Viscous Fluid Flow
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Module - 02
Steady One-dimensional Rectilinear Flows
Lecture - 03
Plane Poiseuille Flow with Slip and Thin Film Flow

Hello everyone, today we will continue with the exact solutions of Navier-Stoke equations in Cartesian coordinates. In today's lecture, we will consider first Plane Poiseuille Flow with a Slip at the wall and then we will consider plane Poiseuille flow. First, we will start with plane Poiseuille flow with slip and we will consider the same slip in both the plates.

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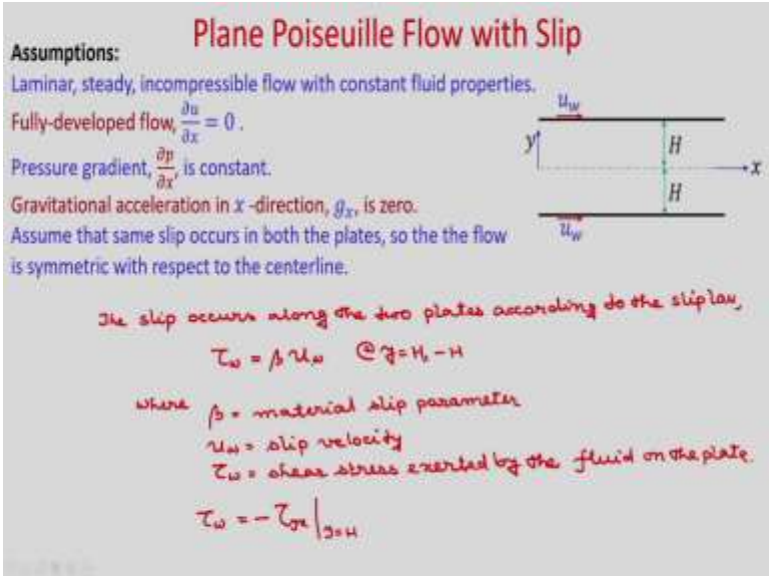

So, we have the assumptions of laminar, steady, incompressible flow with constant fluid properties. Fully developed flow, so $\frac{\partial u}{\partial x}$ will be 0, pressure gradient is constant, gravitational acceleration in x-direction, g_x is 0 and assume that same slip occurs in both the plates, so the flow is symmetric with respect to the centerline.

So, these are the 2 plates infinite parallel plates, this is the x-direction and y is measured from the center line and the distance between 2 parallel plates is 2H and slip is there in both the walls. The slip occurs along with the 2 plates according to the slip law, τ_w which is your wall shear stress is equal to βu_w at y is equal to H and -H; that means, at walls.

Where, β is known as the material slip parameter, material slip parameter u_w is your slip velocity at the wall.

So, you can see your material slip parameter is inversely proportional to the slip velocity u_w and τ_w is your shear stress, exerted by the fluid on the plate. Obviously, τ_w then will be just negative of τ_{yx} at y is equal H or -H. So, you can see that we can apply now the boundary condition either at y is equal to H or -H, u is equal to u_w or you can apply this shear stress star w in terms of β and u_w.

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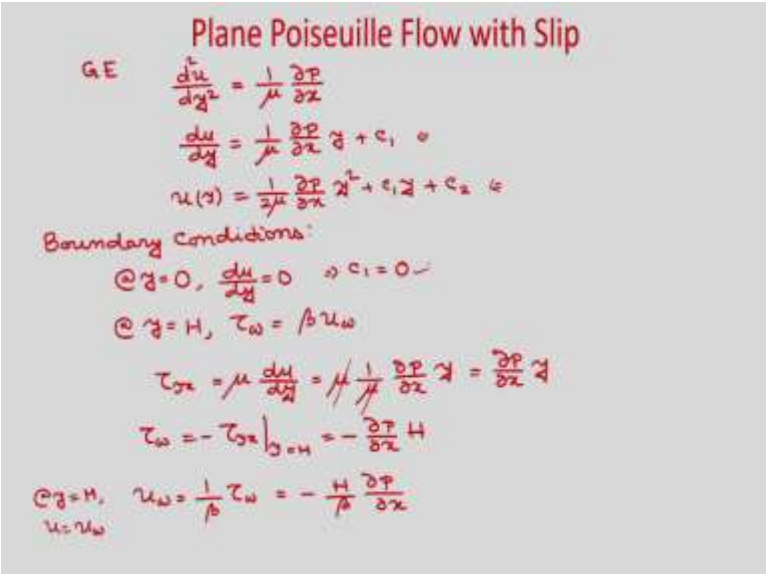

So, considering this fully developed flow with other assumptions we can write the governing equation as;

$$\frac{\partial^2 u}{\partial y^2} = \frac{1}{\mu}\frac{\partial p}{\partial x}$$

$$\frac{\partial u}{\partial y} = \frac{1}{\mu}\frac{\partial p}{\partial x}y + C_1$$

If you integrate then you will get the velocity profile which is your

$$u(y) = \frac{1}{2\mu}\frac{\partial p}{\partial x}y^2 + C_1 y + C_2$$

You see at y is equal to 0 we have assumed that it is symmetric because if you have a slip at both walls is same then you can use symmetry boundary condition at y is equal to 0 and that will give $\frac{\partial u}{\partial y}$ is equal to 0. And at y is equal to H, you can apply the wall shear stress τ_w is equal to βu_w.

