Viscous Fluid Flow
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Module - 02
Steady One-dimensional Rectilinear Flows
Lecture - 04
Combined Couette – Poiseuille Flow

Hello everyone. So, in this lecture, we have already solved flow between two parallel plates where the upper plate is moving with a constant velocity u and the lower plate is stationary. We have also solved plane Poiseuille flow in this case we considered flow between two infinite parallel plates and plates are stationary.

Today, we will consider Combined Couette and Poiseuille Flow, where we will consider flow inside two infinite parallel plates where the lower plate is stationary and the upper plate is moving with a constant velocity u and also we have an imposed pressure gradient. So, you can see in this case the flow will be due to shear where the upper plate is moving as well as it is due to the imposed pressure gradient.

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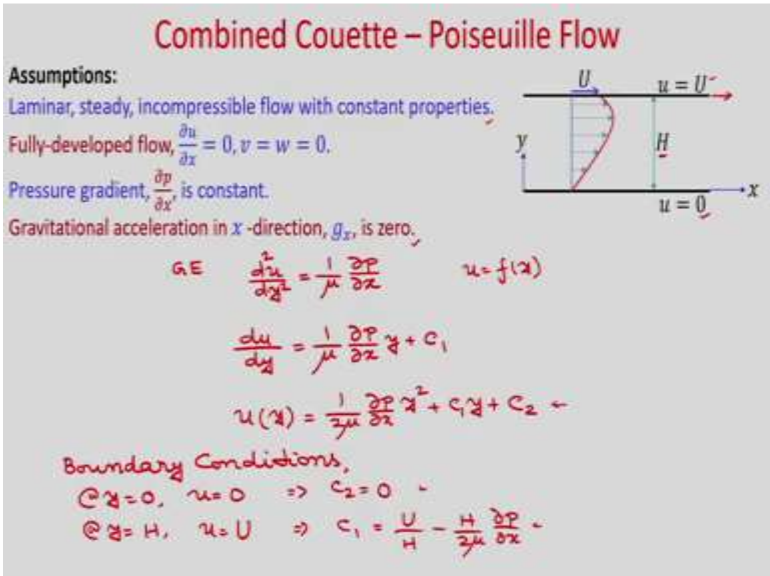

The following assumptions we will take while solving these combined Couette and Poiseuille flow. We will consider laminar steady incompressible flow with constant properties it is a fully developed flow. So, $\frac{\partial u}{\partial x}$ is 0 and; obviously, the plates are infinite.

So, v and w are 0 we have a pressure gradient $\frac{\partial p}{\partial x}$ which is constant and gravitational acceleration in x direction is 0.

So, you can see these are two infinite parallel plates where the lower plate is stationary velocity is 0 and the upper plate is moving in positive x-direction with a constant velocity u and y is measured from the bottom plate. So, invoking these assumptions we can write the ordinary differential equation which is your governing equation. So, it will be

$$\frac{d^2u}{dy^2} = \frac{1}{\mu}\frac{\partial p}{\partial x}$$

So; obviously, u is function of y only so that is why we have written ordinary differential, now if you integrate twice then you will get the velocity profile. So, if you integrate this differential equation you will get

$$\frac{du}{dy} = \frac{1}{\mu}\frac{\partial p}{\partial x}y + C_1$$

$$u(y) = \frac{1}{2\mu}\frac{\partial p}{\partial x}y^2 + C_1 y + C_2$$

So, now, we need to find these constants C_1 and C_2 invoking the boundary conditions. So, we have two unknown C_1 and C_2. So, we have two boundary conditions; one is at y is equal to 0, velocity is 0, u is equal to 0 and at y is equal to H the upper plate is moving at a constant velocity u so u is equal to U. So, we have boundary conditions.

So, the bottom plate is stationary and the upper plate is moving with a constant velocity U. So, now, if you invoke this condition at y is equal to 0, u is equal to 0; obviously, you are going to get C_2 as 0.

And, if you put y is equal to H, u is equal to U and you will get the constant C_1 as

$$C_1 = \frac{U}{H} - \frac{1}{2\mu}\frac{\partial p}{\partial x}$$

Here H is the distance between two parallel plates. So, now if you put this constant C_1 and C_2 in this equation we will get the velocity profile.

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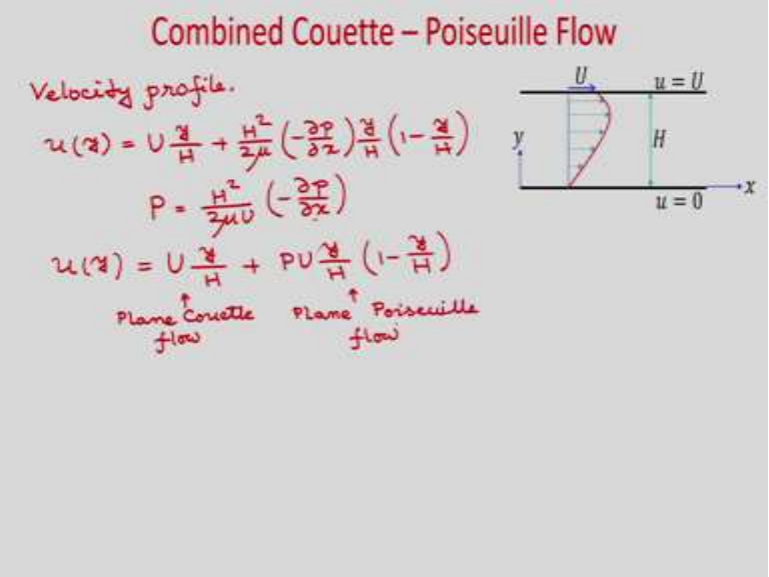

So, velocity profile is

$$u(y) = U\frac{y}{H} + \frac{H^2}{2\mu}\left(-\frac{\partial p}{\partial x}\right)\frac{y}{H}\left(1 - \frac{y}{H}\right)$$

So, you carefully see this velocity profile. So, you can see the 1st term in the right-hand side that is the solution from the plane Couette flow and if you see the 2nd term in this velocity profile that is the solution from the plane Poiseuille flow.

