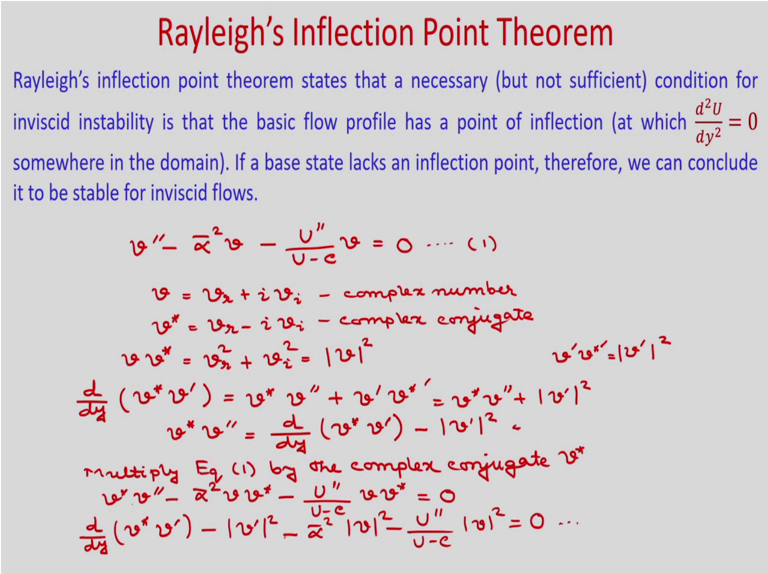

So, Rayleigh's inflection point theorem states that a necessary, but not sufficient condition for inviscid instability is that the basic flow profile has a point of inflection at which d 2 U by d y square will be 0 somewhere in the domain. If a base state lacks an inflection point, therefore, we can conclude it to be stable for inviscid flows.

So, you can see that from this Rayleigh's equation, we will show that to have the instability in the domain there must be d 2 U by d y square is equal to 0 somewhere in the domain. So, let us consider first this Rayleigh's equation whatever we have written. So, that is v double prime minus alpha square v minus U double prime divided by U minus C v will be 0 ok.

So, now, we can represent. So, any complex eigenvalue in conjugate pairs. So, we can write v is equal to v r; which is your real part and i into v i; so, this is your imaginary part. So, this is your complex number. Now, this complex conjugate we can write as v star ok is equal to v r minus i v i ok. So, this is your complex conjugate ok.

So, from here, you can see if you write v v star, then we can write you can see from here, it will be v r square it will be i square right. So, i square will be minus 1. So, it will be plus v i square. So, this we can write mod v square ok. And another simplification will do for this d of d y v star v prime is equal to. So, what we can write? We can write v star v double prime ok plus v prime v star prime ok.

So, now, you can see that from this complex conjugate if we use for v prime. So, you can see v prime and v star prime. So, we can write v prime v star prime will be just. So, this is the derivative. So, it will be v prime square ok. So, this we can write. Now, you can see we can write v star v double prime plus v prime square ok.

And from here, we can represent v star v double prime is equal to d of d y v star v prime minus v prime mod square ok. So, now, let us consider this equation ok and multiply this equation by complex conjugate v star ok. So, this is the equation number let us say, 1 then, multiply equation 1 by the complex conjugate v star ok.

So, if you multiply here, what you will get? You see; you will get v star v double prime minus alpha bar square v v star minus U double prime divided by U minus c v v star is equal to 0.

So, now, from this relation we can this represent by this and v v star in these two places we can write mod v square. So, if you write this. So, you will get d of d y v star v prime minus v prime square.

And now, this we are writing minus alpha square mod v square minus U double prime U minus C mod v square is equal to 0 ok. So, now let us integrate this equation in the within this boundary layer ok and we will invoke the boundary condition whatever we discuss in last side.

