Kinematics of Machines
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Module - 4 Lecture No - 01

In the last module, we have discussed the first step of the kinematics analysis under the title displacement analysis. Today, we shall start the second step of kinematics analysis which you call velocity analysis.

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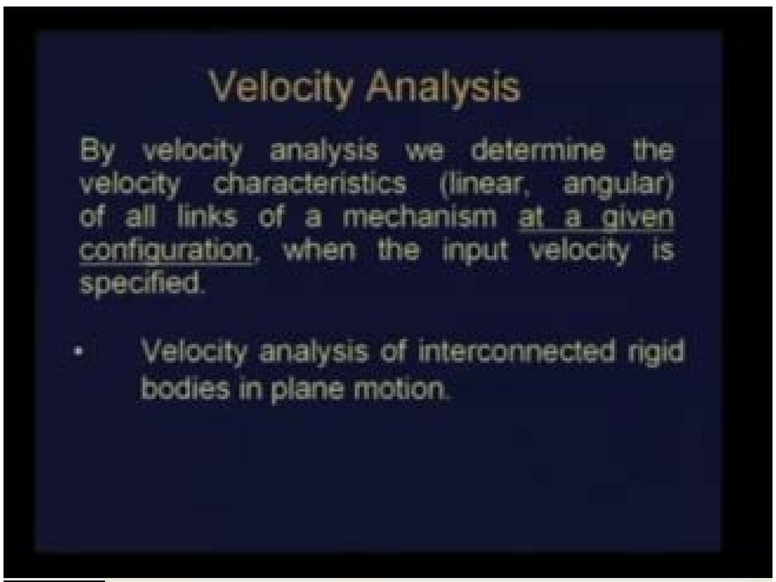

By velocity analysis, we mean that for a given mechanism at a particular configuration, if the input velocity characteristic is specified, we should be able to determine the velocity characteristics of all other links of the same mechanism. Let me repeat, by velocity analysis we determine the velocity characteristics which may be linear velocity of a point or angular velocity of a link of all the links of a mechanism at a given configuration, when the input velocity is specified. So, by velocity analysis basically we mean the velocity analysis of interconnected rigid bodies in plane motion. Because we are discussing on the planar linkage but before getting in to the details of this velocity analysis of interconnected rigid body, let me recapitulate some basic points of kinematics of one rigid body which is under plane motion.

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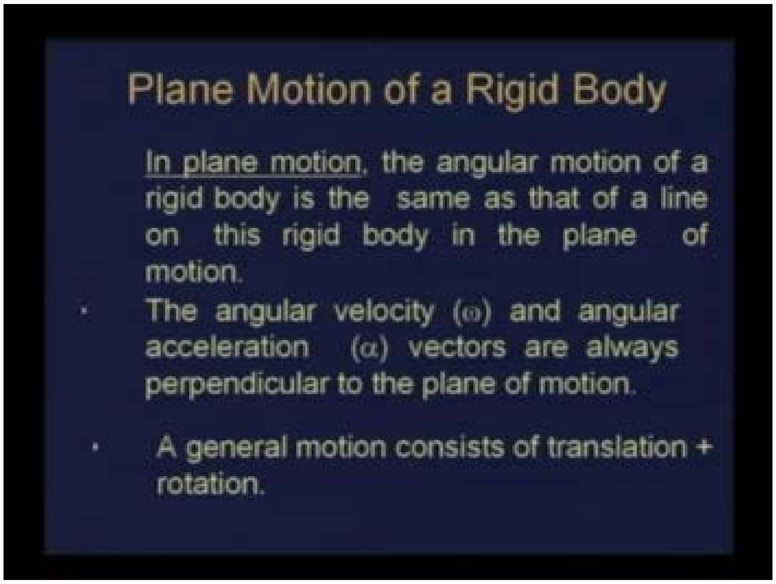

First let me recapitulate some basic points of the plane motion of a rigid body. We should remember that in plane motion the angular motion of a rigid body is the same as that of a line on that rigid body in the plane of motion. This I will explain a little later. The second point is that the angular velocity and angular acceleration vectors of that rigid body are always perpendicular to the plane of motion. Because the rigid bodies in plane motion, all angular quantities will be represented by the vectors which are perpendicular to this plane of motion. We should also remember that a general plane motion consists of translation of the rigid body plus the rotation of that rigid body. In a mechanism, a particular link may be undergoing pure rotary motion and another link may be undergoing pure translatory motion, but there may be some links which undergo both translation and rotation simultaneously.

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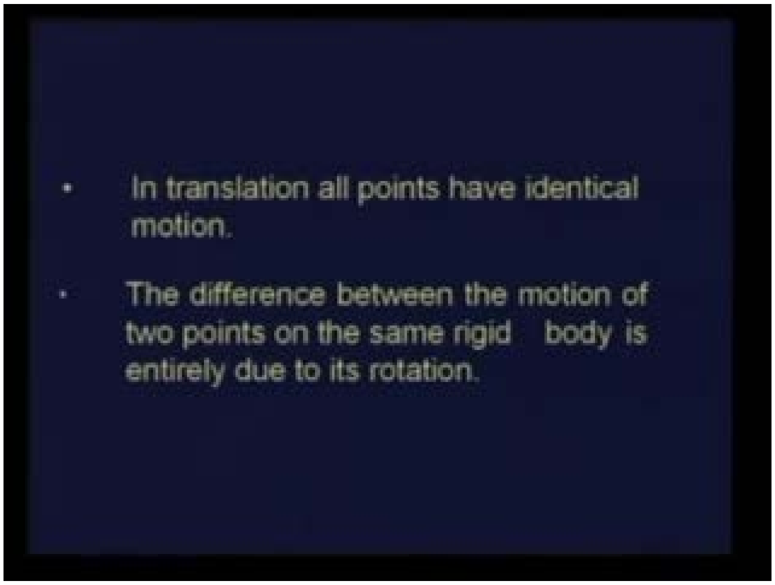

What do you mean by translation is that, if a body is under pure translation, then all points on that body have identical motion. The difference between the motion of two points on the rigid body is entirely due to its rotation. So translation means identical motion of all points and if there is rotation then there will be difference in the motion between various points of the same rigid body.

