Kinematics of Machines
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Module – 12 Lecture - 2

Today, we begin our lecture with the advantages of involute profile. Because of these advantages, we find involute profiles are most commonly used in mass produced gears.

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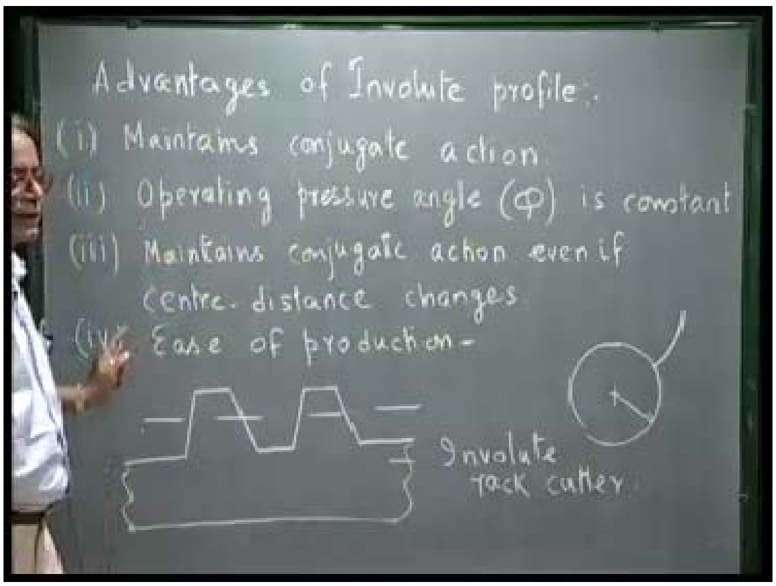

Advantages of involute gear tooth profile: In our last lecture, we have already seen that a pair of involute profiles can maintain the conjugate action. That is, it satisfies the fundamental law of gearing, that is, it maintains constant angular velocity ratio between the two shafts.

First and foremost is, it maintains conjugate action. However, as I said earlier, this is not so much of importance because, if one is given any smooth curve as one profile, one can find another profile which will be conjugate to the given profile. So, this is not the unique choice of involute profiles which maintains conjugate action. There are other advantages of involute profile, over and above that it maintains the most fundamental requirement that is, maintaining conjugate action.

The second advantage is that, operating pressure angle is constant. If we remember in our last lecture, we denoted this operating pressure angle by the symbol phi which is the angle between the line of action and the common tangent to the pitch circles. Operating pressure angle is constant. We will get back to the same figure, which we discussed last time to show, what is the advantage or what is the consequence if the pressure angle remains constant?

The third advantage is that, a pair of involute gears maintains conjugate action, even if the centre distances between the gears are changed. That means, we have a pair of gears, we can mount then with little different centre distance, even then this conjugate action will be maintained. Even if, centre distance between the two gears that is centre distance between the pair of gears changes. These two points as I said, I will explain with reference to the figures that we discussed last time.

The last but not the least, the most important reason, why involute profiles are so common is that, ease of production. Why it is easy to produce involute gears? Because, gear teeth are produced by a process called generation, that is, again we use the conjugate action but not between a pair of gears but between a rack and a pinion and it is a conjugate rack of an involute profile is a straight tooth rack.

If we have an involute profile, we see this later that the conjugate rack tooth, if this is a rack whose tooth profile is straight, then it can maintain conjugate action with an involute gear and it is such a straight sided rack cutter is used to produce the involute tooth profile on a circular gear blank. This is I can say it is an involute rack which is used as the cutter and on the cutter, it is much easier to produce straight side than any complicated curve and it is this involute rack cutter which can maintain conjugate action with an involute tooth profile of a gear. So this is called involute rack cutter.

Why the involute of a rack is a straight line? This point again will be clear in today’s lecture a little later, that the involute of a straight line is another straight line. Like we have seen involute of a circle is the curve. So, if this radius of the base circle goes to infinity, when the gear gets converted into a rack, then this also becomes a straight line. All these points, we will make clear in today's lecture. So let me get back to that figure, where I showed the pressure angle and now we will show, what is the advantage of this pressure angle remaining constant, in a pair of involute gear teeth profile?