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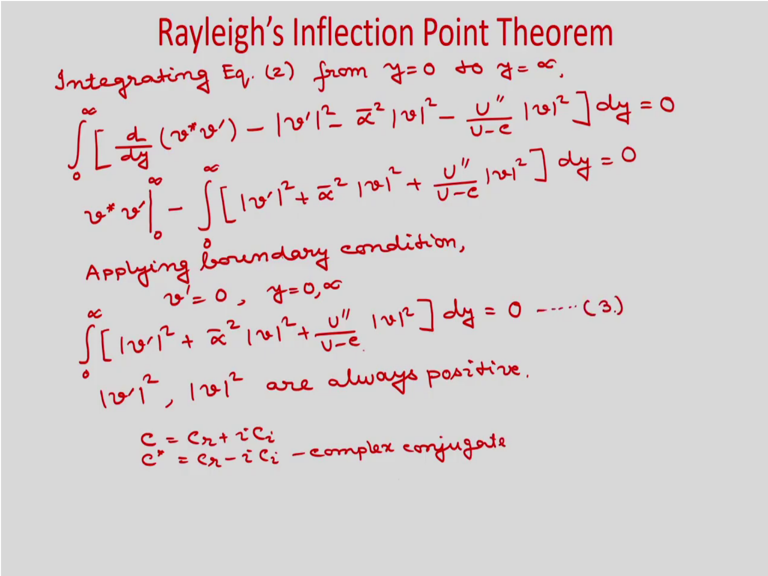

So, if this is equation number 2 then, we can write integrating equation 2 from y is equal to 0 to y is equal to infinity. So, if we integrate this. So, what we will get? So, we will get integral 0 to infinity d of d y v star v prime minus v prime square alpha square v square minus U double prime divided by U minus C mod v square d y is equal to 0 ok.

So, now, we can see the first term if you integrate so; obviously, it will be integral d v star v prime. So, this we will get v star v prime and the limits 0 to infinity and minus we will have integral 0 to infinity v square minus. It will be plus alpha square v square plus U double prime divided by U minus C mod v square d y is equal to 0.

So, now, you can see the first time in the left-hand side. So, if you put the boundary conditions ok v prime at y is equal to 0 it is 0 and v prime at y tends to infinity it is 0. So, the first term will be 0 ok. So, applying boundary condition that v prime is 0 at y is equal to 0 and infinity, then this term will become 0. So, we can write that integral 0 to infinity v prime square alpha square v square plus U double prime U minus C v square d y is equal to 0.

So, please look carefully these first two terms in the integral. So, you can see that v prime mod square ok and mod v square are always positive ok. So, these are always positive, but the third term ok we need to now carefully examine ok. So, for from this term we have to see that the when the flow will become unstable ok. So, so what we will do you can see in the third term, we will just in the numerator we will multiply with U minus C star, where C star is the complex conjugate C ok.

So, if C is C r plus i C i ok, then C star is C r minus i C i. So, this is your complex conjugate. So, now, what we will do? So, let us say that this is the equation number 3. Now, multiply numerator and denominator by U minus C star, where C star is the complex conjugate of C in the third term of equation 3 ok.

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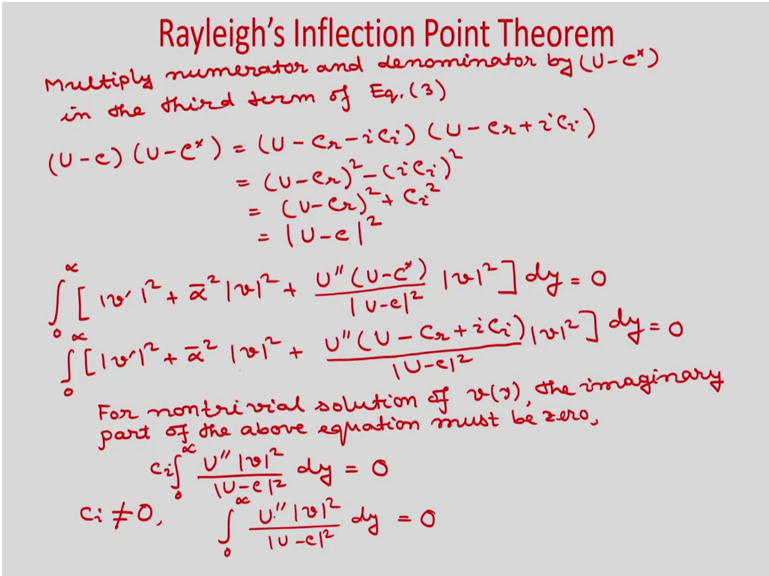

So, if you multiply then, first let us write U minus C and U minus C star. So, what you will get this. So, you can see it will be U minus C r minus i C i ok and C star you can write U minus C r plus i C i ok. So, if you do that multiplication, then you will get U minus C r square minus i C i square. So, it will be just U minus C r square plus C i square. So, it will be U minus C mod square ok.