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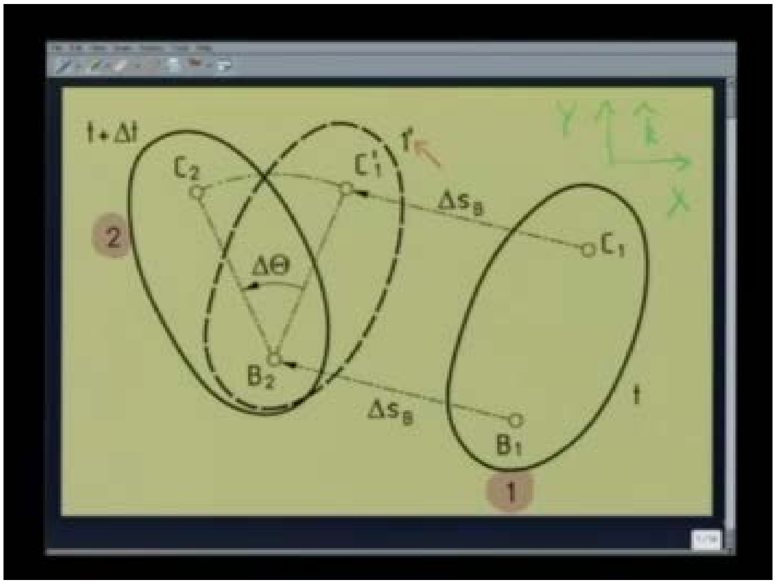

Let me now explain these points with reference to this figure. Here we have a rigid body in plane motion and the plane of motion is the plane of the screen. Let at the time instant t, the rigid body is in this configuration point. After the small interval of time, say delta t, that is at time t plus delta t, the rigid body comes to this configuration which is given by 2. So during time delta t the rigid body moves from configuration 1 to configuration 2. You consider two points on the rigid body namely B and C. We say B_1 C_1 at the locations of B and C at configuration 1 and B_2 and C_2 at the locations of same points B and C in configuration 2. Now, this is a general plane motion of this rigid body. As I said, the plane of motion is the plane of the screen. Let us say X and Y. XY is the plane of motion. So all angular quantities like angular velocity or angular acceleration of this rigid body will be perpendicular to the plane of motion that is, perpendicular to the screen, which we will denote by the Z axis with unit vector K. As I said earlier, for this plane motion, the angular motion of the rigid body will be same as the angular motion of a line on this rigid body for example, the line B_1 C_1. So what do you see? That the line B_1 C_1 has come to the position B_2 C_2.

The line B_1 C_1 has moved and also rotated and occupied the position B_2 C_2. So if I draw B_2 C_1 prime which is parallel to B_1 C_1, then we can see rotation of this line B_1 C_1 during this time delta t is given by delta theta. So let us consider this general plane motion of this rigid body during the time interval delta t when it moved from configuration one to configuration two. This can be decomposed under two headings namely, a translation which takes from configuration one to configuration one prime, when all the points moved by the identical amount and in identical direction. That means, B_1 goes to B_2, C_1 goes to C_1 prime by the same amount. All points of the rigid body move identically and the rigid body goes from one to one prime that is what we call translation. Then from one prime to two, the point B_2 does not move, it undergoes rotation by an amount delta theta.

So, the angular velocity vector of this rigid body can be defined as: omega is limit delta t tending to 0, delta theta by delta t. The direction of this vector is along Z which I write as the unit vector k. Angular velocity vector comes out as theta dot that is the derivative of theta with respective to time in the direction k. Similarly, angular acceleration of this can be written as alpha which is second derivative of theta with respect to time that is theta double dot and in the direction k. So this is what we mean by the general plane motion of the rigid body in the XY plane with the angular velocity vector omega and the angular acceleration vector alpha.

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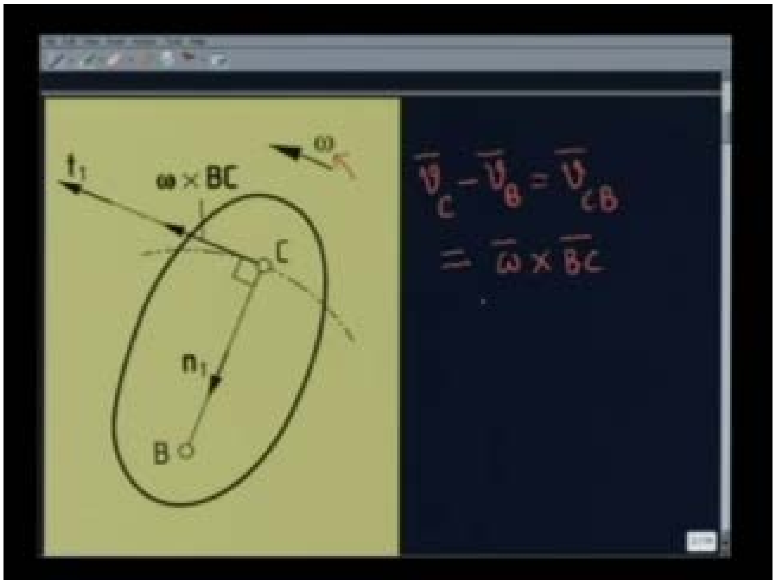

