Viscous Fluid Flow

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Module - 12
Turbulent Flows - II
Lecture - 01
Integral Solution for Turbulent Boundary Layer Flow

Hello everyone. So, in today's lecture, we will use momentum integral equation to have the integral solution for turbulent boundary layer flows. As you know that the exact solution of this turbulent boundary layer flow is not possible, but we can use the momentum integral method to solve this boundary layer flow assuming the velocity profile and the wall shear stress from the existing correlation based on the experimental values.

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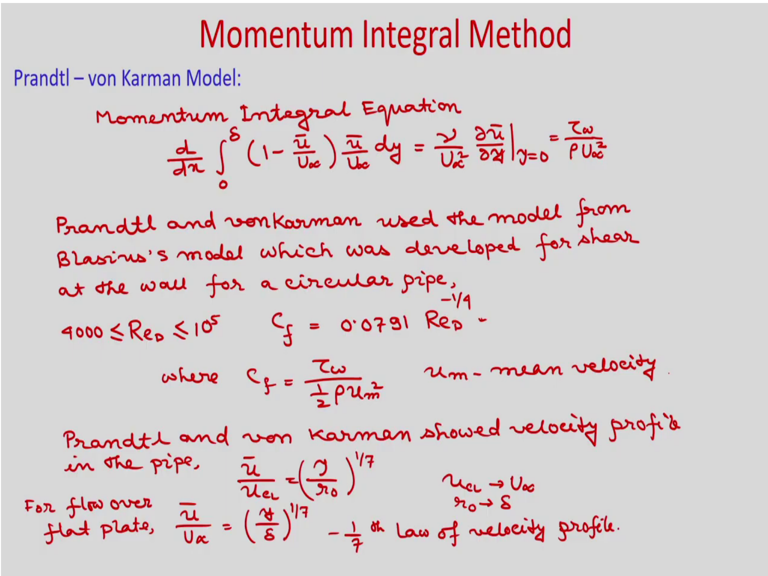

The momentum integral equation which we derived for laminar boundary layer flow that is also valid for turbulent boundary layer flow. This solution was actually first solved by Prandtl and von Karman. So, this model is known as Prandtl-von Karman model. So, let us write first the momentum integral equation which we derived for the laminar boundary layer flows.

So, d of d x integral 0 to delta 1 minus in place of velocity u, we will write time average velocity u bar U infinity u bar by U infinity dy is equal to nu by U infinity square del u bar by del y at y is equal to 0 that means right hand side term is tau w by rho U infinity square.

To use this equation, we need to find the velocity profile or we need to approximate the velocity profile for turbulent flows Prandtl and von Karman used the velocity profile knowing from the Blasius correlation for the pipe flow. So, you can see that Prandtl and von Karman used the model from Blasius model which was developed for shear at the wall for a circular pipe.

So, Blasius proposed that for the Reynolds number based on diameter in the range of 4000 to 10 to the power 5, c f is 0.0791 Re D to the power minus 1 by 4, where c f is tau w by half rho u m square where u m is the mean velocity. So, you can see that here this is based on the correlations ok.

So, from the experimental values, Blasius actually developed this model for the shear stress for a pipe flow. And Prandtl and von Karman showed that velocity profile in the pipe, so Prandtl and von Karman showed velocity profile in the pipe ok. It will be u bar by u CL is equal to y by r naught to the power 1 by 7 ok.

So, obviously, r naught is the radius of the pipe, and u CL is the centre line velocity. So, this was actually showed by Prandtl and von Karman for the pipe flow turbulent pipe flow. And this we can use for this turbulent boundary layer flow for a flat plate.

So, in this equation, you can see that obviously, for the pipe flow, we have this radius r naught and the centre line velocity of u CL, but in the case of flow over flat plate this radius r naught is replaced with the boundary layer thickness delta. And the centre line velocity is replaced with the free stream velocity U infinity ok. So, with that when Prandtl and von Karman used this velocity profile it gave reasonably good result for the boundary layer thickness and the wall shear stress.

So, if you replace this u CL with U infinity, and r naught with boundary layer thickness delta, then we can use u bar by U infinity is equal to y by delta to the power 1 by 7 ok. So, this is for flow over flat plate ok. And this is known as 1-7 law of velocity profile. So, now, you can see that we have the velocity profile for this turbulent boundary layer over a flat plat.