So, if it is so, then you can see that we can if we multiply the third term with U minus C star in denominator and numerator. Then we can write integral 0 to infinity v prime mod square alpha square mod v square plus U double prime U minus C star right. And in the denominator so it will be U minus C into U minus C star. So, these we can write U minus C mod square ok and we have mod v square d y is equal to 0 ok.

Now, this C star if you write as mod v prime square alpha square v square plus U double prime. So, you can see this C star now, we can write U minus. So, it will be C r plus i C i divided by U minus c square v square d y is equal to 0. So, you can see that in this equation we are looking for C r and C i, so that; the nontrivial solution of v y exists right.

So, for this now, you can see that this is positive this is positive and this also you can see that here for the nontrivial solution. We have to see that imaginary part of this equation ok should be 0 ok.

So, imaginary part of this equation must be 0 for the nontrivial solution for v y. So, for nontrivial solution of v y, the imaginary part of the above equation must be 0 right. So, if it is so, then we can see that the imaginary part will be just integral 0 to infinity. So, you have C i and U double prime mod v square divided by U minus C mod square d y is equal to 0.

So, now, you can see that Rayleigh's is inflection point theorem follows from this above equation ok. So, you can see that this equation to be satisfied; either we must have C i is equal to 0 or this integral should be 0, but C i cannot be 0, because we have already assumed that it is a greater than 0. So, C i not equal to 0 ok.

So, we should have integral 0 to infinity U double prime v prime square divided by mod U minus C square d y should be 0 ok. So, because we have assumed that C i greater than 0 corresponding to unstable flow. So, this term should be 0 ok.

So, obviously, for C i not equal to 0, the flow may be stable or unstable and from this condition now, let us find the Rayleigh's inflection point theorem. So, in this equation now, you see this term is greater than 0 U minus C whole square this is also greater than 0. So, these are positive quantity, but this integral should be 0 so; that means, this U double prime should be 0 somewhere inside the domain to make this integral 0 right.

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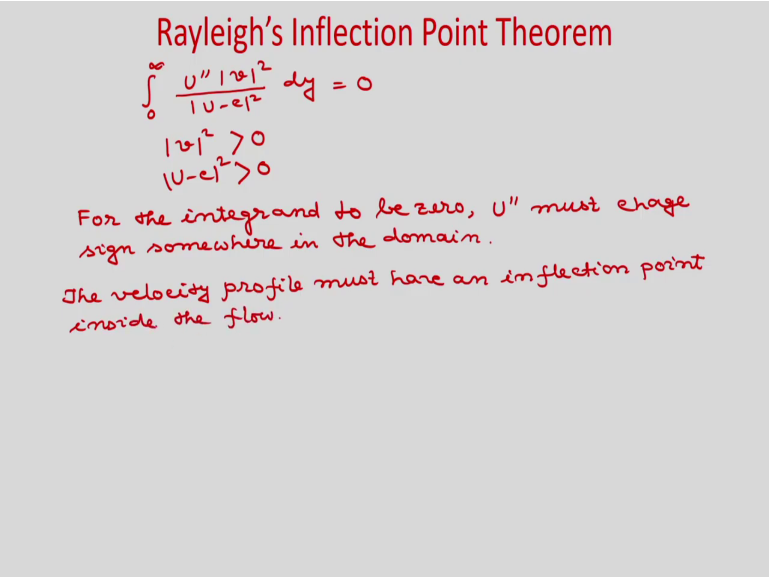

So, you can see that. So, if you write down this equation again 0 to infinity U double prime mod v square divided by mod U minus C square d y is equal to 0. So, from here, you can see this mod v square is greater than 0 ok U minus C mod square greater than 0. So, these are positive. So, this integrand should be 0. So, U double prime somewhere it should changing its sign from positive to negative ok.

So; that means, for the integrand to be zero, U double prime must change sign somewhere in the domain. So; that means, the velocity profile must have an inflection point inside the flow ok. So, this is the Rayleigh's necessary condition for instability.