Now as we see during the translation, there is no difference between the motion of various points of the rigid body like B and C. They are under identical motion. The difference in the motion between C and D is entirely due to rotation that is due to this angular velocity vector omega. Positive k means counter clockwise direction. Now, the difference of velocity between the points C and D, let me write it as difference of velocity of point C minus velocity of point B. This is the difference of velocity we use the symbol, V_{CB}. As we have noticed, this is entirely due to the rotation of the rigid body that means the point C with respect to B moves in a circle and the angular velocity of line BC is omega. So, from our basic knowledge of dynamics or kinematics of rigid bodies, we know that for the circular motion, this difference of velocity is given by omega cross the vector BC. So, this is one very basic result which we will be using very frequently in our subsequent velocity analysis, that if we consider two different points B and C on the same rigid body whose angular velocity is omega, then the difference of velocity V_C minus V_B is given by omega cross BC.

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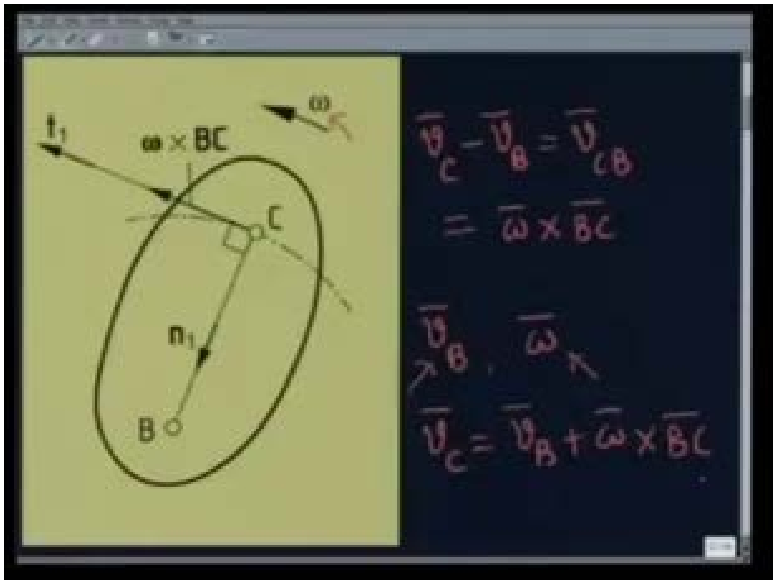

Now we consider the difference of acceleration between these two points namely C and B. As you have already noticed with respect to B, the C is going in a circle and the angular velocity because BC never changes, so C goes on the circle with B as center and the angular velocity of the line BC is omega and the angular acceleration of line BC is alpha, which is nothing but the angular velocity and angular acceleration of the rigid body.

So now we can write the difference of acceleration between these two points namely a_C minus a_B that as before I used the symbol, a_{CB} which is the different between the acceleration of the two points namely C and B on the same rigid body. There is the centripetal acceleration due to the circular motion which is given by omega cross omega cross BC. There is a tangential component which along the vector t_1 that is given by alpha cross BC. Because omega vector is along the k direction and the BC vector is in the XY plane, it can be easily shown that this will always turn out to be, the first term will be omega square CB that is towards B from C, that is the vector CB and the magnitude will be omega square into BC. Plus the tangential component will be alpha cross BC. That means, it is be perpendicular to the vector BC is the sense of alpha, if alpha is counter clockwise then it will be this direction, if alpha is clockwise then it will be in the opposite direction. So we get the relationship between the difference of two points of the velocity like B and C given by omega cross BC.

At this stage, we should know that if we know the velocity of any point say namely V_B which is a two dimensional vector that is the two components of V_B and the vector omega that is the angular velocity of the rigid body. Then I can determine velocity of any other point namely C using this relation that is velocity of any other point C, V_C can be obtain from V_B plus omega cross BC. You should know that there can be only three independent angles namely, two components of V_B and omega. Then velocity of all other points on that same rigid body is completely determined by this equation.

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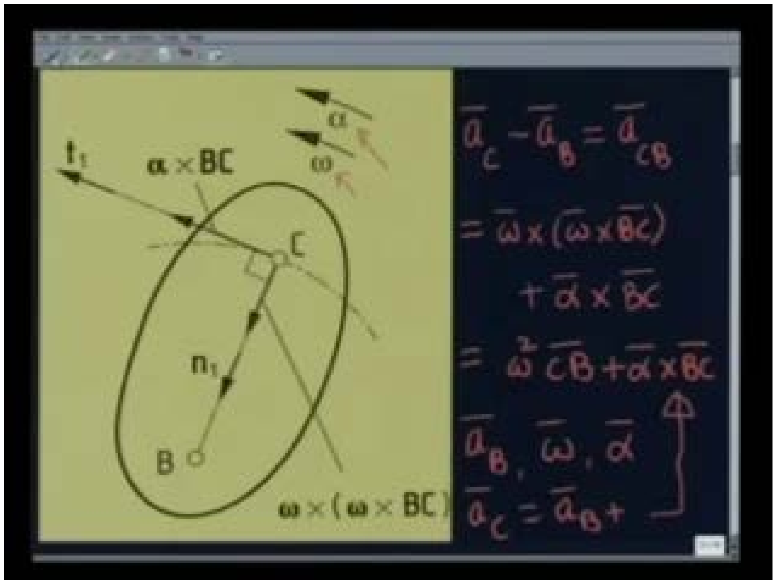

Similarly, if we are given acceleration of a particular point a_B which has two components that is two unknowns and the angular velocity vector omega and the angular acceleration vector alpha then acceleration of any point C can be determined using this equation, that a_C will be a_B plus these two components. So there can be four unknowns, namely two components of a_B and omega and alpha. Then acceleration of all other points on that same rigid body is completely known. This information as we shall see later will be very useful for velocity analysis of mechanisms.