So, you can see that the governing equation is linear and homogeneous so we are getting the superposition of these two velocities as velocity for the combined Couette and Poiseuille flow. Now, we will introduce one pressure parameter P which is $\frac{H^2}{2\mu}\left(-\frac{\partial p}{\partial x}\right)$.

So, you can see that will be considered a favorable pressure gradient then $-\frac{\partial p}{\partial x}$ will be positive and in that case P will be positive and if we consider adverse pressure gradient then $-\frac{\partial p}{\partial x}$ will be negative and P will be negative. So, if you rearrange this term then you can write the velocity profile in terms of this pressure parameter P as

$$u(y) = U\frac{y}{H} + PU\frac{y}{H}\left(1 - \frac{y}{H}\right)$$

So, t is the solution for plane Couette flow and this is the solution for plane Poiseuille flow.

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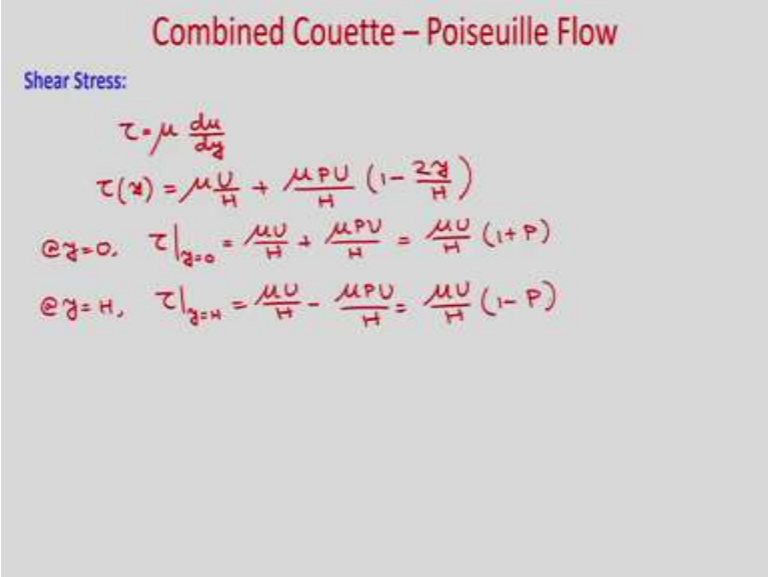

Now, let us calculate the shear stress distribution inside the fluid domain. So, you know in this case shear stress τ is

$$\tau = \mu \frac{\partial u}{\partial y}$$

So, we have the velocity profile this one so if you take the derivative du/dy so; obviously,

$$\tau(y) = \mu \frac{U}{H} + \frac{\mu P U}{H}\left(1 - \frac{2y}{H}\right)$$

So, now if we want to find the shear stress at the bottom wall, then at y is equal to 0 we will get

$$\tau|_{y=0} = \frac{\mu U}{H} + \frac{\mu P U}{H} = \frac{\mu U}{H}(1 + P)$$

And at top wall so you will get the shear stress at y is equal to H as

$$\tau|_{y=H} = \frac{\mu U}{H} - \frac{\mu P U}{H} = \frac{\mu U}{H}$$

Now, we are interested to find the average velocity. So, first we will find the volume flow rate, then we will find the average velocity inside the flow domain.

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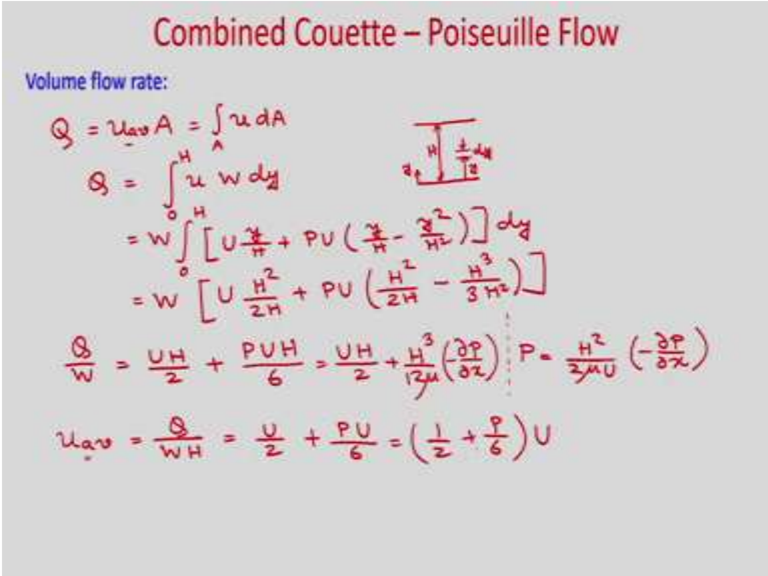

So, you know volume flow rate Q is equal to average velocity into flow area right that you can write as

$$Q = u_{av}A = \int_A u\,dA$$

$$= \int_0^H uWdy$$

$$= W\int_0^H \left[U\frac{y}{H} + PU\left(\frac{y}{H} - \frac{y^2}{H^2}\right)\right] dy$$

$$= W\left[U\frac{H^2}{2H} + PU\left(\frac{H^2}{2H} - \frac{H^3}{3H^2}\right)\right]$$

$$\frac{Q}{W} = \frac{UH}{2} + \frac{PUH}{6} = \frac{UH}{2} + \frac{H^3}{12\mu}\left(-\frac{\partial p}{\partial x}\right)$$

So, now, if you look into this expression carefully you can see the first term; obviously, the volume flow rate per unit width for the case of plane Couette flow, and the second term is for plane Poiseuille flow.