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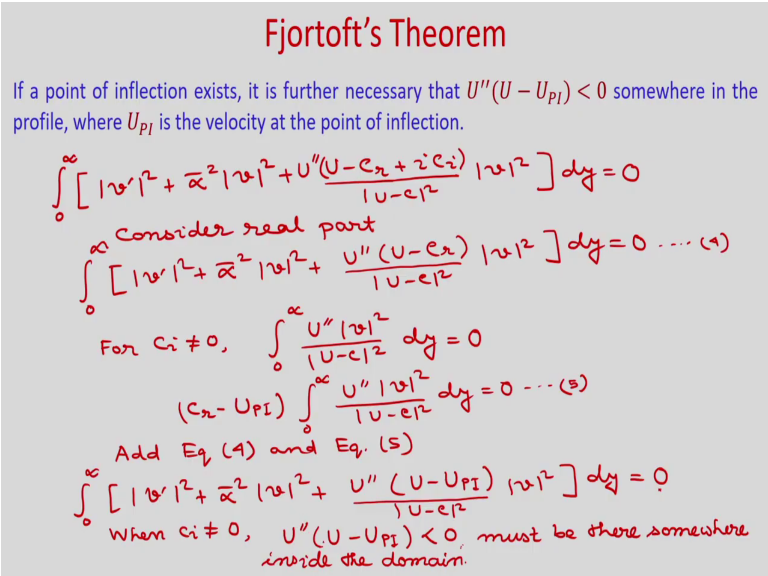

So, now, let us discuss another theorem, which is known as Fjortoft's Theorem for this inviscid stability analysis. So, you see if a point of inflection exists, which is we have seen from this Rayleigh's inflection point theorem, it is further necessary that U double prime into U minus U P I less than 0 somewhere in the profile, where U P I is the velocity at the point of inflection. So, if you consider the equation whatever we have written.

So, you can see it is 0 to infinity v prime square plus alpha square mod v square plus U minus C r plus i C i divided by U minus C square mod v square. Here, U double prime d y is equal to 0. So, this we have already derived. So, now, you consider the real part of this equation. So, consider real part.

So, you can write integral 0 to infinity mod v prime square plus alpha bar square mod v square plus U double prime U minus C r mod U minus C square mod v square d y is equal to

0. So, we are considering this real part of this equation and you have seen that from Rayleigh's inflection point theorem, we have already put this imaginary part as 0 and C i not equal to 0.

For that we have written that for C i not equal to 0, we have written integral 0 to infinity U double prime v square U minus C square d y is equal to 0 ok. So, this we have already written. So, now, what we will do? Now, you multiply this equation with C r minus U P I. So, where U P I is the velocity at the point of inflection. So, we will write C r minus U P I; these we are multiplying with this integral 0 to infinity U double prime mod v square divided by mod U minus C square d y is equal to 0 ok.

So, if this equation is 4 and this equation is 5 what you do; you just add these two equations ok. So, add equation 4 and equation 5 ok. So, you can see what you will have. So, if you add these two equations. So, here C r is there here, minus C r is there and the other terms are a same.

So, this will get cancelled and you will get integral 0 to infinity mod v prime square alpha bar square mod v square plus. Now, you can see here you will get U double prime U minus U P I divided by mod U minus C square mod v square d y is equal to 0. Now, you carefully look this equation.

Again, you can see the first two terms in this equation are positive right, because this is mods v prime square and mod v square these are positive greater than 0. And these integral then, you can see that here, this U double prime into U minus U P I should be less than 0 somewhere in the domain. So, you can see in this equation U double prime into U minus U P I should be less than 0 somewhere in the domain to make this integral as 0 ok.

Because these are positive. So, this has to be negative somewhere to make this integrand 0. So, for that you can see that when C i not equal to 0; obviously, we have assumed that C i greater than 0. So, U double prime into U minus U P I ok less than 0 ok must be there somewhere inside the domain ok.

So, you can see that these are positive. So, this has to be less than 0 somewhere in the domain to make this integrand equal to 0. So, you can see that this is the another necessary condition ok for C i not equal to 0, just having an inflection point inside the boundary layer is not enough.

So, you can see that U double prime should change its sign ok inside the boundary layer. So; that means, we should have some inflection point and along with that we should have U double prime U minus U P I should be less than 0.