Now, in the momentum integral equation, we need to find the shear stress because right hand side we have tau w by rho infinity square. So, obviously, you can see that for this velocity profile this one-seventh law of velocity profile, if you find the shear stress at the wall it will become infinity ok.

So, for that reason the shear stress in the momentum integral equation is not calculated from this velocity profile, rather the value of this shear stress was taken based on the wall shear stress of the pipe flow. So, already we have shown the drag coefficient for the pipe flow that was used for the shear stress calculation of this flow over flat plate.

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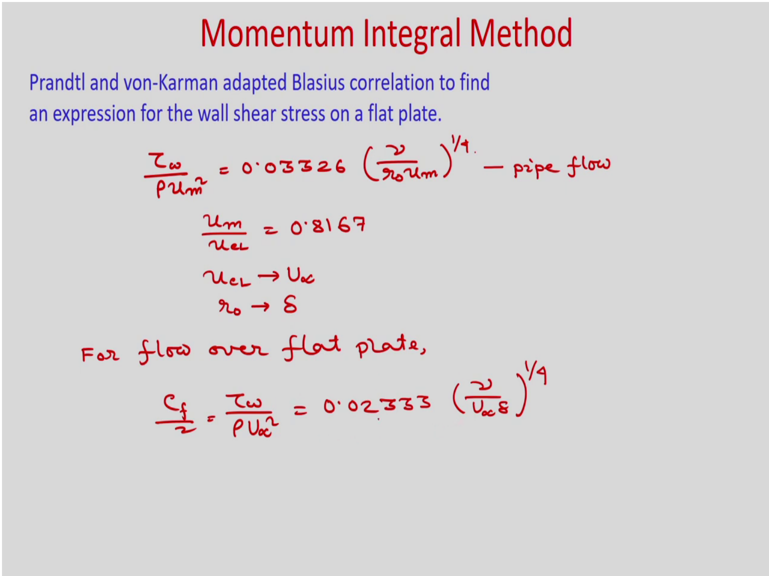

So, Prandtl and von Karman adapted Blasius correlation to find an expression for the wall shear stress on the flat plate. So, tau w by rho u m square is equal to 0.03326 nu by r naught u m to the power 1 by 4 ok.

So, and you can see that this is for pipe flow, and this correlation was used for the solution of this turbulent boundary layer flow over a flat plate with a condition that u m by u CL is equal to 0.8167. And obviously, u CL is replaced with the U infinity and r naught is replaced with boundary layer thickness delta.

So, if you put it here, then for flow over flat plate we can write c f by 2 is equal to tau w by rho U infinity square is equal to 0.02333 nu by U infinity delta to the power 1 by 4. So, now, we use one-seventh law of velocity profile in the momentum integral equation, and this shear stress expression we use in the momentum integral equation.

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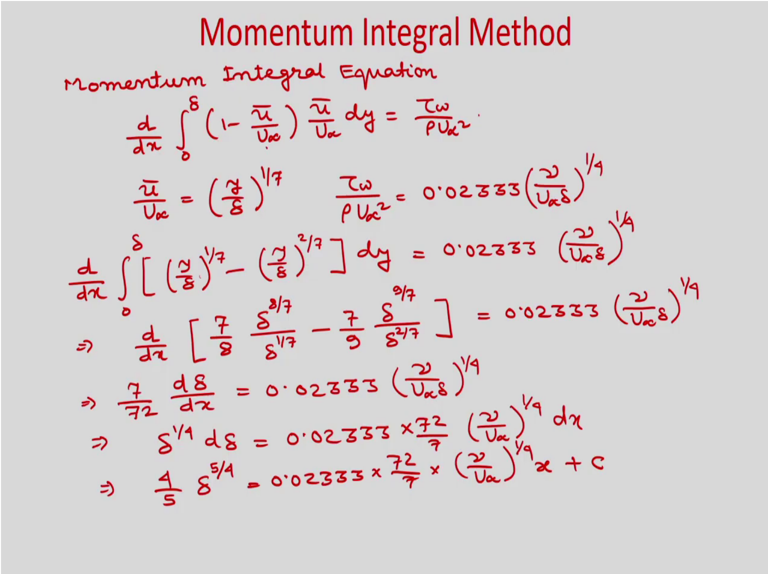

So, we have this momentum integral equation d of d x integral 0 to delta 1 minus u bar by U infinity u bar by U infinity dy is equal to tau w by rho U infinity square. So, you can see in this velocity profile, we use one-seventh law of velocity profile. And this tau w we use the Blasius correlation for the pipe flow with some modification.