Now that it is clear, how to correlate the velocity and acceleration of two points on the same rigid body, let us develop another very important concept for velocity analysis that is called instantaneous center of velocity or in short IC.

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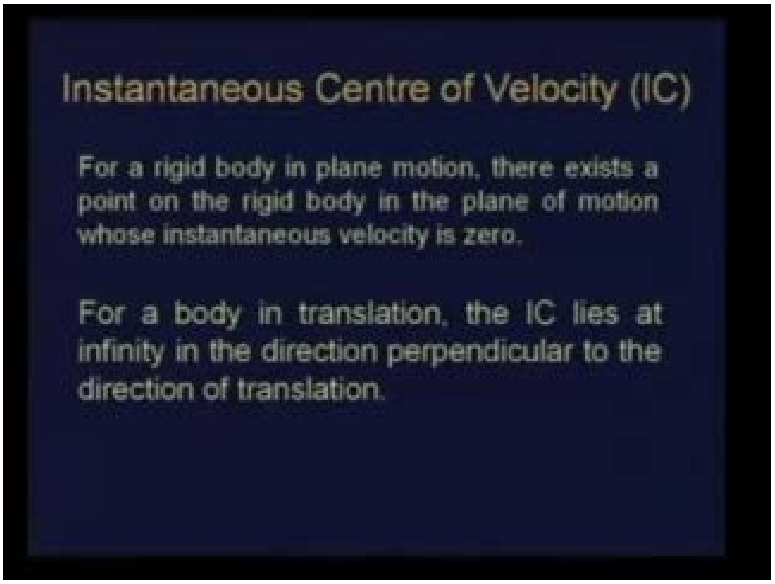

For a rigid body in plane motion, we will be able to show that always there exists a point on the rigid body of course in plane of motion whose instantaneous velocity is 0 and this point is called the instantaneous center of velocity or IC. It must be mentioned that this IC may lie outside the physical boundary of the rigid body but it lies in the plane of motion of that rigid body. If a rigid body is under pure translation, then obviously no point has 0 velocity, in that case we say that the IC lies at infinity in the direction perpendicular to the direction of translation. Just like a prismatic pair, we consider a revolute pair in a direction perpendicular to the direction of sliding. Let me now show that there does exist such a point that which we call the instantaneous center of velocity. To prove the existence of instantaneous center of velocity for a rigid body under plane motion, let us consider this figure.

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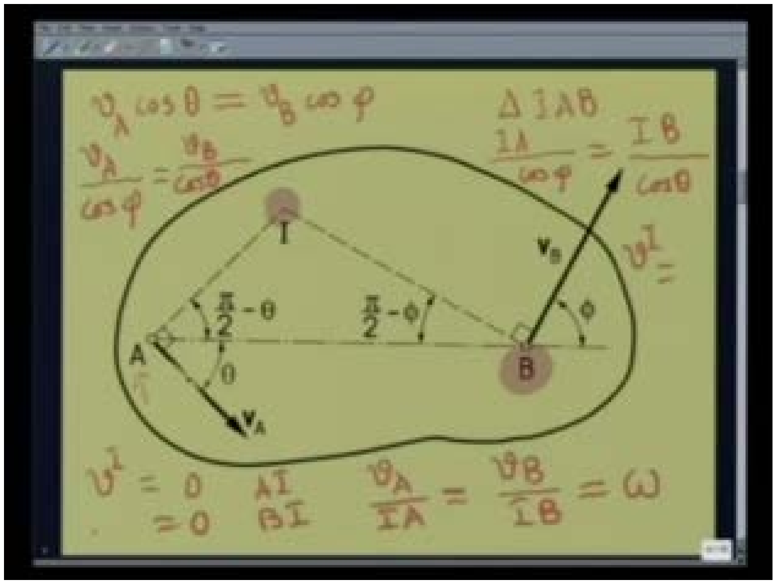

We consider two points namely, A and B whose velocities are given by V_B and V_A. We have already seen that if V_A is given and omega is given, then V_B is completely determined. That means both V_B and V_A having four components cannot be independent. Now, let me draw perpendicular to velocity vector. At B I draw the line B_I which is at 90 degree to $V_{B.}$ At A, I draw a B_I line A_I which is at 90 degree to the velocity vector V_A. These two perpendicular lines drawn at A and B meet at I. We will show that the velocity of this point I of the rigid body at this configuration must be zero and that is what is called the instantaneous center of velocity for this configuration. To show that V_I is 0, let us consider the component of the velocity V_A in the direction of AB. That will be V_A cos theta. V_A cos theta is the velocity A in the direction AB. The component of V_B in the direction of AB is V_B cos phi where phi is the angle between the direction V_B and the direction AB. Now the distance AB never changes that means, these two velocity components must be same. That is the component of V_A along AB must be same as the component of V_B along AB because the distance AB never changes. So I can put V_A cos theta is same as V_B cos phi.