Now, if you want to find the average velocity. So, you can write

$$u_{av} = \frac{Q}{WH} = \frac{U}{2} + \frac{PU}{6} = \left(\frac{1}{2} + \frac{P}{6}\right)U$$

So, now depending on the value of P ok you can find the average velocity; obviously, if it is a favorable pressure gradient PU will be positive, but if it is an adverse pressure gradient so P will be negative. So, depending on that you will get u average whether it is positive or negative.

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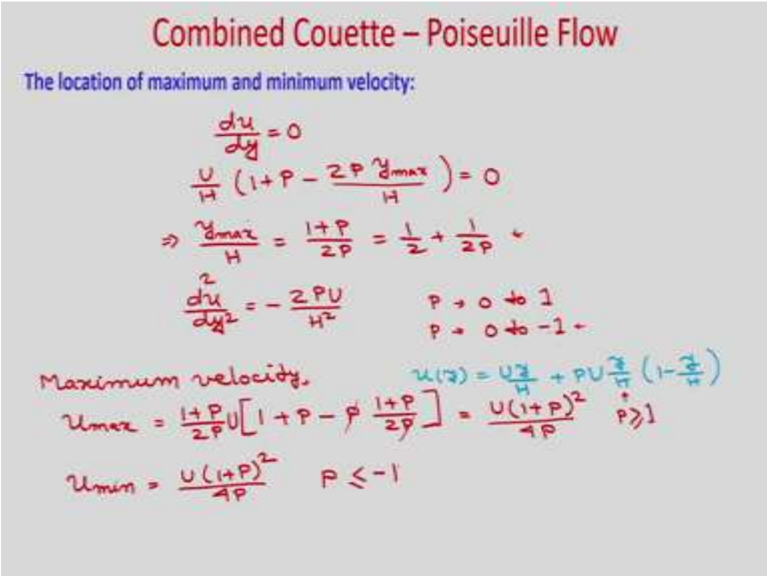

So, now you want to find the location for maximum or minimum velocity, because you can see that the flow is occurring due to shear-driven flow as well as pressure-driven flow. So, you can see that maximum velocity always it will not occur at the top plate.

So, it may occur inside the fluid domain depending on the value of the pressure parameter P or the pressure gradient $-\frac{\partial p}{\partial x}$. And if you have an adverse pressure gradient, so there will be minimum velocity also minimum velocity it will not be 0 it may be negative.

So, first to find the location where the minimum and maximum velocity will occur so; obviously, we will put du/dy is equal to 0. So, to find the location for maximum and minimum velocity we need to put du/dy is equal to 0 and if you put it so you will get

$$\frac{du}{dy} = 0$$

$$\frac{U}{H}\left(1 + P - \frac{2Py_{max}}{H}\right) = 0$$

$$\frac{y_{max}}{H} = \frac{1+P}{2P} = \frac{1}{2} + \frac{1}{2P}$$

So, you can see this y_{max} is the y corresponding to the location of maximum or minimum velocity. You can see that if P is equal to 0 then; obviously, it is plain Couette flow because it is a purely shear-driven flow. So, in that case, maximum or minimum velocity will not occur inside the flow domain right, because you can see that your upper plate is moving with a velocity U and that is the maximum velocity. And inside the fluid domain, you will not get the maximum or minimum velocity.

Now, if you consider that P is equal to 1 then what will happen? So, we put P is equal to 1 then you can see y_{max}/H will be 1; that means when you vary P from 0 to 1 the maximum velocity will not occur inside the flow domain. So, always the velocity will be maximum at the plate. So, and if you consider P is equal to minus 1 so, what will happen? So, if you P is equal to minus 1 then; obviously, you can see y_{max}/H will be 0.

That means in that case when you have an adverse pressure gradient and P is equal to minus 1, then you will not get the minimum velocity inside the fluid domain only at the bottom plate you will get the minimum velocity. And in that case, you can see that when P varies 0 to 1, so, velocity will monotonically vary from 0 at the bottom plate to U to the upper plate.

And when P varies between 0 to -1, so in that case the velocity will monotonically decrease from the upper plate at U to 0 at the lower plate so, if you; now if you want to find whether the maximum velocity or minimum velocity will occur. So, you can find $\frac{d^2u}{dy^2}$ which is

$$\frac{d^2u}{dy^2} = -\frac{2PU}{H^2}$$

So, you can see depending on the value of P, we will get either maximum velocity or minimum velocity inside the flow domain.

So; obviously, you can see P greater than 0; that means, it is a favorable pressure gradient. So, you will get the maximum velocity inside the flow domain and if P less than 0 then; obviously, it will be positive quantity and minimum velocity you will get the flow domain, but between P when it varies 0 to 1; obviously, the maximum or minimum velocity will not occur inside the flow domain and when P varies between 0 to minus 1 then also you will not get the maximum or minimum velocity inside the flow domain.