So, now you can see if you apply this boundary condition at y is equal to 0, du/dy is equal to 0 from the first expression you can get c_1 is equal to 0; so that will give c_1 is equal to 0. Now, let us find the shear stress

$$\tau_{yx} = \mu\frac{\partial u}{\partial y} = \frac{\partial p}{\partial x}y$$

Now, you can calculate the wall shear stress τ_w is equal to $-\tau_{yx}$ and then we will apply the second boundary condition. So,

$$\tau_w = \tau_{yx}|_{y=H} = -\frac{\partial p}{\partial x}H$$

So, now, apply this boundary condition at y is equal to H,

$$u_w = \frac{1}{\beta}\tau_w = -\frac{H}{\beta}\frac{\partial p}{\partial x}$$

So, this u_w expression you can put it here and find the other constant C_2; C_1 is 0 anyway.

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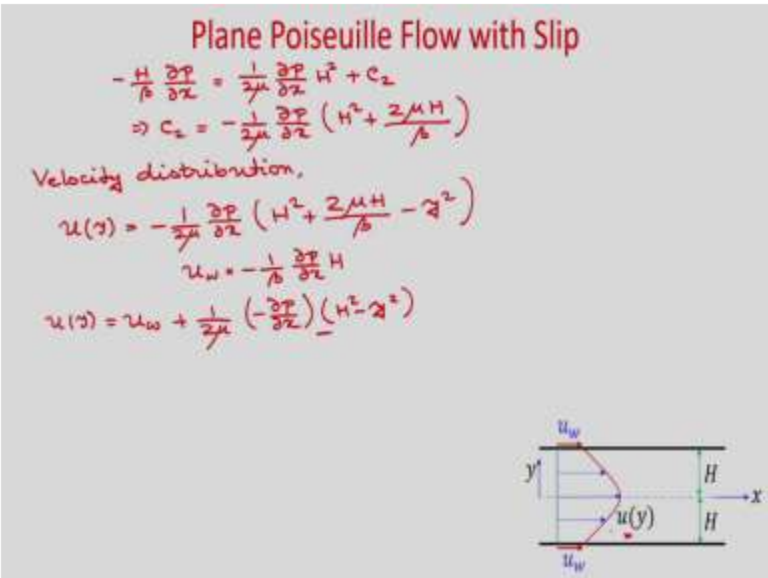

So, if you put in the left hand side u_w expression then it is

$$-\frac{H}{\beta}\frac{\partial p}{\partial x} = \frac{1}{2\mu}\frac{\partial p}{\partial x}H^2 + C_2$$

So, from here you can find the value of C_2 as

$$= \frac{1}{2\mu}\frac{\partial p}{\partial x}\left(H^2 + \frac{2\mu H}{\beta}\right)$$

So, if you substitute this constant C_2 in the expression of velocity then you will get the velocity profile, velocity distribution u(y) is

$$u(y) = -\frac{1}{2\mu}\frac{\partial p}{\partial x}\left(H^2 + \frac{2\mu H}{\beta} - y^2\right)$$

$$u(y) = u_w + \frac{1}{2\mu}\left(-\frac{\partial p}{\partial x}\right)(H^2 - y^2)$$

So, you can see here that u is just the superposition of velocity u_w and the velocity from the plane Poiseuille flow without slip.

So, u_w is your slip velocity and this is the velocity profile of plane Poiseuille flow without slip. So, this is the superposition of these two. So, what will be the velocity profile? So, obviously you can see it is parabolic. You have this from plane Poiseuille flow whatever solution you are getting so that is your parabolic in nature with a constant velocity u_w at the wall.

So, you can see that at the wall you have u_w and this is your velocity profile which is function of y.

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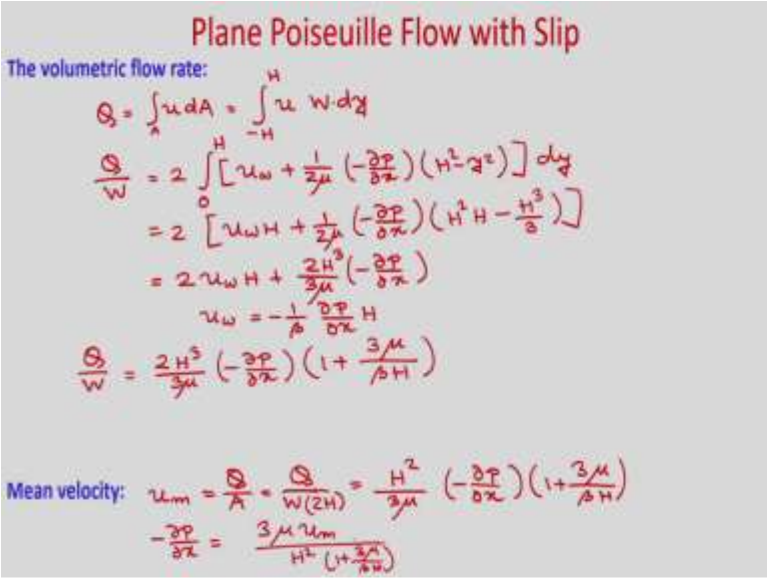