Let us consider the triangle IAB, because IA is perpendicular to V_A and IB is perpendicular to V_B, we can easily see that this angle is pi by 2 minus phi and this angle that is between IA and AB is phi by 2 minus theta. Considering this triangle IAB if we use the sin law, I can write, IA divided by sin of IBA that is sin of pi by 2 minus phi that is cos phi is same as IB divided by sin of the angle IAB that is sin of phi by 2 minus theta which is cos theta. So we get from this equation V_A divided by cos phi is same as V_B divided by cos theta. From the geometry IA by cos phi is same as IB by cos theta. So I can easily write, V_A by IA is equal to V_B by IB. This ratio I call omega that is the angular velocity of the rigid body. So V_A is IA into omega and V_B is IB into omega and the velocity vectors V_B and V_A are by construction perpendicular to the direction of IA and IB, which means I is the center of rotation. At this instant the body is rotating about the point I with angular velocity omega.

In fact, you can also prove that velocity of I is zero by considering the velocity of point I in two different directions namely, IA and IB. To find the velocity of point I along IA, I come from the point A. Velocity of A is along $V_{A,}$ so the component along IA is zero and I and A are two points from the same rigid body. So velocity of I along IA is also 0, because the relative velocity of A and I along A_I is 0. Velocity of I with respective to A along A_I is also 0, so velocity of I along this direction A_I is 0. Coming from the point V, I see that the velocity of the point V is perpendicular to IB. So component of V_B along B_I is 0.Velocity of I along VI is same as the velocity B along B_I. So velocity of I along B_I is also zero. Thus the velocity of the point I is zero along two directions, along A_I and it is also zero along B_I. So if VI is a two dimensional vector and if it is zero in two different directions, then the velocity of I must be zero. We get, velocity of the point I which I write VI is zero at this configuration and this is what is called the instantaneous center of velocity.

Now that we have explained the concept of instantaneous center of velocity, let us extend this concept between two moving bodies. Previously we said one rigid body which is moving and there exists an instantaneous center of velocity. Now let us consider two moving bodies say number two and three.

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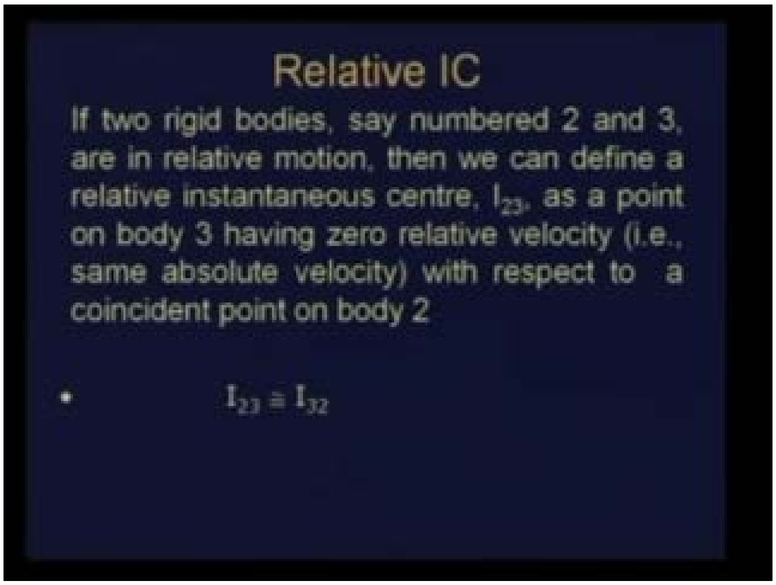

Then we can define a relative instantaneous center say, I_{23} as a point on body three having zero relative velocity, that is same absolute velocity with respect to a coincident point on body two. Let me repeat, we are talking unto two moving rigid bodies say, number two and three. We define a relative instantaneous center between these two bodies two and three, we call it I_{23}. What is I_{23}? This is a point on body three having zero relative velocity with respect to a coincident point on body 2. So from this definition it is very clear that I_{23} is same as I_{32}. In light of this relatives instantaneous center, we can redefine the absolute instantaneous center of velocity which we discussed earlier as the relative IC with respective fixed link one.

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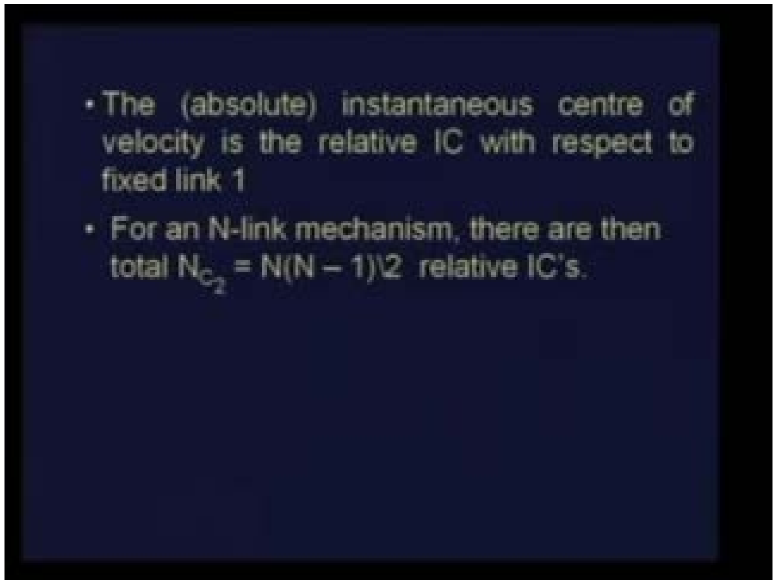