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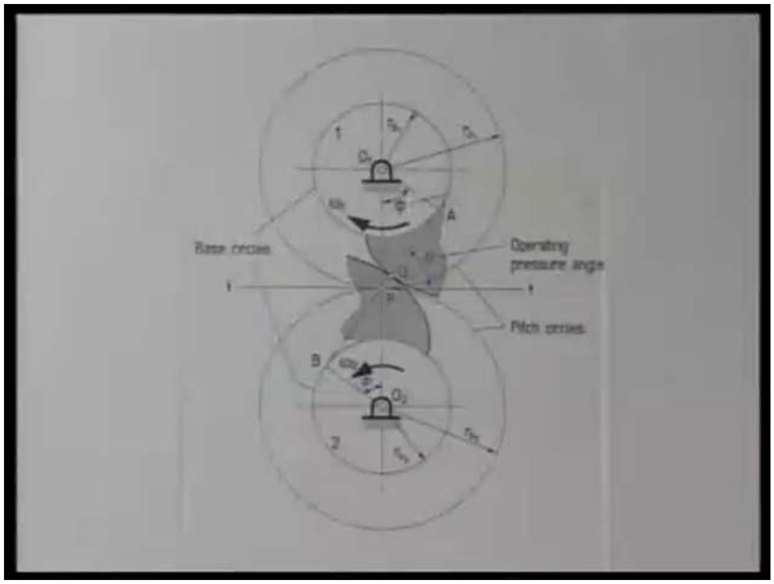

Let us get back to this figure, which we discussed in our last class. These are the two base circles of radius r_{b1} and $r_{b2.}$ These are the pitch circles, of radius r_{p1} and r_{p2} and it is the common tangent to the base circle, which we call the line of action. And this line of action AB makes an angle phi with this common tangent to the pitch circles. This tt is the common tangent to this pitch circles of radius r_{p1} and r_{p2} and this angle phi, we call it as operating pressure angle. Now as we are already noted, the line of action just remains AB, this pressure angle does not change as if the profiles are involute. The pressure angles remain constant during the entire interval. What is the advantage that the pressure angle remains constant? If we neglect the friction force between the gear teeth, which if it is well oiled and well lubricated, friction force is not very large. The driving effort is along the common normal. If this gear is driving this gear, then most of the force is acting along this common normal. So if this normal force, we call say N, then the torque that is transmitted is N times the base circle radius, because this line of action N is along AB and this angle is 90 degree. So, the torque acting on this gear 2 is N into r_{b2}. This torque is transmitted to gear 2.

If this torque remains constant, then N remains constant because r_{b2} is constant, which is the base circle radius. If N remains constant, then we see both the magnitude and direction of this force N is remaining constant, which means the bearing reactions here and here is the same N which is acting on this gear also. So the direction and magnitude of N both remains constant, because phi does not change and if torque is steady torque, if we are transmitting a constant torque, then this torque is also constant which means magnitude of N is constant, because phi is constant, this means the direction of N is constant, which means the bearing reactions, while the gears are transmitting a steady torque remains the same, which means the bearings are not subjected to any direct dynamic reaction. The magnitude and direction both of the reactions remain same. So long a steady torque is being transmitted. This enhances the life of the bearings, which are used to mount these two shafts on the foundation. Thus, the pressure angle remaining constant implies that under steady torque, the bearing reactions are not dynamic. They are static. That is the advantage of pressure angle remaining constant.

The third advantage, we discussed was that, suppose this pair of gears, right now the centre distance is $1O_2$. Suppose, this gear remains where it is, only this gear is shifted a little bit upward. We will see that, it is the same involute profiles will be useful to maintain the conjugate action.