So, if you find the maximum velocity, so u_{max}. So, you have the velocity profile, what is your velocity profile? So, the velocity profile you can write as

$$u(y) = \frac{Uy}{H} + PU\frac{y}{H}\left(1 - \frac{y}{H}\right)$$

So, now, to get the maximum velocity you can put the value of y_{max}/H this expression in this velocity profile.

So, you will get

$$u_{max} = \frac{1+P}{2P}U\left[1 + P - P\frac{1+P}{2P}\right] = \frac{U(1+P)^2}{4P}$$

So, and this u_{max} you will get when P greater than equal to 1. And minimum velocity also you will get the same expression if you put it here, but minimum velocity will occur when you have an adverse pressure gradient.

And we have seen that if P varies 0 to minus 1; obviously, the maximum velocity will occur at P is equal to minus 1 at the wall and velocity monotonically decreases from capital U from the upper plate to the 0 in the lower plate. So, this minimum velocity will occur when P is less than equal to minus 1.

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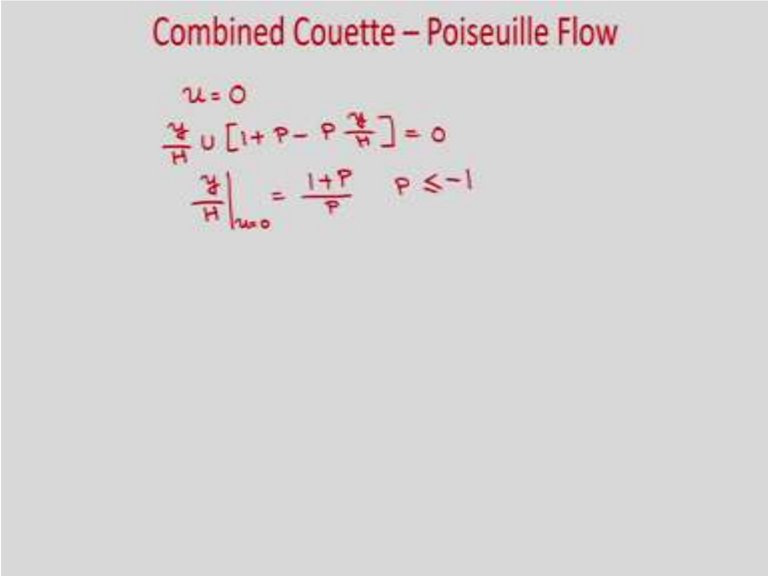

So, now let us find that, where the velocity will become 0 in the case of adverse pressure gradient inside the fluid domain. Because you can see that there will be minimum velocity inside the fluid domain and you have the maximum velocity at the top plate where y is equal to H which is moving at a constant velocity U so; obviously, somewhere inside the flow domain velocity will become 0.

So, we can write when u is equal to 0, so, from the expression of velocity profile you can see you can write

$$\frac{y}{H}U\left[1+P-P\frac{y}{H}\right]=0$$

So from here you can see

$$\left.\frac{y}{H}\right|_{u=0}=\frac{1+P}{P} \qquad P\leq -1$$

Now, we will consider first favorable pressure gradient; that means, $-\frac{dp}{dx}$ is greater than 0; that means, the pressure parameter is greater than 0. If p is equal to 0; obviously, you know that it is a case for plane Couette flow and velocity profile will be linear because it varies linearly from 0 to U.

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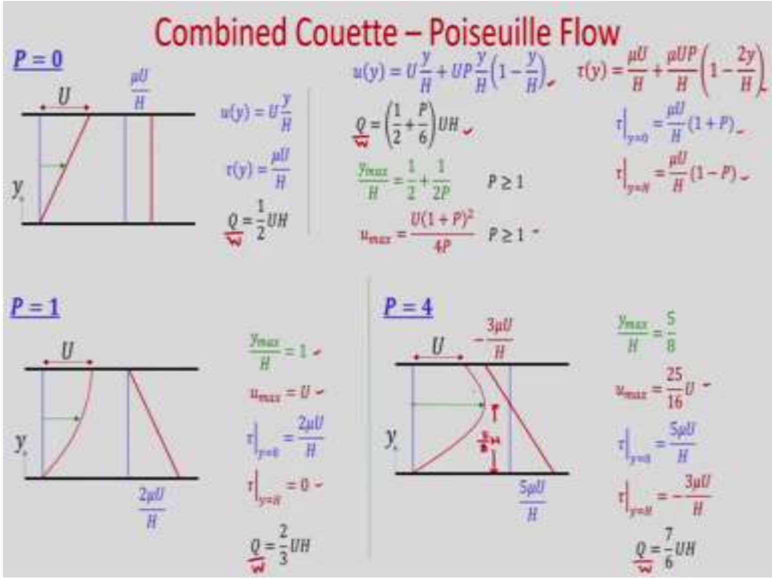

So, you can see this is our velocity profile this is the shear stress profile, and the volume flow rate is per unit width so it is Q/W is this. And bottom plate shear stress is this one upper plate shear stress is this one and for maximum velocity P greater than equal to 1 you will get

$$\frac{y_{max}}{H} = \frac{1}{2} + \frac{1}{2P}$$

And this is the expression for maximum velocity

$$u_{max} = \frac{U(1+P)^2}{4P}$$

So, at if P is equal to 0 we know this is a plane Couette flow so; obviously, velocity will vary from 0 to u. So, you can see u is equal to U y/H and shear stress is constant and this is the shear stress distribution. And volume flow rate per unit width will be just UH/2. In this case when we consider plane Couette flow the pressure gradient was 0, now let us consider favorable pressure gradient; that means, P greater than 0.