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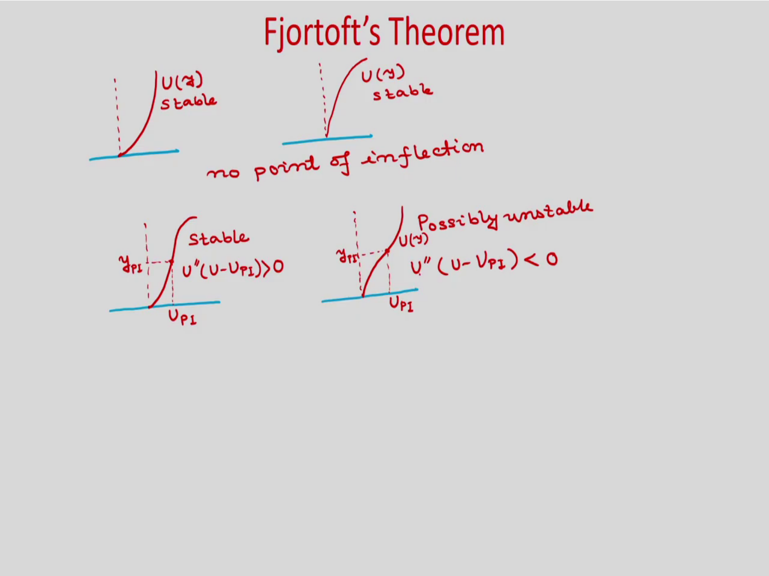

So, let us demonstrate this in a boundary layer flow. Let us say, that this is your plate. So, we should have the velocity profile. So, if it is looks like this ok where U is function of y, then it should be stable ok. And if the velocity profile look like this ok, so, that case also it is stable ok there is no point of inflection you can see here ok. So, these two cases; if you find the velocity profile U y like this then, it will be stable as there is no point of inflection.

Now, you considered the velocity profile like this ok. So, let us say that you have the profile like this ok. So, you can see somewhere here, it is changing its sign ok. So, this is your at y this is the inflection point. So, there we have U P I. So, this is the U P I this is the U velocity at inflection point.

So, you can see in this case your U double prime into U minus U P I is greater than 0 ok it is greater than 0, because if you see the U double prime gradient. So, this will be greater than 0 so; obviously, at this point you can see this is the case for stable ok. And another case, if you consider that you have the velocity profile like this. So, you can see this is the inflection point. So, at y P I, we have the velocity U P I ok. So, in this case you can see that U double prime U minus U P I is less than 0.

So, this is possibly unstable, because we know that this is a necessary condition ok this is not a sufficient condition. So, if this is a necessary condition. So, if you have this type of flow. So, this is the U y. So, in this case; obviously, U double prime into U minus U P I is less than 0. So, this is the case for possible unstable flow.

So, whatever we have studied today. So, let us apply for a simplified flow which is your parallel flow ok. So, if you consider a parallel flow then, we can use the Rayleigh's equation and let us see that whether in this case the flow will be stable or unstable. If it is unstable what is the condition.

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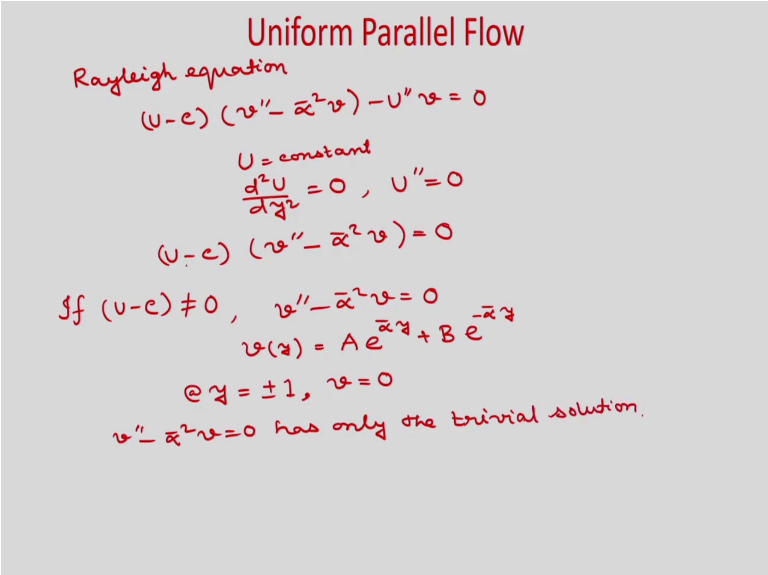

So, we have the Rayleigh equation as U minus C v double prime minus alpha square v minus U double prime v is equal to 0 ok. As you are considering uniform parallel flow so, you can see that U is constant right. So, in this case, d 2 U by d y square that will be 0 ok; that means, U double prime is 0 ok.