Now, let us calculate the volume flow rate. So, volume flow rate you can calculate Q as

$$Q = \int_A u(y)\, dA = \int_{-H}^{H} uW dy$$

$$\frac{Q}{W} = 2\int_0^H [u_w + \frac{1}{2\mu}\left(-\frac{\partial p}{\partial x}\right)(H^2 - y^2)]\, dy$$

$$= 2\left[u_w H + \frac{1}{2\mu}\left(-\frac{\partial p}{\partial x}\right)\left(H^2 H - \frac{H^3}{3}\right)\right]$$

$$= 2u_w H + \frac{2H^3}{3\mu}\left(-\frac{\partial p}{\partial x}\right)$$

$$u_w = -\frac{1}{\beta}\frac{\partial p}{\partial x} H$$

$$\frac{Q}{W} = \frac{2H^3}{3\mu}\left(-\frac{\partial p}{\partial x}\right)\left(1 + \frac{3\mu}{\beta H}\right)$$

Now, we want to find the average velocity or mean velocity and from the volume flow rate easily you can calculate it because if you divide the flow area then you will get the mean velocity.

So, mean velocity u_m you will get Q/A, where A is the flow area. In this case, you have W and flow area is 2H. So, from here you will get average velocity as

$$u_m = \frac{Q}{A} = \frac{Q}{W(2H)} = \frac{H^2}{3\mu}\left(-\frac{\partial p}{\partial x}\right)\left(1 + \frac{3\mu}{\beta H}\right)$$

So, in the flow situation if average velocity is given then the pressure gradient you can find from this expression or if pressure gradient is provided then you can find the average velocity. So, the pressure gradient will be

$$-\frac{\partial p}{\partial x} = \frac{3\mu u_m}{H^2\left(1 + \frac{3\mu}{\beta H}\right)}$$

So, here you can see that mean velocity is positive, other quantities are positive so; obviously, $-\frac{\partial p}{\partial x}$ is greater than 0.

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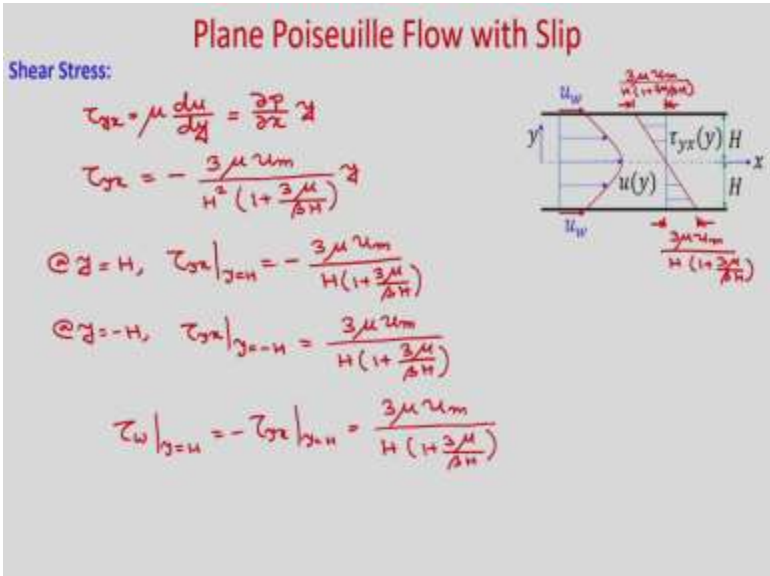

Now, let us find the shear stress distribution inside the wall and from there you will be able to calculate the wall shear stress and if you are interested you can find what is the force acting on the flat wall. So, shear stress τ_{yx} you will just find as

$$\tau_{yx} = \mu\frac{\partial u}{\partial y} = \frac{\partial p}{\partial x}y$$

$$\tau_{yx} = -\frac{3\mu u_m}{H^2\left(1+\frac{3\mu}{\beta H}\right)}y$$

So, at y is equal to H, your τ_{yx} will be

$$\tau_{yx}|_{y=H} = -\frac{3\mu u_m}{H^2\left(1+\frac{3\mu}{\beta H}\right)}$$

At y is equal to –H τ_{yx} will be

$$\tau_{yx}|_{y=-H} = \frac{3\mu u_m}{H^2\left(1+\frac{3\mu}{\beta H}\right)}$$

So, how it will vary inside the flow domain? So, y is equal to -H you have a positive value. So, on the top wall, you have a negative value of τ_{yx} and it linearly varies.

So, from here to here it linearly varies; obviously, at the center we have already assumed that shear stress is 0 due to symmetry, so τ is equal to 0 at the center. And, if you want to calculate the wall shear stress at y is equal to H then it will be just $-\tau_{yx}$ at y is equal to H.

So, this will be

$$\tau_w|_{y=H} = -\tau_{yx}|_{y=H} = \frac{3\mu u_m}{H^2\left(1+\frac{3\mu}{\beta H}\right)}$$

So, if you are interested to calculate the skin friction coefficient and friction factor. So you will be able to do so just using these expressions. Let us consider a fully developed flow of a thin film that is flowing in that downward direction.