So, first let us consider P is equal to 1. So, if P is equal to 1 then you can see from this expression y_{max}/H will be 1, u_{max} will be U, and τ at y is equal to 0, if you put in this expression you will get 2μU/H and τ at the upper plate this will be 0. And the volume flow rate per unit width it will be $\frac{2UH}{3}$.

So, you can see the shear stress at the top plate it will be 0. So, shear stress distribution is this one and bottom plate is $\frac{2\mu U}{H}$ and as τ is equal to 0; that means, at the upper plate velocity gradient is 0; that means, velocity profile will look like this, and here du/dy will we 0. Here at this location the velocity gradient du/dy will be 0.

Now, let us increase the pressure gradient. So, let us consider P is equal to 4. So, if you consider P is equal to 4, then y_{max}/H you will get $5u_{max}$ /8 you will get 25U/16 at the bottom plate shear stress is 2μU/H and at the top plate τ is equal to -3μU/H and volume flow rate per unit width is 7 UH /6.

So, you can see; obviously, at location y_{max}/H 5/8 you will get the maximum velocity; that means, this distance, so this distance is 5H/8. And shear stress distribution is like this and maximum velocity is occurring at y is equal to 5H/8 and maximum velocity is this.

So, you can see the maximum velocity is occurring inside the fluid domain and for favorable pressure gradient you can see the location y_{max} by H is $\frac{1}{2}+\frac{1}{2P}$ is; obviously, positive quantity. So, the maximum velocity always will occur just near to the top plate because it is half plus something right. So, it will be near to the upper plate.

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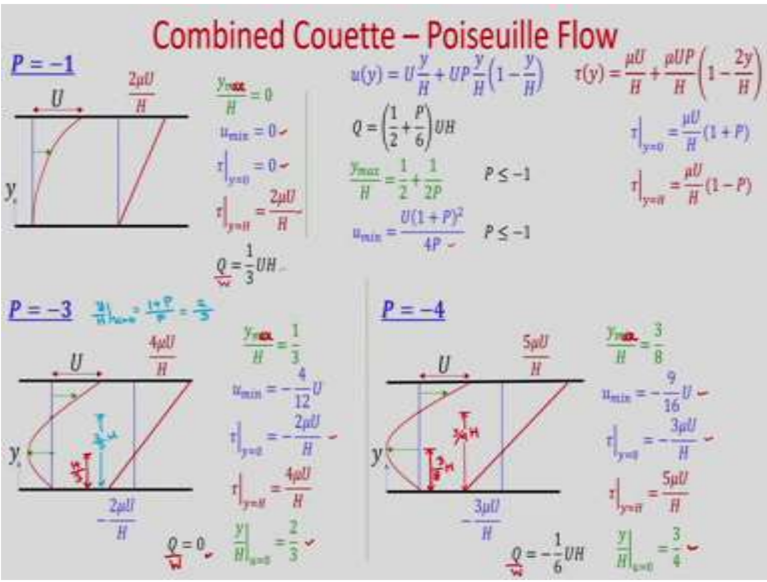

Now, let us consider adverse pressure gradient where P is negative. So, first let us consider when P is equal to -1. So, if we consider P is equal to -1. So, here u minimum is expression is this and P for P less than equal to minus 1 and from the expression here it will be y_{min}/H it is you can write y_{max}/H from this expression, so it will be 0.

So, you can see that at the bottom plate you are going to get the minimum velocity and that is 0 and shear stress at bottom plate is 0 upper plate is $2\mu U/H$ and volume flow rate per unit width is UH/3. So, you can see the shear stress varies from 0 to $2\mu U/H$ linearly and the velocity decreases monotonically from U to 0 and at y is equal to 0, du/dy is 0 because shear stress is 0.

So, the velocity gradient du/dy is 0 at this location and this is the onset of pro-reversal. So, you can see when shear stress is becoming 0 near to the bottom plate this fluid which is sitting over this plate will not experience any shear stress. So, if you slightly decrease the pressure gradient. So, there will be flow reversal. So, this is the point for onset of flow reversal.

So, at this point you can see this is the onset of flow reversal for P is equal to -1 and if you decrease P then; obviously, flow reversal will take place. Now, if you consider P is equal to -3, then y_{max}/H you will get 1/3, u_{min} you will get -4U/12 at the bottom plate you will get the shear stress $-2\mu U/H$ and at the upper plate τ you will get $4\mu U/H$.

So, you can see shear stress is varying linearly from $-2\mu U/H$ to $4\mu U/H$. And so you can see at this location which is your y_{max}/H is 1/3 so; that means, H/3. At this location, you will get minimum velocity so you can see flow reversal has taken place.

So, in this area near the upper plate you will get the flow velocity in the positive x-direction. And near the bottom plate you can see there is flow reversal and velocities in negative x-direction. And where the velocity will become 0? So, you can see y/H, where U is equal to 0, is, (1+P)/P. So, you can see P is equal to -3 so; obviously, you are going to get 2/3.