As you know in mechanism, there will be a number of rigid bodies, interconnected rigid bodies in plane motion and there is a fixed length which we always number one. So out of this any N-link mechanism, we will have number of relative instantaneous centers and the relative instantaneous centers with respective to the fixed link one are nothing but the absolute instantaneous center of velocity that is at that instant that particular body is rotating about that instantaneous center. So if you have the N-link mechanism then according to this kind of definition of relative IC's, we will have N_{C2} that is 2 links taken at a time. How many different combination we can make? That is N_{c2} or N into (N minus one) by two relative IC's. So for an N-link mechanism, we will have N into N minus one by two so many number of relative instantaneous centers. At this stage, let us discuss how by looking at the interconnections between various rigid bodies in a mechanism, we can determine the relative instantaneous centers. As you know, in a planer mechanism, rigid bodies will be connected either by revolute pair or by prismatic pair or by some kind of higher pair.

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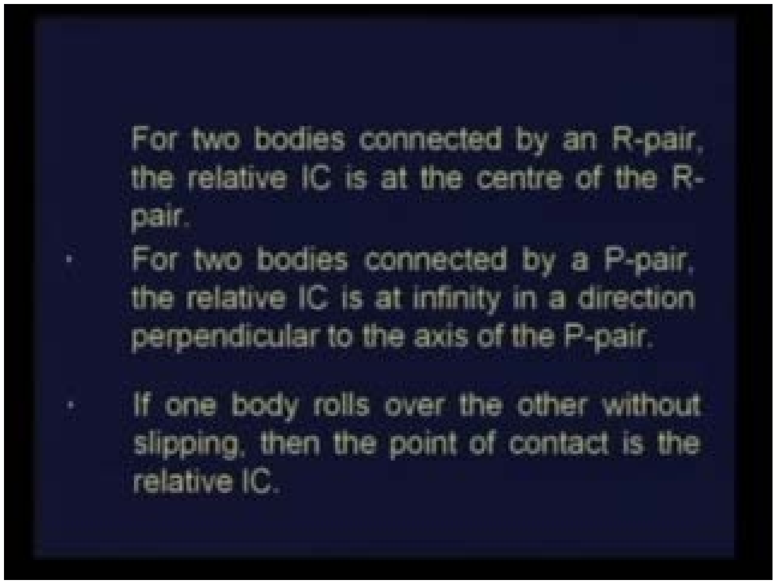

If the two bodies are connected by a revolute pair, then the relative instantaneous centers between those two bodies is obviously at the center of the R-pair because as the mechanism moves, the center of the R-pair between these two bodies always move together. So there is no relative velocity between those two coincident points of the two rigid bodies connected by the R-pair. The center of the R-pair would be the relative IC.

If the two bodies are connected by a prismatic pair, then the relative instantaneous center is at infinity in a direction perpendicular to the axis of the P-pair. Because if the two bodies are the connected by a prismatic pair, then as we know the relative motion is a pure translation and in case of translation, the instantaneous center lies at infinity in a direction perpendicular to the direction of sliding.

We can recall that a P-pair is nothing but an R-pair at the infinity in a direction perpendicular to the direction of relative sliding. So these two sentences imply that for a P-pair the relative IC is at infinity in a direction perpendicular to the axis of the P-pair.

Suppose we have a higher pair with a kinematic constant that, one body rolls over the other without slipping. That means it is not a normal higher pair, it is the higher pair with a kinematic constant that there is no slip at the point of contact. So from no slip condition, I know that velocity of the two points of contact belonging to two different bodies must be the same. That is the condition for no slip. Consequently, the point of contact itself will be the relative IC between those two bodies which are connected by a higher pair without slip. But for general higher pair where slipping is allowed, that means if one body simultaneously rolls and slides over the other when it constitutes a general higher pair, then the relative IC lies somewhere on the common normal at the point of contact.

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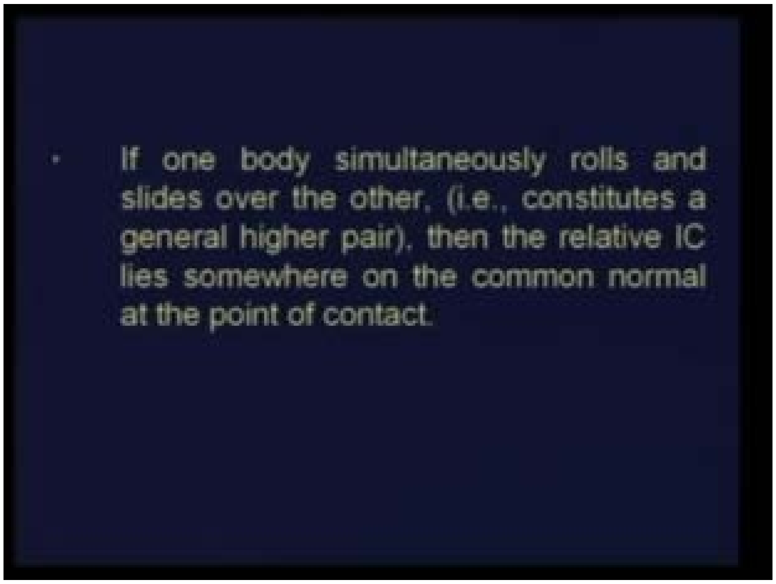

We cannot determine it exactly where on this line, common normal will lie that required some other information, but we know that it must lie on the common normal at the point of contact. At this stage, let me say that there is a very important theorem called Aronhold-Kennedy Theorem of three centers, which will be very useful to determine all the relative IC's of a mechanism along with the information that we have just now discussed. That is, at the R-pair the relative IC is at the center of the R-pair. At the P-pair, the relative IC is at infinity in a direction perpendicular to the direction of relative sliding. If it is a higher pair without any slip, then it is at the point of contact and if it is a general higher pair with slip, then it is somewhere along the common normal at the point of contact. With this information plus the application of the Aronhold-Kennedy Theorem, we should be able to determine all the relative IC's of planar mechanism having up to six links which is sufficient for all purposes.