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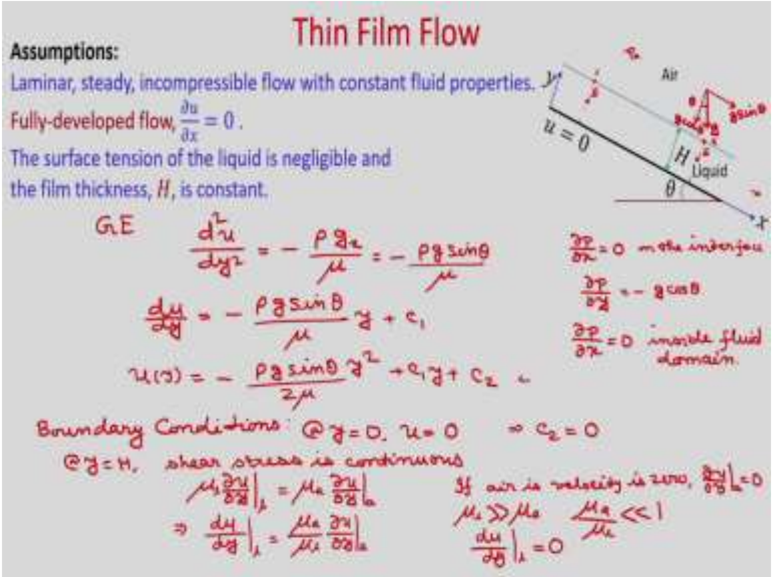

So, if you consider this is an inclined flat plate ok. So, a liquid is just flowing down due to gravity in the downward direction. So, in this direction it is going down and the film thickness is H and it is very small and we are neglecting the surface tension. So, that your surface will be straight; that means, the interface between this air and liquid, this interface will be straight and thickness will be constant as it is going down.

So, we will take for convenience x along the plate, and y is measured perpendicular direction from the plate. And at the bottom wall; obviously, u is equal to 0 and the plate is inclined with the horizontal with angle θ. So, you can see that gravity will be acting in this direction and it will have two components.

So, one is in this direction, another is along the x-direction and this is your θ so; obviously, this is your gsinθ and in this direction you have gcosθ. So, when you will consider the x momentum equation you need to consider this gravity gsinθ right.

So, these are the assumptions we are taking; laminar, steady, incompressible flow with constant fluid properties, it is a fully developed flow, so; obviously, $\frac{\partial u}{\partial x}$ will be 0. The surface tension of the liquid is negligible and the film thickness H is constant. So, with this you can write the governing equation as

$$\frac{\partial^2 u}{\partial y^2} = -\frac{\rho g_x}{\mu}$$

And here we are not considering the pressure gradient in the x-direction because $\frac{\partial p}{\partial x}$ is 0, because outside the film you have the atmosphere, so; obviously, everywhere you have pressure p atmosphere and at the free surface if you consider and consider 2 points on the free surface. So, you can see at this position or at this position you will have the same pressure because it is just the atmospheric pressure. So, $\frac{\partial p}{\partial x}$ will be 0 on the interface.

If it is 0 now you see $\frac{\partial p}{\partial y}$ is constant, you can see from here that your $\frac{\partial p}{\partial y}$ is nothing but -gcosθ and it is constant. So, if this is constant then your pressure inside if you take at any point ok. So, and at the same distance from the top if you consider, so it will also have the same pressure at this location.

So, if this is distance δ and if you consider at this point pressure and this point pressure as $\frac{\partial p}{\partial y}$ is equal to -gcosθ and it is nothing but the hydrostatic pressure. So, there will be the same pressure at this location.

So, again you will get $\frac{\partial p}{\partial x}$ is equal to 0 inside fluid domain ok. So, hence we are not considering the pressure gradient here because $\frac{\partial p}{\partial x}$ will be 0. And you know the value of g_x. So, g_x is just gsinθ, so we can write

$$\frac{\partial^2 u}{\partial y^2} = -\frac{\rho g_x}{\mu} = -\frac{\rho g \sin\theta}{\mu}$$

So, if you integrate once you will get

$$\frac{\partial u}{\partial y} = -\frac{\rho g \sin\theta}{\mu} y + C_1$$

Another term if you integrate you will get

$$u(y) = -\frac{\rho g \sin\theta}{\mu} y^2 + C_1 y + C_2$$

Now, let us discuss about the boundary conditions.

One boundary condition is very simple because at the wall you will have the velocity u is equal to 0. So, boundary conditions at y is equal to 0, you have u is equal to 0. So, from here you can see if you put this in this expression then you will get the integration constant c 2 as 0.

Now, what about the other boundary condition at y is equal to H? So, at y is equal to H we can apply the shear stress continuity. So, we will apply shear stress continuity, and if we assume that at the air side if it is a stationary fluid then your shear stress at the interface will be 0 ok.