So, if you put all these values, u bar by U infinity as y by delta to the power 1 by 7, and tau w by rho U infinity square as 0.02333 nu by U infinity delta to the power 1 by 4 ok. So, if you put it in this expression, what you will get? d of dx integral 0 to delta ok. So, if you multiply here, so you will get y by delta to the power 1 by 7 minus u bar by U infinity whole square, so y by delta to the power 2 by 7 dy is equal to 0.02333 nu by U infinity delta to the power 1 by 4.

So, if you integrate it and put the limits, then you will get d of dx 7 by 8 delta to the power 8 by 7 to the power divided by delta to the power 1 by 7 minus 7 by 9 delta to the power 9 by 7 divided by delta to the power 2 by 7 is equal to 0.02333 nu by U infinity delta to the power 1 by 4. So, you can see that this will be 7 by 72 d delta by dx is equal to 0.02333 nu by U infinity delta to the power 1 by 4.

So, now, we need to integrate this. So, if you rearrange, you will get delta to the power 1 by 4 d delta is equal to 0.02333, so 72 by 7 nu by U infinity to the power 1 by 4 dx. So, if you integrate it, we will get 4 by 5 delta to the power 5 by 4 is equal to 0.02333 72 by 7 nu by U infinity to the power 1 by 4 x plus integration constant c.

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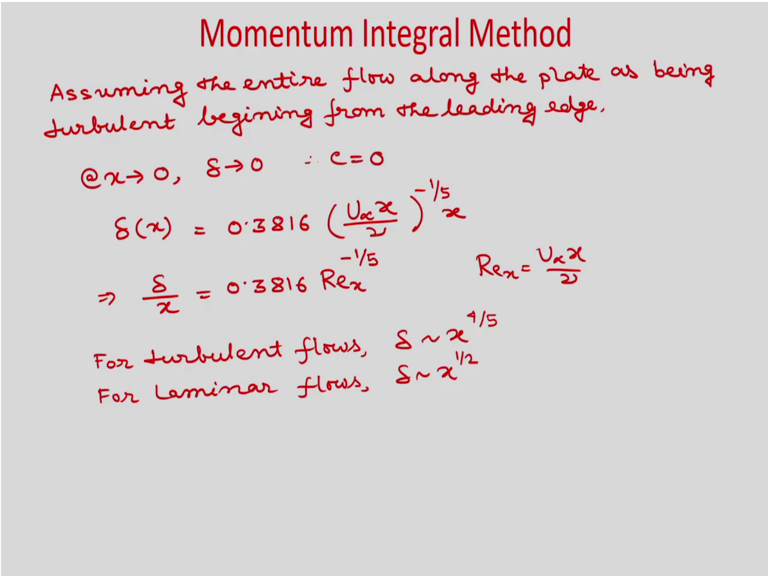

So, to find this integration constant, just we will assume that the turbulent flow exist in the entire flat plate. And at the leading edge as x tends to 0, obviously, the boundary layer thickness delta tends to 0. So, if you invoke this condition, then the integration constant will become 0. So, assuming the entire flow along the plate as being turbulent beginning from the leading edge ok. So, we can write as x tends to 0, delta tends to 0, so c will become 0. So, this assumption was first proposed by Prandtl.

And now you can find after rearrangement delta x as 0.3816 U infinity x by nu to the power minus 1 by 5 x. So, you will get delta by x is equal to 0.3816. And you can see this is the Reynolds number based on x. So, it is U infinity x by nu. So, we can write Re x to the power minus 1 by 5.

So, from here you can see that delta for turbulent flows, it varies as x to the power 4 by 5 ok. For turbulent flows, delta varies as x to the power 4 by 5. And for laminar flows, we have already shown that delta which is your boundary layer thickness varies as x to the power half. So, for turbulent flow, this delta varies as x to the power 4 by 5 assuming the one-seventh law of velocity profile.

We have already used the Blasius correlation for the pipe flow with some modification to find the shear stress near to the wall for this flow over plat plate. Now, if you remember that those that c f we have represented in terms of the boundary layer thickness delta. Now, we know the value of boundary layer thickness delta. So, we can put this delta in that expression, and we can find the wall shear stress in terms of the x coordinate ok.