Let us see what is the centre distance? Centre distance O_1O_2 is r_{p1}, the pitch circle radius of the first gear plus r_{p2}, pitch circle radius of the second gear. We have already noted that, base circle radius r_b is nothing but r_p cosine of phi, where phi is the operating pressure angle. So, we know r_{p1} by r_{p2} is same as r_{b1} by r_{b2} because r_{b1} is r_{p1} cos phi, r_{b2} is r_{p2}cos phi, cos phi cancels. So this ratio remains same.

So long the gears are same that is, the base circle radius remains same, r_{p1} by r_{p2} remains same. With the centre distance changing, it is r_{p1} plus r_{p2}, this quantity is varying. If the centre distance varies, the pitch circle radius r_{p1} and r_{p2} both will change, but the ratio of r_{p1} and r_{p2} that remains same. So it is the same angular velocity ratio $omega_1$ to $omega_2$ is maintained. Only thing that changes because r_{p1} and r_{p2} is changing means, phi is changing because r_{b1} is not changing and r_{b1} is r_{p1} cosine phi, where phi is the pressure angle and r_{b1} is not changing because the gears are not changing but with changing centre distance r_{p1} is changing which means phi is changing because of this common tangent, if this circle is shifted a little bit upward, the common tangent between these base circles will change, that means this angle phi will change. So pressure angle changes a little bit, but the same constant angular velocity ratio is maintained by the pair of gears, even when the centre distance changes a little bit, that was the third advantage.

And the fourth advantage as I said, will be explained later, when we will be able to show, that the involute of a straight line is a straight line that is for a rack to maintain conjugate action with an involute profile, the tooth profile on the rack will be straight. Like trapezium, as we have shown earlier on the board. Involute profiles have all this advantages as I said earlier, they are almost universally used in mass produced gears and as a consequence of this mass production of involute gears, they have been standardized. There are of various standards, but I can mention some typical standards which are more commonly used like British standard for involute profile.

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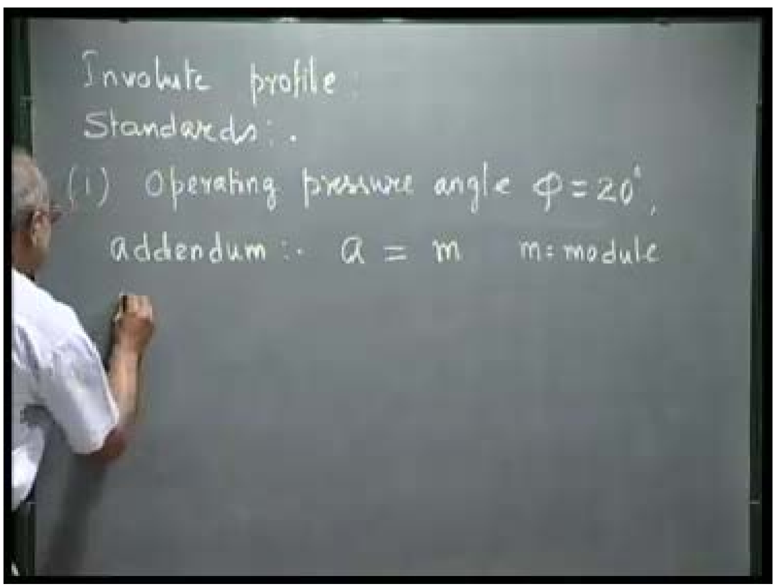

So in involute tooth profile, some standard dimensions are: The most common value of the pressure angle, operating pressure angle phi is 20 degrees. Some old gears which were cast also had a value phi equal to 14 and 1/2 degree, but nowadays many involute gears had an operating pressure angle equal to 20 degrees. Similarly, the value of the addendum which we defined earlier is equal to the module, where m is module. The gear teeth are described in terms of the module of the gear teeth and the standard value of the addendum is equal to module.