So, let us see it. So at y is equal to H, we are applying shear stress is continuous. So, you can write

$$\mu_l \left.\frac{\partial u}{\partial y}\right|_l = \mu_a \left.\frac{\partial u}{\partial y}\right|_a$$

$$\left.\frac{\partial u}{\partial y}\right|_l = \frac{\mu_a}{\mu_l}\left.\frac{\partial u}{\partial y}\right|_a$$

In this expression if you assume that air side velocity is 0; that means, it is stationary here then obviously, the velocity gradient will be 0 and velocity gradient 0 means just $\frac{\partial u}{\partial y}$ at the interface at y is equal to H it will be 0.

Another way you can explain that liquid side viscosity is much much greater tha air. So, if μ_l is much much greater than μ_a then the right-hand side term, obviously, will be much much smaller than the left hand side term. So, you can see here so, if air velocity is 0 then $\frac{\partial u}{\partial y}$ in air side it will be 0.

So, you can see that $\frac{\partial u}{\partial y}$ in the liquid side, it will be 0. Another way you can see that if μ_l is much much greater than μ_a. So, you can see that $\frac{\mu_a}{\mu_l}$ will be much much less than 1; if it is so it will be

$$\left.\frac{\partial u}{\partial y}\right|_l = 0$$

So; obviously, from either of these two conditions you can use and you can put the velocity gradient is 0; that means, shear stress is 0 on the interface.

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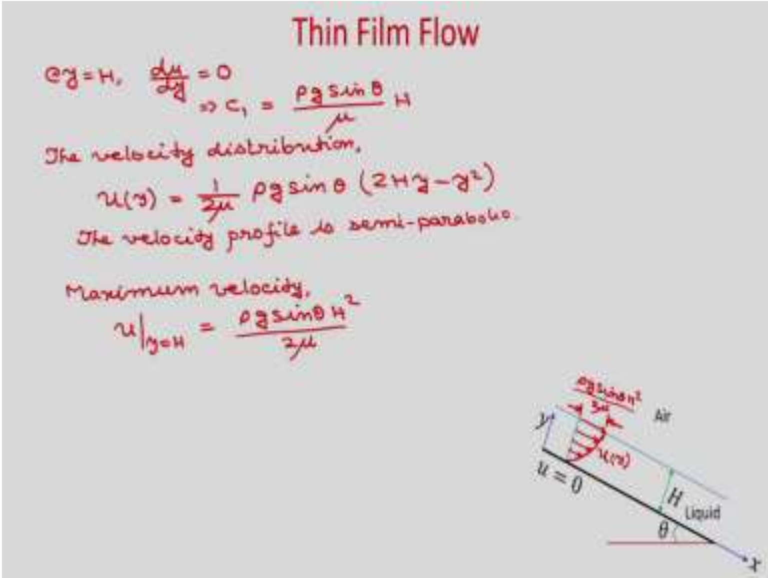

So, we are using at y is equal to H, shear stress is 0; that means, $\frac{\partial u}{\partial y}$ is equal to 0 ok. So, if you put this you will get

$$C_1 = \frac{\rho g \sin\theta}{\mu} H$$

So, if you put these constants in the velocity distribution you will get the velocity distribution as

$$u(y) = \frac{1}{2\mu} \rho g \sin\theta \, (2Hy - y^2)$$

So, you can see that the velocity profile is semi parabolic.

So, you can see; you will get the maximum velocity at y is equal to H. So, if you plot the velocity profile so your velocity profile will be semi-parabolic and maximum velocity will occur at the interface so you will get a velocity profile like this.

So, it will cut perpendicularly because $\frac{\partial u}{\partial y}$ is equal to 0 at y is equal to H so, this is your velocity profile. So, the maximum velocity you will get at y is equal to H and the value is

$$u|_{y=H} = \frac{\rho g \sin\theta \ H^2}{2\mu}$$

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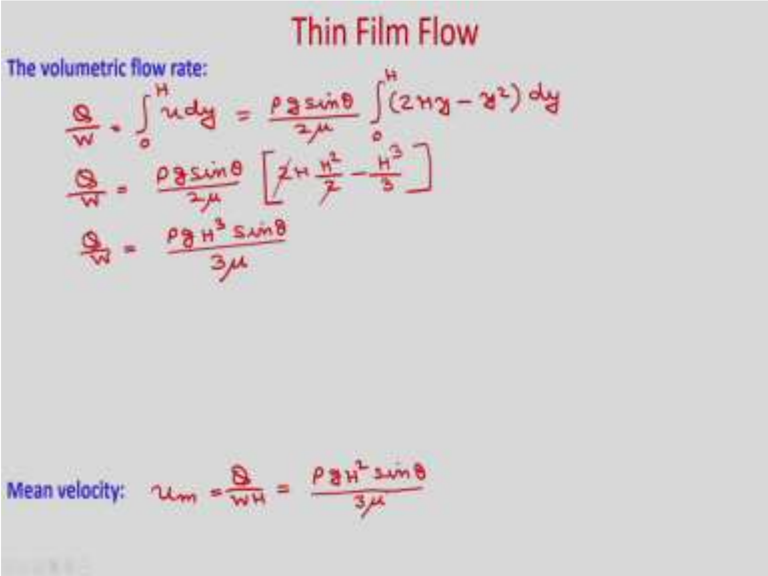