So; that means, this location where U is becoming 0 because this is becoming from velocity is changing from positive value to negative value. So, you will get this as 2H/ 3, so at this location, velocity is becoming 0. And, if you find the volume flow rate per unit width from here you can see P is equal to -3, then volume flow rate will become 0. So, you can see volume flow rate Q /W is equal to 0.

So, you can see it is a special case when P is equal to - 3; that means, you have an adverse pressure gradient, when P is equal to -3 the volume flow rate is 0; that means, average velocity is 0. So, what is happening here? So, here you can see that your pressure gradient balances the shear. So, in this case, you will get re-circulating flow inside the flow domain such that your average velocity will become 0.

Now, if you decrease the pressure gradient further, now let us consider P is equal to - 4 then what will happen? So, P is equal to -4 so y_{max}/H you will get 3/8, the minimum velocity you will get -9U/16 and shear stress at the bottom plate -3µU/H at the top you will get -5µU/H and y/H; obviously, you will get 3/4, here 2/3 and here 3/4 and Q, so it will be minus U/6H.

So, you can see shear stress distribution looks like this and velocity distribution; obviously, near to the upper plate you will get velocity in the positive x-direction due to the shear of this upper plate and due to adverse pressure gradient you will get the flow reversal. And this location where maximum velocity will occur so that is 3H/8 and where velocity 0 you will get this location this is 3H/4.

So, here you can see here velocity when P is equal to -1 the volume flow rate per unit width is positive when P is equal to -3 it is becoming 0 and if you further decrease the pressure gradient volume flow rate per unit width becomes negative; that means, your net flow will occur in the negative x-direction.

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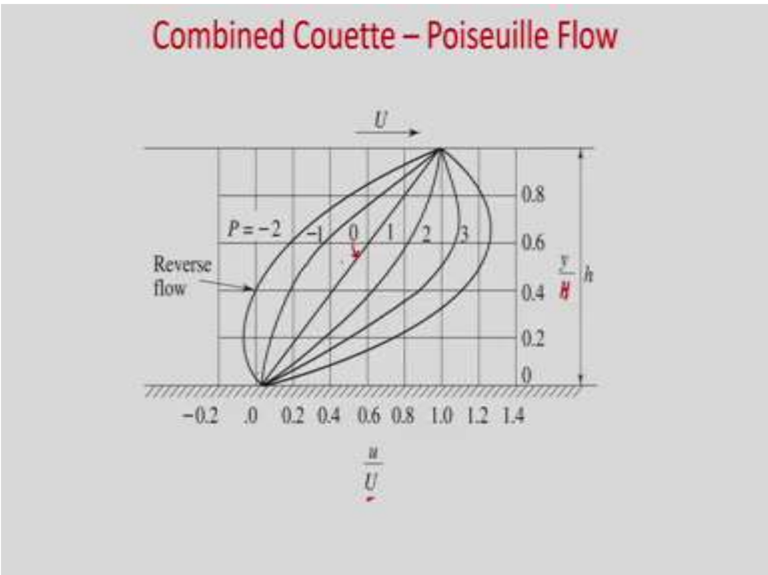

So, if you plot for different values of P this velocity profile. So, this is the non-dimensional velocity u/U and this is the non-dimensional y coordinate y/H. So, if you plot it for a different pressure gradient so; obviously, P greater than 0 it is favorable pressure gradient P is so you will get you see this is the P is equal to 0 for p is equal to 0.

And it is a linear profile if you see P is equal to 1, 2, 3 maximum velocities will occur and if your pressure gradient is adverse, then you can see P is equal to -1 this is the onset of flow reversal and if you further decrease the pressure gradient, then flow reversal take place.

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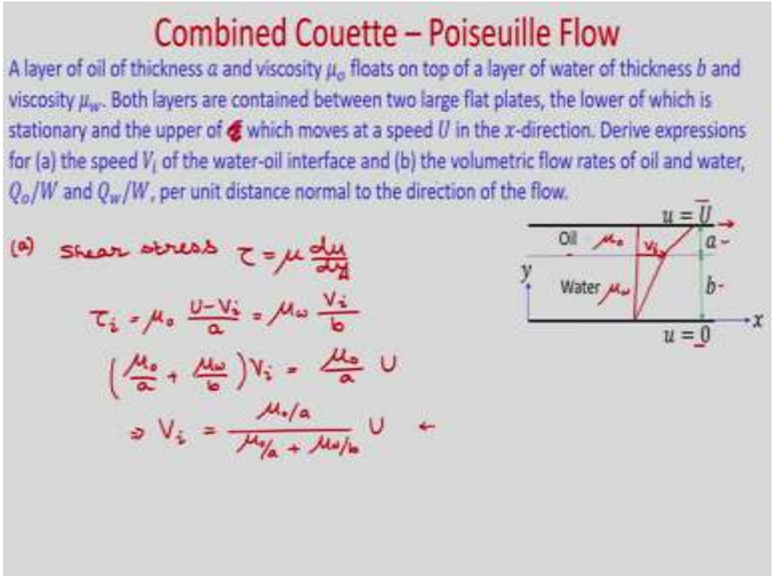

Now, let us solve two problems. First problem we will consider as 2 layer Couette flow. Already you have found the velocity profile as well as the volume flow rate per unit width. So, first problem is a layer of oil of thickness a and viscosity μ_o floats on top of a layer of water of thickness b and viscosity μ_w.

Both layers are contained between two large flat plates, the lower of which is stationary and the upper of which moves at a speed U in the x-direction. Derive expressions for the speed V_i of the water-oil interface and b the volumetric flow rates of oil and water Q_o/W and Q_w/W per unit distance normal to the direction of the flow.