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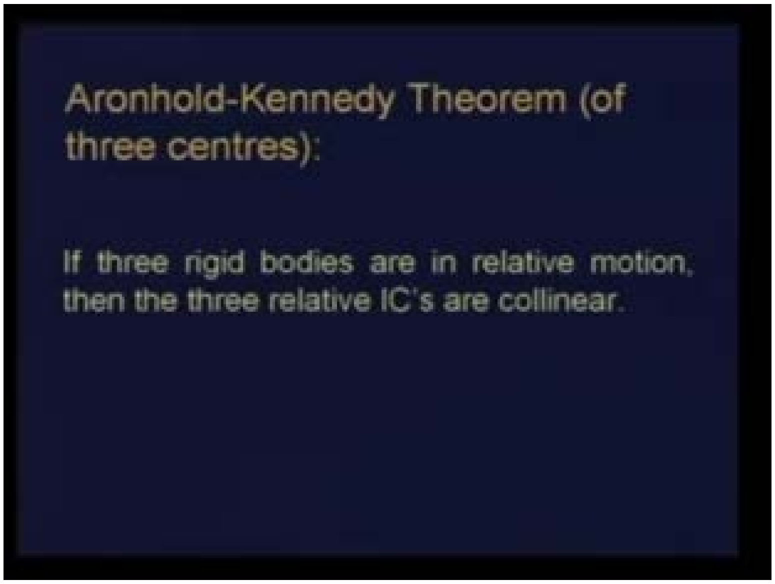

So next, let me state and prove the Aronhold-Kennedy Theorem. Aronhold-Kennedy Theorem of three centers is the following, three rigid bodies are in relative motion. We are talking now three rigid bodies which are in relative motion but they may not be interconnected, but with three bodies we have three relative IC's because $3C_2$, that is C into C minus one divided by two gives you three. So three rigid bodies in relative motion, there are the three relativities the IC's. The Aronhold-Kennedy Theorem says that these three relatives IC's must lie on a line. That is three relative IC's are collinear. We shall prove the Aronhold-Kennedy Theorem by contradiction. That means, we will assume them to be non-collinear and show that is impossible, which means they have to be collinear.

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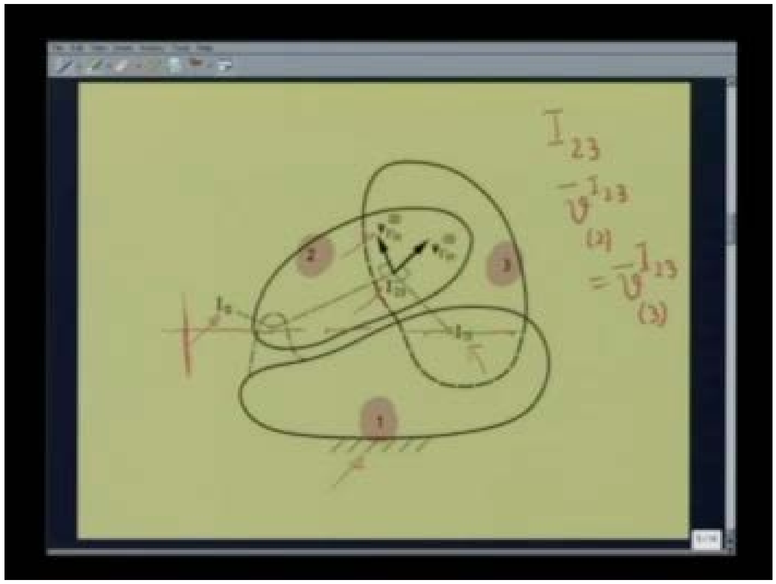

To prove Aronhold-Kennedy Theorem, let as consider three rigid bodies namely, one, two and three which are in relative planar motion. Because we are considering relative motion, we do not lose any generality, if I assume one of the bodies to be fixed. So let me assume that body one is fixed as indicated by this hatched lines. So to consider the relative planar motion between three rigid bodies namely, one, two, and three, I consider the body one to be fixed. Let us say that, I_{12} is the relative instantaneous center for body number one and two. Similarly, I_{13} is the relative instantaneous center between body number one and three. That means, at this instance, body two is under pure rotation about the point I_{12} with respect to body one. Similarly, body three is under pure rotation about the point I_{13} with respect to body one.

Now let us say that Aronhold-Kennedy Theorem is wrong. That means, the instantaneous center I_{23} need not lie on the line joining I_{12} and $I_{13.}$ Let us assume it is at this location I_{23} prime. Now if I_{23} prime is the relative instantaneous center between body number two and three, then it must have the same velocity whether I consider it to be a point on body two or I consider it to be a point on body three. That is the definition of relative IC I_{23}. If I consider it to be a point on body two, it must have the same velocity if I consider it to be a point on body three. That is the definition of I_{23}.