Now, let us calculate the volumetric flow rate as well as the average velocity. So,

$$\frac{Q}{W} = \int_0^H u\, dy = \frac{\rho g \sin\theta}{2\mu} \int_0^H (2Hy - y^2)\, dy$$

$$\frac{Q}{W} = \frac{\rho g \sin\theta}{2\mu} \left[\frac{2HH^2}{2} - \frac{H^3}{3}\right]$$

$$\frac{Q}{W} = \frac{\rho g H^3 \sin\theta}{3\mu}$$

And from here you will be able to calculate now the average velocity or mean velocity over a cross-section of the film as

$$u_m = \frac{Q}{WH} = \frac{\rho g H^2 \sin\theta}{3\mu}$$

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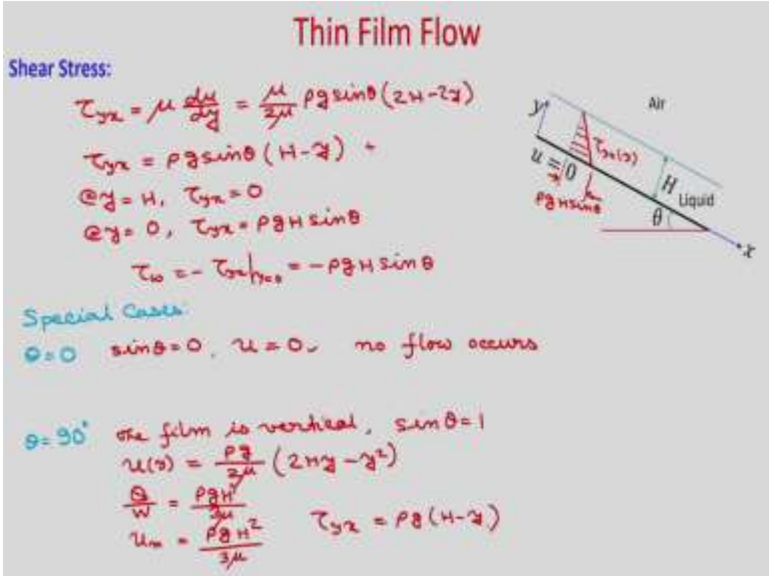

Now, let us calculate the shear stress distribution inside the flow domain at y is equal to H; that means, at the interface already we have invoked the boundary condition shear stress as 0. So, shear stress will be 0 at the interface. So, let us find the expression

$$\tau_{yx} = \mu\frac{\partial u}{\partial y} = \frac{\mu}{2\mu}\rho g \sin\theta\,(2H - 2y)$$

$$\tau_{yx} = \rho g \sin\theta\,(H - y)$$

So, at y is equal to H, τ_{yx} is 0, at y is equal to 0, τ_{yx} is equal to $\rho gH \sin\theta$. And, you can see from this expression that it will vary linearly, interface it is 0 to a value of $\rho gH \sin\theta$. And this is the linear variation.

So this is your shear stress distribution inside the fluid domain and at the bottom wall if you want to calculate the shear stress it will be

$$\tau_w = -\tau_{yx}|_{y=0} = -\rho gH \sin\theta$$

Now, let us consider two special cases. One is your horizontal plate; that means θ is equal to 0 and vertical plate where θ is equal to 90 degree. So obviously, you can see if it is a horizontal plate sinθ will be 0 and if it is a vertical plate then sinθ will be 1 and in that case if it is placed horizontally; obviously, velocity will be 0 because sinθ is 0.

So, special cases; at θ is equal to 0 and θ is equal to 90 degree. So, if θ is equal to 0 then sinθ is equal to 0, then you will get u as 0. So, everywhere velocity will be 0 because the flow is just driven by the gravity. If it is a horizontal plate then there will be no driving force for the fluid flow that's why u is equal to 0 so, no flow occurs.

If the film is vertical; the film is vertical; that means, sinθ will be 1. So, velocity distribution will be

$$u(y) = \frac{\rho g}{2\mu} \sin\theta\,(2Hy - y^2)$$

Volume flow rate per unit width will be

$$\frac{Q}{W} = \frac{\rho g H^3}{3\mu}$$

And

$$\tau_{yx} = \rho g(H - y)$$

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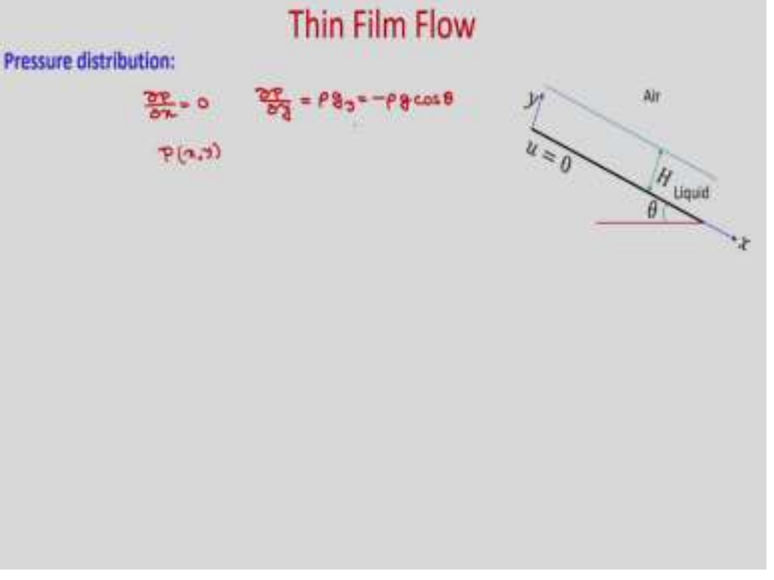

Now, let us discuss about the pressure distribution inside the fluid domain; obviously, there is no pressure gradient in the axial direction that we have already discussed that $\frac{\partial p}{\partial x}$ is 0. And from the y momentum equation, you will get

$$\frac{\partial p}{\partial y} = \rho g_y = -\rho g \cos\theta$$

.And we know that p is a function of x, y and let us say that g_z is equal to 0 ok.