So, you can see bottom plate is stationary upper plate is moving with a constant velocity U, oil of thickness a and water of thickness b. So, oil viscosity is μ_o , water viscosity is μ_w.

So, in this case you know that at the interface; the oil-water interface; obviously, shear stress will be continuous. So, you know the shear stress τ is equal to μdu/dy and at this location if you see from the water side and oil side this shear stress will be same.

So, shear stress if you calculate τ this is μdu/dy and in this case we can write the shear stress τ_i at the interface; that means, at this position. So, we know the velocity profile is linear for this case, but it will vary, depending on the values of these viscosities. So, this is linear so you can see that the velocity at this location let us say V_i interface velocity, then you can write τ_i from oil side if you see so it will be μ_o. So, du/dy, so it will be

$$\tau_i = \mu_o \frac{U - V_i}{a} = \mu_w \frac{V_i}{b}$$

So, from here if you take V_i in the left hand side then you will get

$$\left(\frac{\mu_o}{a} + \frac{\mu_w}{b}\right) V_i = \frac{\mu_o}{a} U$$

$$V_i = \frac{\frac{\mu_o}{a}}{\frac{\mu_o}{a} + \frac{\mu_w}{b}} U$$

So this is the interface velocity.

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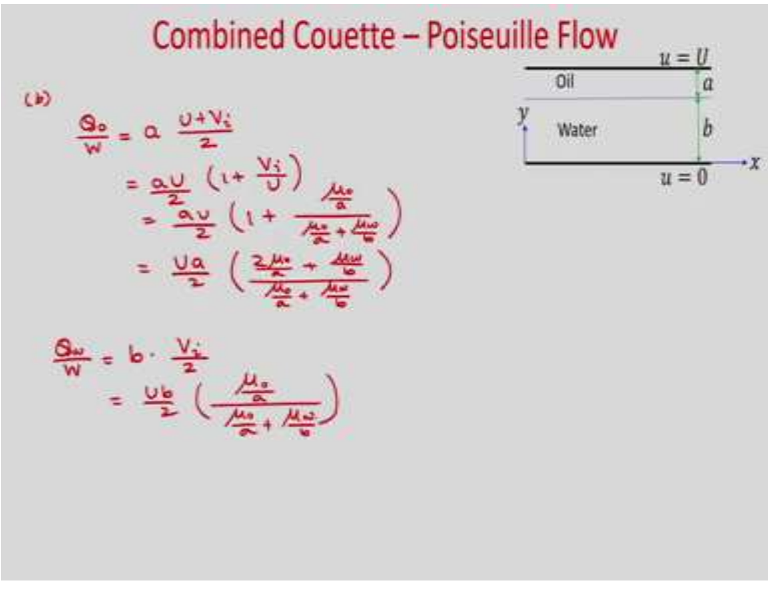

Next, we need to calculate the volumetric flow rates. So, you can see that as the velocity profile is linear, so you can write the volumetric flow rate in oil per unit width is equal to the flow area is

$$\frac{Q_o}{W} = a\frac{U + V_i}{2}$$

$$= \frac{aU}{2}\left(1 + \frac{V_i}{U}\right)$$

$$= \frac{aU}{2}\left(1 + \frac{\frac{\mu_o}{a}}{\frac{\mu_o}{a} + \frac{\mu_w}{b}}\right)$$

$$= \frac{Ua}{2}\left(\frac{\frac{2\mu_o}{a} + \frac{\mu_w}{b}}{\frac{\mu_o}{a} + \frac{\mu_w}{b}}\right)$$

So, you can see this is the volumetric flow rate inside the oil layer. Now, similarly you can calculate the volumetric flow rate inside the water. So,

$$\frac{Q_w}{W} = b\frac{V_i}{2}$$

$$= \frac{Ub}{2}\left(\frac{\frac{\mu_o}{a}}{\frac{\mu_o}{a} + \frac{\mu_w}{b}}\right)$$

So, next we will consider another problem where you have combined Couette and Poiseuille flow.

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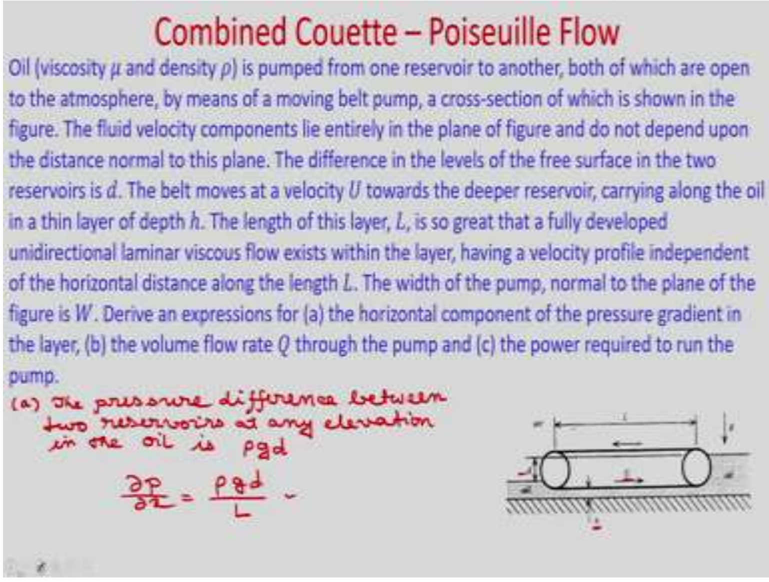

Oil viscosity μ and density ρ is pumped from one reservoir to another both of which are open to the atmosphere by means of a moving belt pump, a cross-section of which is shown in the figure. The fluid velocity components lie entirely in the plane of figure and do not depend upon the distance normal to this plane, the difference in the levels of the free surface in the two reservoirs is d.