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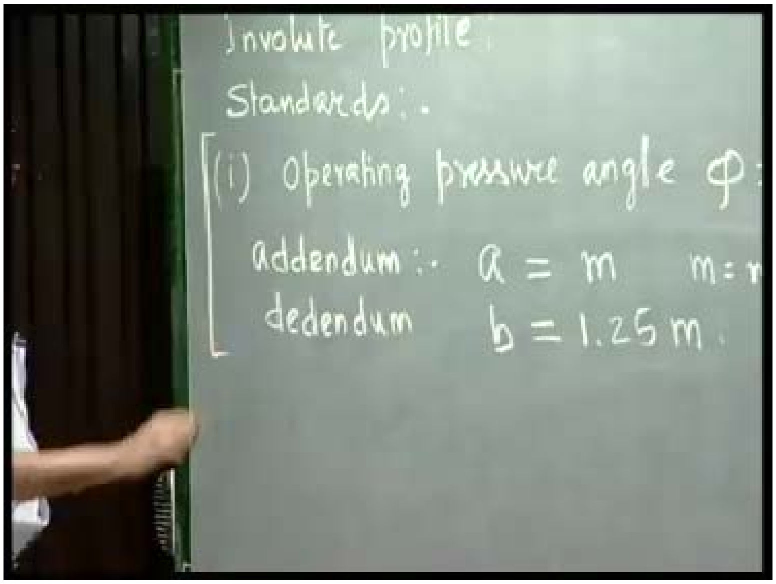

Similarly, dedendum which we denoted by b and which is always more than a, the most common value is 1.25 times the module, m is the module which is expressed in millimeter per metric gears, a is equal to the module m and b is equal to 1.25 m. These three standard values we may take if unless otherwise specified. As I said, involutes are most commonly used and geometry of involute teeth is a very vast subject. We are not going to get into all the details of involuted geometry, but I will give you a little glimpse or a little basic idea of this curve involute, which we called involutometry.

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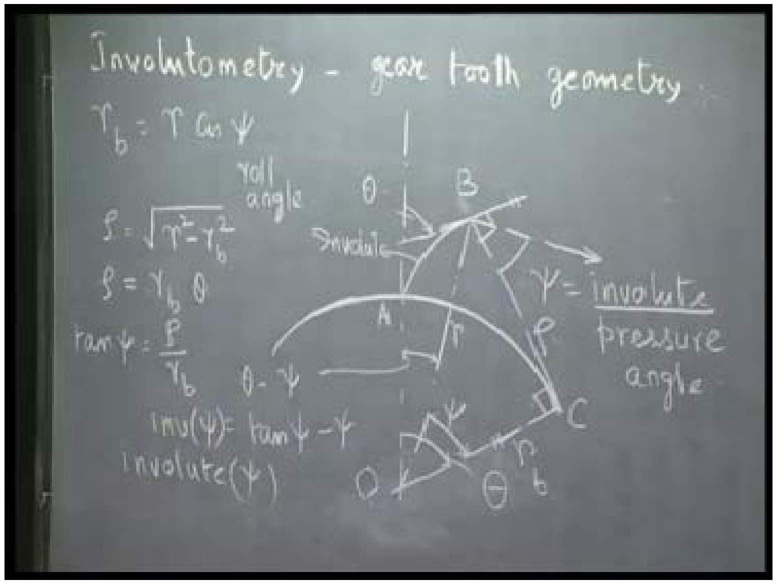

The result we obtained from this involutometry will be very useful to determine various proportions of the gear teeth, which are involutes. So we can say application of involutometry in gear tooth geometry, I repeat we will discuss of course only the very basics of this gear tooth geometry. We can always refer to hand books for the details that we may like to know.

So let me first say, what is involutometry? Suppose, this is the base circle with the centre here and we start unwinding the string from this base circle from this point A and by unwinding the string, I generate this involute. This is the string length, which is same as this arc length because, this string was wound on to this cylinder and now it has unwound up to this point and this is the involute from this base circle. So the string is tangent to the base circle radius r_b, which is perpendicular to the string.