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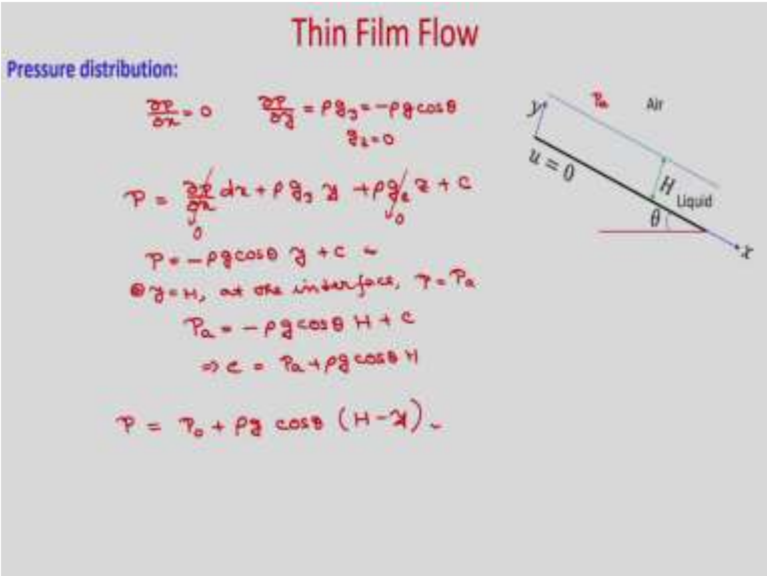

So, if g_z is 0 then from this expression we have already derived

$$p = \frac{\partial p}{\partial x}dx + \rho g_y y + \rho g_z z + c$$

$$p = -\rho g \cos\theta \;\; y + c$$

And from here you can see that at y is equal to H; that means, at the interface p is equal to p_a ok. So, if you put here then you will get

$$p_a = -\rho g \cos\theta \;\; H + c$$

So, now integration constant c you can find

$$c = p_a + \rho g \cos\theta \ H$$

Now you put the value of this in this equation then you will get the pressure distribution as

$$p = p_o + \rho g \cos\theta \ (H - y)$$

So obviously, this is hydrostatic pressure distribution so it linearly varies.

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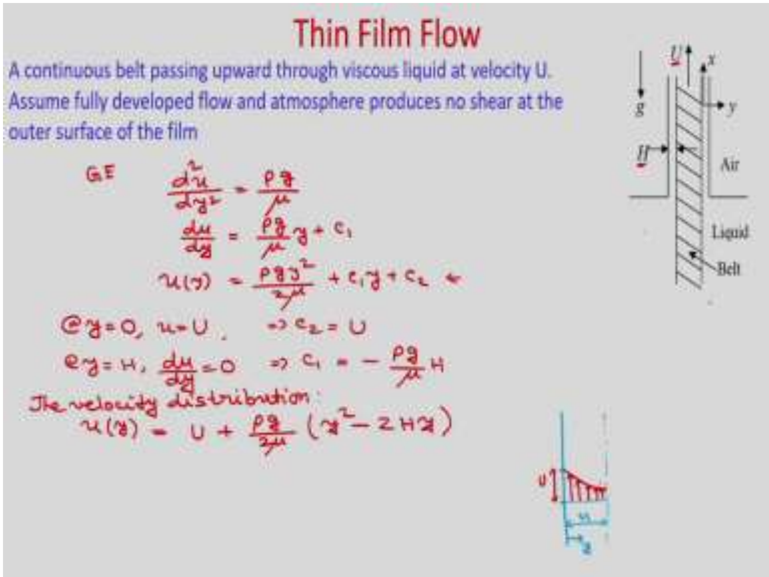

The application of this thin film flow in the vertical direction whatever special case we have discussed let us take one example. Let us consider that one belt vertically moving upwards in a viscous liquid. So, if you consider here that liquid pool is here and one belt is moving upward with a constant velocity u.

When this belt will move upwards due to the viscous effect this liquid will form a thin film over this belt ok. And this thin film let us consider that it is very thin and the thickness is H and gravity is acting; obviously, in the downward direction and we are taking the coordinates x in vertically upward direction, and y is measured from the belt.

So, for this case now we assume that a continuous belt passing upward through viscous liquid at velocity u and assume fully developed flow and atmosphere produces no shear at the outer surface of the film. So; that means, shear stress is 0 at the interface and;

obviously, you can neglect the surface tension effect, so and assume that the film is maintaining a constant thickness H.