The belt moves at a velocity U towards the deeper reservoir carrying along with the oil in a thin layer of depth h. The length of this layer L, is so great that a fully developed unidirectional laminar viscous flow exists within the layer, having a velocity profile independent of the horizontal distance along the length L.

The width of the pump normal to the plane of the figure is W. Derive an expression for a the horizontal component of the pressure gradient in the layer and b the volume flow rate Q through the pump and c the power required to run the pump. So, we can see this figure.

So, this is the belt pump moving with a constant velocity U. And you can see; obviously, from here this oil is pumped to a higher level in this reservoir. So, as the belt is moving in this direction so; obviously, the fluid height will increase in the right side reservoir.

So, you can see as there is a difference in oil level so; obviously, there will be a pressure gradient and it is moving with a constant velocity u so it is a shear driven flow. So, it is a combined shear-driven as well as pressure-driven flow. And the distance you can see of this layer is h and W is the width of this plate and we need to calculate a, the horizontal component of the pressure gradient and volume flow rate Q through the pump and the power required to run the pump.

So, the pressure difference between two reservoirs at any elevation in the oil will be ρgd. So; obviously, you can see the pressure gradient will be

$$\frac{\partial p}{\partial x} = \frac{\rho g d}{L}$$

So, this is the constant pressure gradient.

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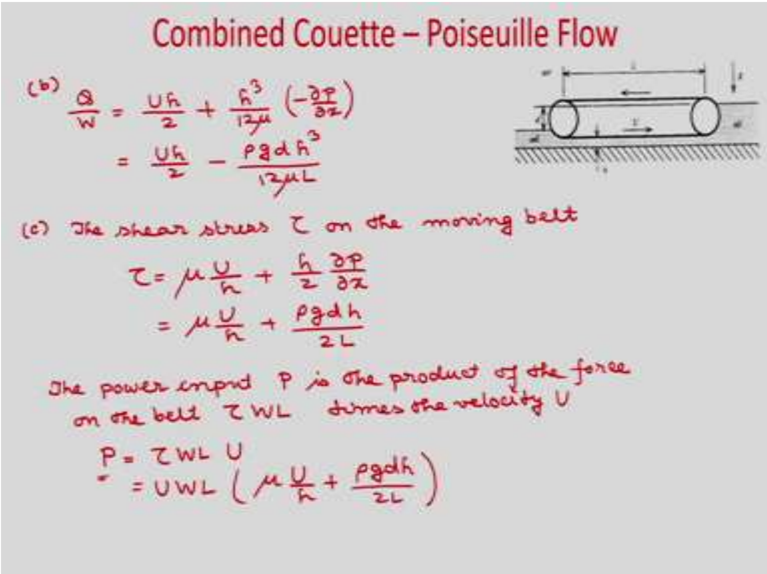

So, next we need to calculate the volume flow rate. So,

$$\frac{Q}{W} = \frac{Uh}{2} + \frac{h^3}{12\mu}\left(-\frac{\partial p}{\partial x}\right)$$

$$= \frac{Uh}{2} - \frac{\rho g d h^3}{12\mu L}$$

Now, the shear stress τ on the moving belt; so, it will be

$$\tau = \mu\frac{U}{h} + \frac{\rho g d h}{2L}$$

Next we need to calculate the power input.

So, the power input P is the product of the force on the belt. What is that? Obviously, it will be τ into area. So, τ W L times the velocity u. So, P will be shear stress, so shear stress is τWL that is the force into velocity U. So, it will be so τ expression is this one so if you put it here so you will get

$$P = \tau WLU$$

$$= UWL\left(\mu\frac{U}{h} + \frac{\rho g d h}{2L}\right)$$

So, in today's class we considered combined Couette and Poiseuille flow. You have seen that this is a combined pressure driven and shear driven flow in this case you have seen as the governing equation is linear and homogeneous the velocity profile is the superposition of 2 velocities one is from plane Couette flow plus the velocity profile from the plane Poiseuille flow.

Then we calculated the shear stress distribution inside the fluid domain and we calculated the volumetric flow rate per unit width. Here, we have expressed one pressure parameter P in terms of pressure gradient $\frac{\partial p}{\partial x}$ and for favorable pressure gradient P is greater than 0 and for adverse pressure gradient p is less than 0.

As it is the flow due to pressure-driven as well as shear-driven flow, in this case, depending on the pressure gradient there may be flow reversal and we have seen that maximum or minimum velocity may occur inside the flow domain. Considering favorable pressure gradient we have seen that if p is equal to 1 then shear stress will become 0 at the top plate and velocity gradient will be 0.

And, if you further increase the pressure gradient then maximum velocity will occur near to the bottom plate; if we consider adverse pressure gradient then for P is equal to minus 1 the shear stress will become 0 at the bottom plate and that is the onset of flow reversal.

If you further decrease the pressure parameter P, then flow reversal will take place and minimum velocity will occur near the bottom plate. There is a special case when P is equal to minus 3, then volumetric flow rate becomes 0 hence average velocity is 0. So, you will get re-circulating flow inside the fluid domain.

Thank you.