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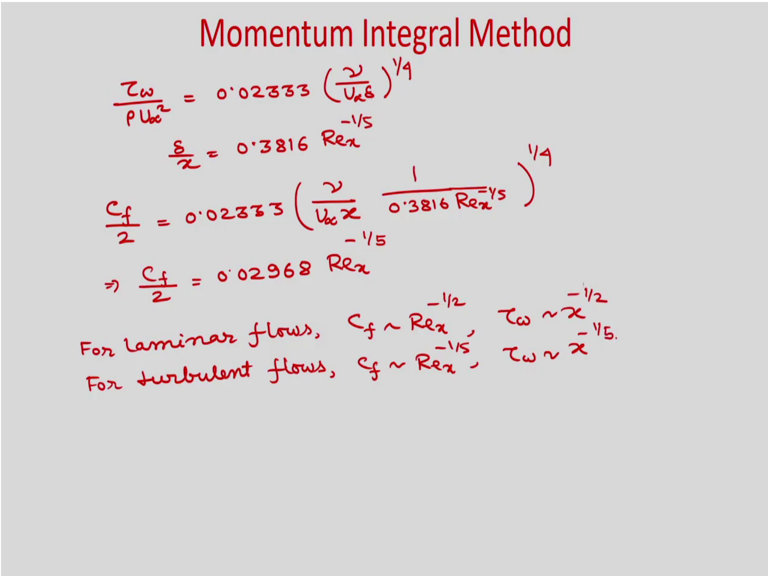

So, you can see that we have written the shear stress tau w by rho infinity square is equal to 0.02333 nu by U infinity delta to the power 1 by 4. So, here you can see that we have expressed in terms of the boundary layer thickness delta. Now, we know that delta by x as 0.3816 Re x to the power minus 1 by 5.

So, this you can see that we can write c f by 2 is equal to 0.02333 nu by. So, you can see that we can write U infinity delta we are putting. So, it will be x and we have 0.3816 Re x to the power minus 1 by 5 to the power 1 by 4 ok. So, from here you can see that c f by 2 we can write, so this will be also 1 by Re x ok. So, if you rearrange, you will get 0.02968 Re x to the power minus 1 by 5 ok.

So, if you remember that for laminar flows ok, c f varies as Re x to the power minus half where tau w is varying with x to the power minus half. And for turbulent flows, from this expression, you can see that c f varies as Re x to the power minus 1 by 5, so tau w varies as x to the power minus 1 by 5.

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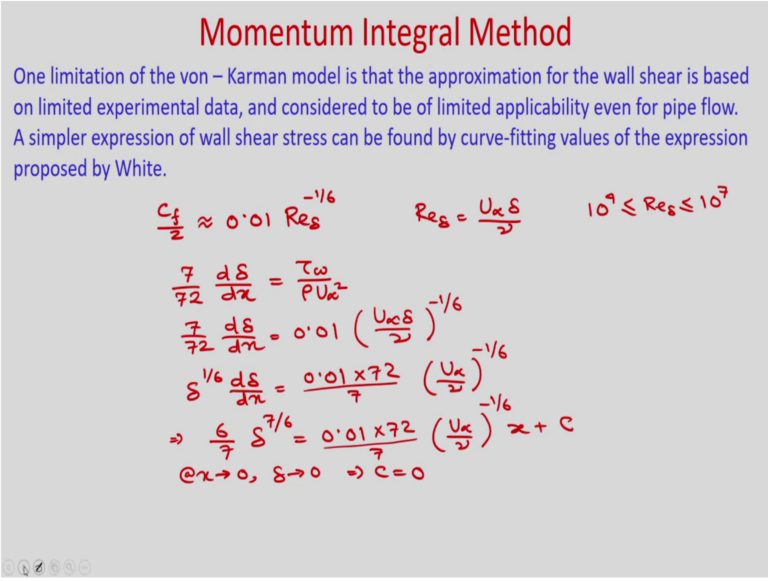

One of the limitation of this Prandtl von Karman solution is that the correlation what we have used that is based on limited experimental data. So, later there are many improvements based on the experimental values, and different scientists proposed correlations which are better than whatever we have just used for the shear stress correlation.