Suppose it lies here, then this vector velocity I_{23} if I consider it to be a point on body two must be in this direction. That is, perpendicular to the line joining I_{12} and I_{23}. This is the line I_{12}, I_{23}, then velocity of this point as a point on body two which is rotating about this point must be in this direction perpendicular to this line. But if I consider this point to be a point on body three which is rotating at this instant about I_{13}, then the velocity must be perpendicular to this line joining I_{13} and I_{23} that is in the direction. These two directions are different so this equation can never hold good. This equation can never hold good because these two vectors V I_{23} as a point on body three and V I_{23} as a point on body two are in different directions. So we immediately get a contradiction. These two vectors can act same direction only if I_{23} lies somewhere on this line, say here or there or there or there. Then only the rotation about I_{12} if I consider to be a point on body two which will be perpendicular to this line and if I consider to be a point an body three which is rotating about I_{13} also is perpendicular to this line. So these two velocity vectors can have the same direction which is necessary condition for it to be the relative instantaneous center I_{23}. So we have proved Aronhold-Kennedy Theorem of three centers by contradiction and now I will demonstrate by use of a module.

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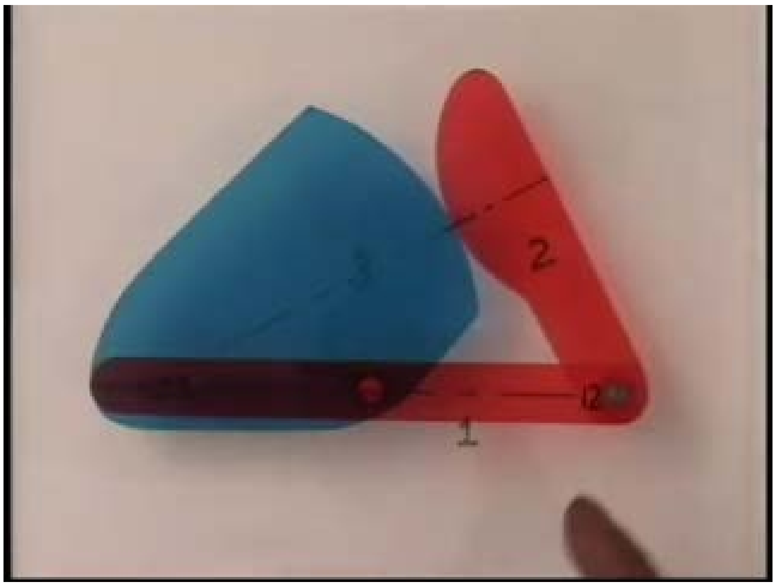

Let me try to explain both the concept of relative IC and the application of Aronhold-Kennedy Theorem with the help of this model. In this model as we see, we are talking of three rigid bodies in relative motion namely one which is the fixed link, two this is this red body and three that is this blue body.

So now, 1 and 2 is connected by a revolute pair here, so immediately I locate the relative instantaneous center 1 2 at this point. Similarly body 3 which is connected by a revolute pair with 1 at this point I consider this as I_C, a relative instantaneous center 1 3. Now 3 and 2 has a higher pair at this point of contact, so we are talking of three rigid bodies namely, 1, 2, and 3. 1 and 2 has a revolute pair here, 1 and 3 has a revolute pair there which means I have located two relative instantaneous centers namely 1 2 and 1 3. Now the relative instantaneous center 2 3, I know must lie along the common normal at this point of contact. This is a general higher pair. We have already discussed that in case of general higher pair the relative instantaneous centers in this case between body two and body three that is I_{23} must lie on this common normal. We also know from Aronhold-Kennedy Theorem, that 1 2, 1 3 and 2 3 must be collinear. That is how we determine the location of 2 3 at the intersection of the common normal and this line joining 1 2 and 1 3. 1 2, 1 3 and 2 3 must be collinear and 2 3 must lie on this common normal that determines this location 2 3. So this is the relative instantaneous center 2 3 between two moving bodies two and three. By definition at this instant, if I consider this point to be a point on body two, that is this red link or a point on body three that is this blue link has the same velocity.

So as we will notice that around this configuration, these two points body 2 and body 3 at the location of I_{23} move with same velocity. However these two points now as we see have moved differently. This is a point 2 3 on body 2 and this is the point 2 3 on body 3. At this location, at this configuration, their velocities are same. So around this configuration, they move almost with same velocity. At this instant, with exactly same velocity around that region, almost with same velocity, so there is not much of separation, however when they move a lot as we see that point two three, which was here as a point of body 3 has come here and as a point on body 2 has come here. I hope, this clears both the concept of relative instantaneous center and the application of Aronhold-Kennedy Theorem which we shall continue subsequently.

Let me now summarize what we have covered in this lecture. We started with the general plane motion of a single rigid body and I have shown that, if we know the velocity of any point on that rigid body and the angular velocity of the rigid body, then we can determine the velocity of all other points that rigid body. Then we have shown that if we know the acceleration of the point on that rigid body and the angular velocity and angular acceleration of the rigid body, then acceleration of all other points on that rigid body can be determined.

Next we have shown the existence of the so called instantaneous center of velocity. That means, for a rigid body in plane motion, there exists a point about which at that instant the body can be considered under pure rotation. Then we extended the concept of instantaneous center to relative motion of between two moving bodies. Then we defined relative instantaneous center exactly the same way.

Finally, we have stated and proved the Aronhold-Kennedy Theorem of 3 centers and finally we have shown the concept of relative instantaneous center and the application of Aronhold-Kennedy Theorem for determining all the relative IC's of a particular mechanism.

In our next lecture, we will continue with the application of this instantaneous center, their determination and their use for continuing the velocity analysis of planar mechanisms.