If U double prime is 0, so from this equation you can see that we can write U minus C v double prime minus alpha square v is equal to 0. So, now, there are two possibilities that either U minus C is 0 or v double prime minus alpha square v is equal to 0. If say U minus C not equal to 0 then, we will have this equation v double prime minus alpha square v is equal to 0 ok.

And if you solve this equation so; obviously, you can see that you will get the solution v y as A e to the power alpha y plus B e to the power minus alpha y.

So, in this case now, if you consider a parallel flow. So, we can have that the boundary conditions at y is equal to plus minus 1 as you are considering non-dimensional flow ok v is equal to 0 ok; obviously, the disturbances will be 0 at y is equal to plus minus 1, where v these are non-dimensional quantity.

So, if it is 0 then, had only the trivial solution you can see. If you apply this, you will have the trivial solution. So, this has only the trivial solution. So; obviously, U minus C not equal to 0. So, if U minus C not equal to 0.

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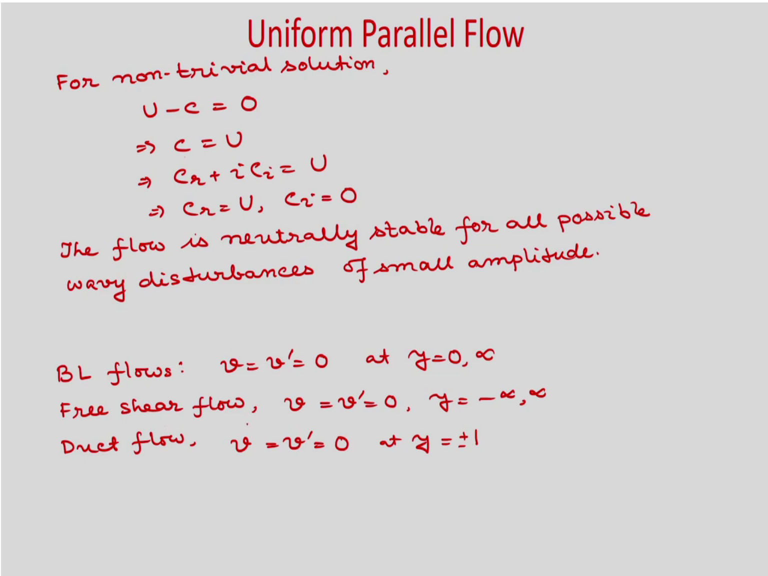

So, for non-trivial solution ok we need U minus C should be 0 so; that means, C should be U. And now, C we can write as C r plus i C i is equal to U. So, from here, you can see that C r will be U and C i is equal to 0 ok. So, if C i is equal to 0 ok. So, it is a neutrally stable flow ok so; that means, the flow is neutrally stable for all possible wavy disturbances of small amplitude.

So, the Rayleigh's equation also, you can apply in different inviscid flow and see the criteria for stability. So, for different flows like if you have free shear flows or you if you have boundary layer flows then, you can write the boundary condition as. So, for boundary layer flows, we have already discussed. These are the boundary conditions ok v is equal to v prime is equal to 0 at y is equal to 0 and infinity ok.

For free shear flow. So, v is equal to v prime is equal to 0 for y is equal to minus infinity to plus infinity and if it is a duct flow ok. So, there v will be v prime is equal to 0 ok at y is equal to ok plus minus 1. So, in non-dimensional form all we have written. So, these will be the boundary conditions.

So, using these for different flows you can solve this Rayleigh's equation. So, in today's class, we considered the inviscid stability analysis dropping the viscous term from the Orr-Sommerfeld equation. And then, we discussed two important theorems one is a Rayleigh's a point of inflection theorem and Fjortoft's Theorem. So, from the analysis we have seen that U double prime; that means, d 2 U by d y square should be negative somewhere a inside the flow.

And another necessary condition is that U double prime into U minus U P I should be less than 0 inside the domain. And then, we consider the parallel flow and we have seen that in this case, C i is equal to 0; that means, this a neutrally stable flow.

Thank you.