So, one of the correlation now we will use which was proposed by White. And later it was actually written simplified way like c f by 2 as 0.01 Re delta to the power minus 1 by 6 where Re delta is U infinity delta by nu, and it is valid in the range of Reynolds number delta 10 to the power 4 and 10 to the power 7 ok. So, this is actually from the curve-fitting values of the expression proposed by White.

And this if you use in the momentum integral equation which we have already derived you can see that after putting the one-seventh law of velocity profile, we have derived 7 by 72 d delta by d x is equal to tau w by rho U infinity square. So, you can see that we can now use this 7 by 72 d delta by dx is equal to 0.01 and Re delta is U infinity delta by nu to the power minus 1 by 6 ok.

So, after rearrangement, we can write delta to the power 1 by 6 d delta by dx is equal to 0.01 into 72 by 7 U infinity by nu to the power minus 1 by 6. So, if you integrate it, we will get 6 by 7 delta to the power 7 by 6 is equal to 0.01 into 72 by 7 U infinity by nu to the power minus 1 by 6 x plus integration constant c.

So, again assuming the turbulent boundary layer flow prevails from the leading edge of the flat plate, we can use that x tends to 0, delta tends to 0. So, putting this condition, you can find the integration constant c as 0. So, as x tends to 0, delta tends to 0, that will give c is equal to 0.

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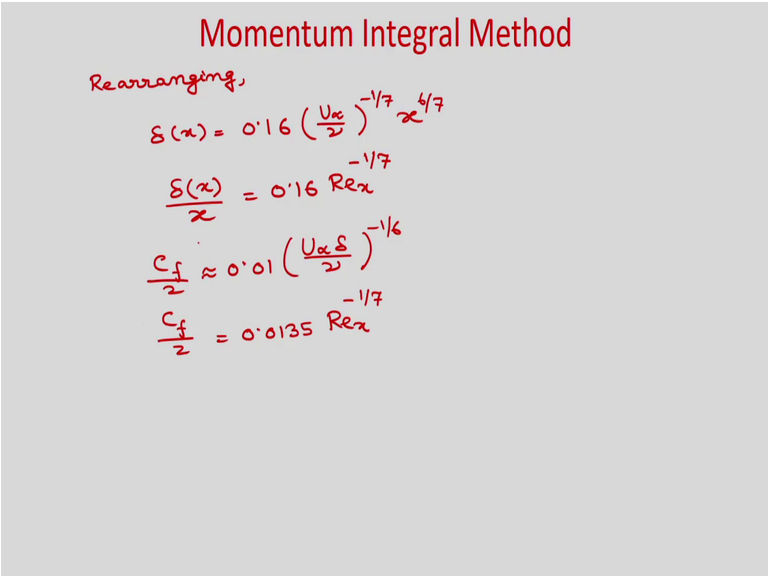

So, if you put c is equal to 0 and rearrange it, then we will get delta x is equal to 0.16 U infinity by nu to the power minus 1 by 7 x to the power 6 by 7. Now, after rearranging, we can write delta by x as 0.16 Re x to the power minus 1 by 7. And now this delta if you put in the expression of c f, then we will get c f by 2 as 0.01 U infinity delta by nu to the power minus 1 by 6, so that we have written now.

So, after putting the value of delta by x ok in this expression, so we can write c f by 2 as 0.0135 Re x to the power minus 1 by 7. So, you can see this expression of this boundary layer thickness and the skin friction coefficient, we have represented in terms of the Reynolds number.

So, in today's class, we use the momentum integral equation which we derived for the laminar boundary layer flow, and used the Blasius correlation for the shear stress in the momentum integral equation. And also we have used one-seventh law of velocity profile as the velocity profile in the momentum integral equation.

So, to solve this turbulent boundary layer flow over a flat plate using momentum integral equation, we use the Blasius correlation for the shear stress in terms of the boundary layer thickness, and then we use the one-seventh law of velocity profile to solve the momentum integral equation.

So, using these correlations, we have found the boundary layer thickness delta. And from there we have expressed the skin friction coefficient in terms of Reynolds number. The one of the limitation for this Prandtl von Karman solution is that the Blasius correlation actually based on limited data.

So, later many scientists proposed different correlations based on experimental data. And using White’s correlation with a curve-fitting, again we use the shear stress correlation and we found the boundary layer thickness and the skin friction coefficient in terms of Reynolds number based on the free stream velocity U infinity the x and the kinematic viscosity nu.

Thank you.