At this point, the involute takes off radially. So, this radius is tangent to the involute at this extent and at this configuration, this is the string which is perpendicular to the involute. So the tangent is perpendicular to the string, it is tangent at this point say B. This radial line is tangent to the involute at A and this line which is perpendicular to the string at the point B is the tangent at the point B and this angle between these two tangents, I call the roll angle theta. Theta is called roll angle. This line doesn’t look like perpendicular, so let me draw it correctly. This is O. So this line is perpendicular to the string, this radius is also perpendicular to this string, which is tangential. This is parallel to this.

If this angle is theta, this is also theta, the roll angles. That is, the string has unwound up to this point B, whereas the roll angle is theta. This particular point on this involute, at this distance OB is the radius vector, let me call it r. If this represents the base circle of the gear and the gear is rotating about the point O, then this particular point has a velocity which is perpendicular to OB. This is the direction of the velocity and this is the normal to the involute. So that is the line of action.

At this point, this angle between the normal and the direction of the velocity and normal indicates the direction of the force at this particular point. This angle is called psi, which we will call involute pressure angle. This is the angle between the line which is perpendicular to OB and the string at this configuration, when I have this up to the point B. Please note that, this is not the operating pressure angle which we discussed in case of a pair obvious. This is only one gear, we are talking of, one involute and this particular angle between the direction of the velocity and the direction of the normal to the involute, I call it involute pressure angle. This line is perpendicular to the tangent and this line is perpendicular to the direction of velocity. So this angle is also psi, involute pressure angle. Theta is the roll angle. Psi is the involute pressure angle. Theta is the angle between these two tangents at A and B, which is same as these two radius at A and corresponding to B and psi is the involute pressure angle.

Let us say base circle radius is r_b and this string length is the radius of curvature of the involute at this point B. If this is the involute, at every instant, the radius of curvature is nothing but the string length, which has been unknown from the base circle. So, rho is if I call this point C, then BC is the radius of curvature rho. Rho is given as: square root of r square minus r_b square, where r is this radius vector of the point B measured from the centre of the base circle and if I use this angle is the polar angle to define the involute curve, what is this angle? This angle is theta minus psi and we also see this rho is same as this arc length AC, because this is the length of the string and this was the original length of the string. So, AC is nothing but rho, which is same as r_b into theta and also we can see tan psi is rho by r_b. If this angle is 90 degree, this is rho. This is r_b and this is psi. So, tan psi is rho by r_b.

From here, we see rho by r_b is nothing but theta. So I can write theta minus psi as: tan psi minus psi. Theta is rho by r_b and rho by r_b is tan psi. So, this angle that the line OB makes with this original radius OA, that angle is tan psi minus psi. This is given a name, which is called involute function of psi and we can also see r_b is nothing but r of cosine psi. So, we can write base circle radius is related to r as r cosine psi. Everything has been found in terms of this involute pressure angle psi, which keeps on changing at various points.

The distance from O, I can get as r_b by cos psi and this angle, I can get as tan psi minus sin psi which is called involute psi. This is very easy to see that, if one gives me the value of psi, I can calculate involute psi, very simple. This is called involute function just like sin, cos, we call it involute of psi. Given the value of psi, it is easy to calculate involute of psi, but not the other way around. So for this, tables are available just like, tables of sine, cosine, tan and log. I can get the value of psi, if you give me the value of involute of psi by consulting the table. Now, at this stage, I will be able to show that what happens to this radius of curvature of the involute profile as r_b increases. As I said, when r_b goes to infinity, the gear gets converted into a rack. So under that situation, what is the radius of curvature of the corresponding involute profile?

Let me now say, what is the radius of curvature of the involute tooth profile of a rack? Rack means, where r_b goes to infinity.