So, whatever we have already discussed in the thin film flow in an inclined plate. So, it is a special case, but the plate is moving upward direction in with velocity u. So, our governing equation will start with

$$\frac{\partial^2 u}{\partial y^2} = \frac{\rho g}{\mu}$$

$$\frac{\partial u}{\partial y} = \frac{\rho g}{\mu} y + C_1$$

So, this is your governing equation and if you integrate twice you will get the velocity profile then you will get the velocity profile as

$$u(y) = \frac{\rho g y^2}{2\mu} + C_1 y + C_2$$

So, the velocity boundary condition at y is equal to 0. So; obviously, it will have the belt velocity and belt velocity is u so u is equal to U. So, from here you will get C_2 is equal to u and at y is equal to H we are assuming that shear stress is 0, so du/dy is equal to 0. So, this will give C_1 is equal to $-\frac{\rho g}{\mu} H$.

So, from here if we put the integration constants in the expression then velocity distribution will be velocity distribution will be

$$u(y) = U + \frac{\rho g}{2\mu}(y^2 - 2Hy)$$

If you see this expression it is just a superposition of two velocities one is u which actually the belt is moving in the upward direction and this expression with a negative sign you have already derived in the earlier case. In that case, actually x was in the opposite direction that’s why you are getting this expression with a minus sign.

So; obviously, from this expression, you will be able to find what is the volume flow rate. So, this is the belt and this is the interface, y is measured from the belt, the distance is H and velocity at the wall is u. So, velocity distribution so this will be u and it will be like this. So, it will cut perpendicularly because du/dy is equal to 0 at y is equal to H. So, this is the velocity profile.

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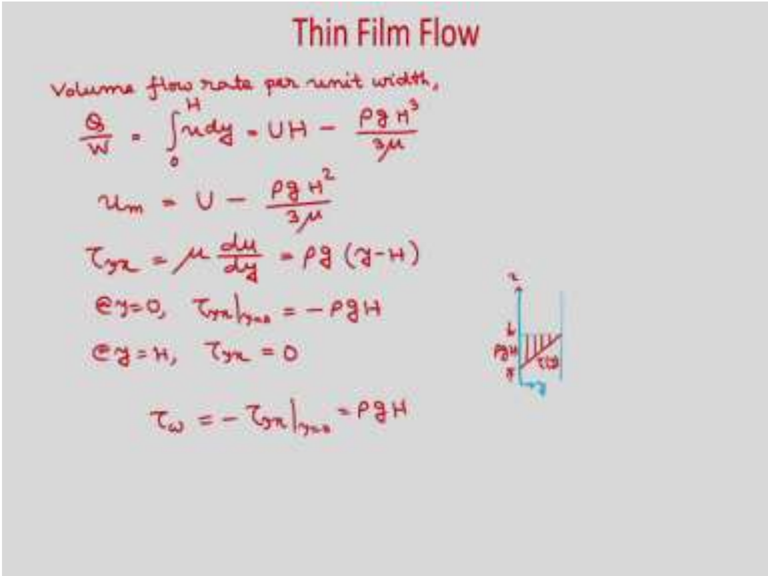

Now, let us calculate the volume flow rate. So, volume flow rate per unit width per unit width will be

$$\frac{Q}{W} = \int_0^H u\,dy = UH - \frac{\rho g H^3}{3\mu}$$

$$u_m = U - \frac{\rho g H^2}{3\mu}$$

$$\tau_{yx} = \mu \frac{du}{dy} = \rho g(y - H)$$

At y=0

$$\tau_{yx}|_{y=0} = -\rho g H$$

And at y=H,

$$\tau_{yx} = 0$$

So, shear stress distribution will look like this. So, it is belt and this is the interface y is measured from the belt so shear stress distribution. And, wall shear stress you can write

$$\tau_w = -\tau_{yx}|_{y=0} = \rho g H$$

So, in today's class first, we considered plane Poiseuille flow with the same slip in both the plates. So, with that we took the expression for slip as τ_w is equal to βu_w where u_w is the slip velocity at the wall then we found the velocity distribution. And you can see that velocity distribution is the superimposition of constant velocity u_w at the wall plus the velocity distribution of the plane Poiseuille flow.

From there we calculate the volume flow rate per unit width, average velocity then we calculate the shear stress distribution. Next, we consider thin-film flow where one liquid is draining down due to gravity and neglecting the surface tension effect we assumed that the thickness of the film is constant.

So, in that case, we found the velocity distribution assuming that the outside you have a stationary fluid and that is air. Then, we applied the boundary condition that y is equal to H at the interface you have the shear stress as 0. Then, we calculated the volume flow rate and the average velocity and the shear stress distribution.

And we took two special cases one is that if it is a horizontal plate then; obviously, velocity will be 0 and if it is a vertical plate then, in that case, we found the velocity distribution. The second special case again we extended to a problem where one belt is moving upward with a constant velocity u.

And, obviously, when it is moving upward due to the viscous effect some liquid will stick to the belt and neglecting the surface tension we assume that the thickness of this thin film is constant. And in that case, we found the velocity distribution and volume flow rate and the shear stress distribution.

Thank you.