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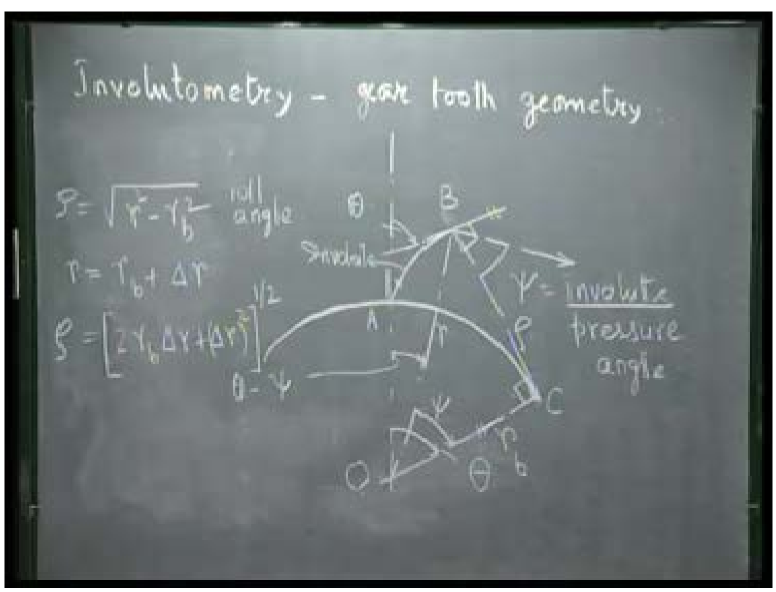

We have already got rho equal to square root of r square minus r_b square, where r is the instantaneous value of this distance. If, we start from A and go up to some point by changing initial value of r is r_b and then I go to r_b plus delta. Let us say, as I start from A, I go a little away from this, when r changes from r_b to r_b plus delta. If I substitute it there, what we get? We get twice r_b into delta r plus delta r squared to the power half. This is the value of rho, as we go out from this point A. If r_b goes to infinity, it is obvious that rho also goes to infinity. And if the radius of curvature goes to infinity, which means this profile has become a straight line.

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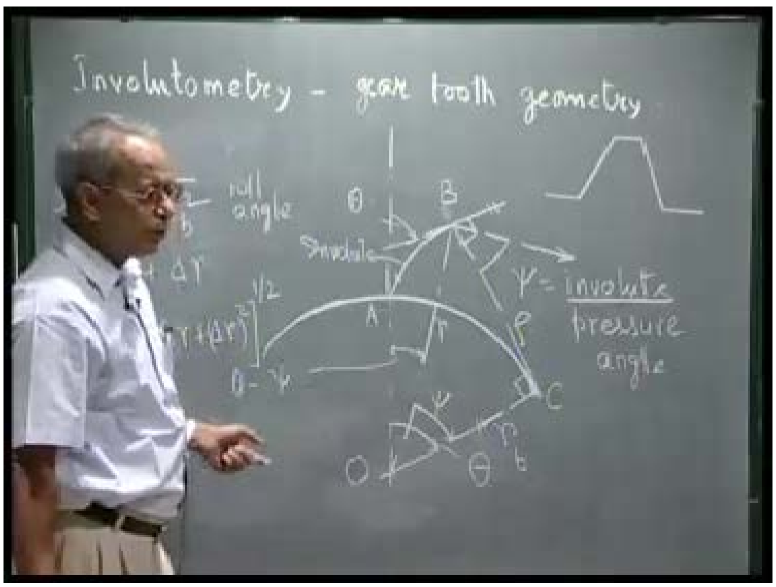

That is what we said that, if we have an involute rack, let the tooth profile from the base circle as it come out, it comes out in the form of a straight line. So, the tooth profile of an involute rack is a straight line and which can maintain conjugate action with an involute gear or involute pinion and that gives the advantage that, I can use such a rack cutter to generate the involute tooth profile on a circular gear blank.

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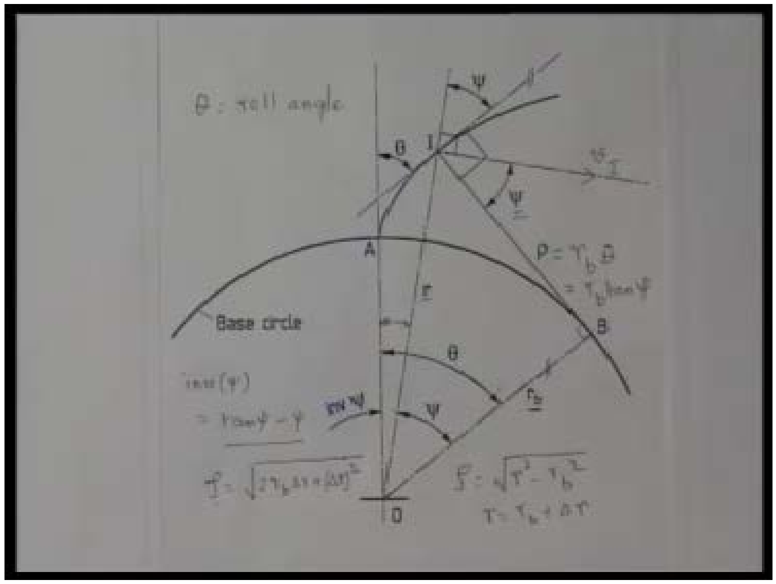

Let me now repeat, what we have just now discussed with reference to this figure. This is the base circle and this is the centre of the base circle. An involute is being generated, starting from this point A. When the involute, if I consider a point I here, then the string at this configuration corresponds to this line IB, which is tangent to the base circle. OB is the radius of the base circle, which we denoted by r_b. The instantaneous polar radius of this point I, I denote by r. The tangent at A to the involute is this radial line and the tangent at I is this line. The angle between this tangent at I and the tangent at A, this angle theta, we call the roll angle. Theta was defined as the roll angle.

If this I take as a gear and the gear is rotating about the point O, then the velocity of this point I is perpendicular to OI. Say, this is the direction of the velocity of the point I, if it happens to be a point on the gear and the string which is normal to this gear tooth profile or the involute profile, that is the normal to the involute at the point I and the angle between this velocity direction and this normal which is nothing but BI, I call psi. This is normal and this is a tangent, so this angle is 90 degree. This is also perpendicular to this string BI which is tangent to the base circle. So this line, the tangent at I and this radius OB are parallel. So, if this angle is psi, then this angle is also psi.

Similarly, if this angle is theta, then this angle is also theta because, this line is parallel to this line. So theta is the roll angle and psi is this angle. Now, the polar angle of this line OI from this vertical line, that is the radius through A initial point, this angle which is theta minus psi, we define as involute of psi. This length AB is nothing but r_b into theta and the string length BI is also equal to AB. This rho, which is the radius of curvature of the involute profile at I, the string length is the radius of curvature rho. This rho is nothing but r_b into theta.

From this triangle, OBI, I can write tan of psi because this angle is 90 degree. Tan of psi is also rho by r_b. We can write, rho is r_b tan psi which means theta is equal to tan psi. So this involute psi, which is theta minus psi, I can write it as tan psi minus psi. Then we also found that expression of rho in terms of r and r_b, which we wrote as rho equal to square root of r square minus r_b square. As we draw, this involute starting from this base circle, the r changes from r_b. It increases from r_b say by a value delta r. Then substitute it in there, canceling r_b. We found the expression of rho turned out to be twice r_b into delta r plus delta r whole square. So this clearly shows, if r_b tends to infinity, then rho also tends to infinity. That means, right from the point A, as soon as r changes from r_b to plus small delta r, the radius of curvature becomes infinity, that is the involute of a straight line. When r_b goes to infinity, this base circle becomes a straight line. That is, the gear is converted to a rack and the involute profile becomes a straight line because, radius of curvature is infinity, which tells us involute rack tooth profile looks like a straight line.

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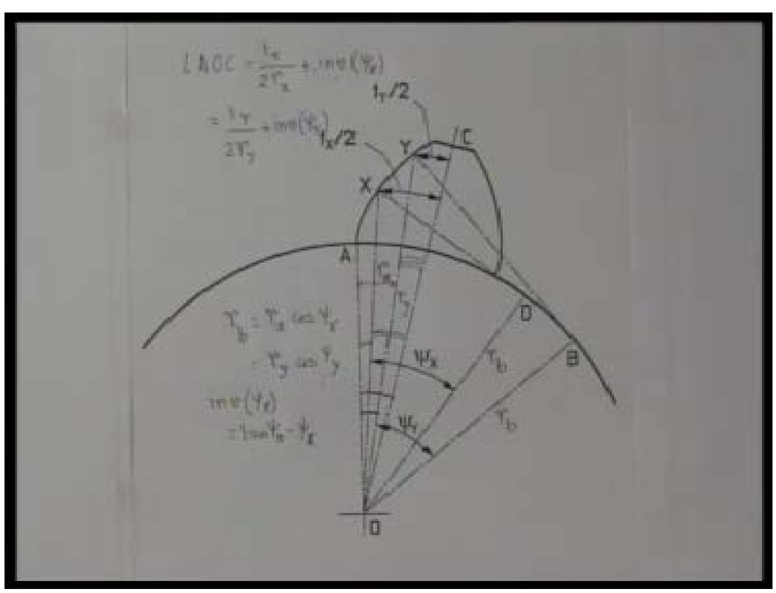

Just now, we have discussed the basic geometrical relationship between an involute profile and its base circle. These relations are very useful for studying gear tooth geometry, as I said earlier and here we discuss an example. Suppose, this is an involute tooth profile, which has been generated from this base circle, suppose the thickness of the tooth as we see keeps on varying from the base circle up to the addendum circle. Suppose, the thickness of the tooth at any point X, which is defined by the polar radius r_X, this point X is defined by this distance OX and the thickness at this level, I denote by $t_{X.}$

When this polar distance changes from r_X to say r_Y, I get to this point Y and I want to find, what is the thickness of the tooth at this level? That is t_Y. So t_X is given. I would like to find t_Y and r_X and r_Y are given. To do this first we note, when the involute towards that X, the string is DX. This is the length AD, which has become DX and when we come to the point Y, the string is represented by BY. This is the tangent, BY is same as the arc length AB. So the angle between this OX and OD, we defined as the psi. So I write it as psi_X corresponding to X, XD is tangent to the base circle and the angle between OD and OX, I call the involute pressure angle at X, which is psi_X. Similarly, the involute pressure angle at Y is the angle between OY and the tangent to the base circle from Y, which is YB that is what I call psi_Y. We have already seen the base circle radius r_b. This is also base circle radius r_b and this can be written as: r_b is nothing but r_X cosine psi_X, which is also equal to r_Y cosine psi_Y. If we were the given the values of r_X and r_Y, I can get psi_X and psi_Y.

How t_X and t_Y are related? For that, let me consider this angle AOC. OC is the mid line or the symmetric line of this gear tooth and the angle that OC makes with OA, I call this angle. So angle AOC, I can write as t_X by two divided by r_X I get this angle. This angle is t_X by two divided by r_X and this angle is nothing but, involute of psi_X. At the point X, the angle that OX mixes with this starting line OA, I defined as involute function of psi_X. So angle AOC is nothing but t_X by two divided by OX, which is r_X plus involute function of psi_X and if we remember, involute function is tan psi_X minus psi_X, involute of psi_X is tan psi_X minus psi_X. AOC the same angle, I can also write as: this angle which is t_Y by two divided by r_Y. So t_Y divided by two r_Y plus this angle. This angle that is, the angle between OA and OY is nothing but involute of psi_Y. This angle is involute of psi_Y. So this plus involute of psi_Y.

If r_X and r_Y are given, I can find psi_X and psi_Y from this relation. Once, psi_X and psi_Y are known, I can find involute of psi_X and involute of psi_Y, because involute function is tan minus the angle. Involute function of any angle psi_X is tan of psi_X minus psi_X. Of course, psi_X must be measured in radian. From this relationship, if t_X is given, r_X is given, r_Y is given, I can find t_Y which is the only unknown because psi_X and psi_Y, I have already found out from here and in using the involute function, I can find involute of psi_X and involute of psi_Y. Thus, for the gear tooth geometry, as we see for the thickness at any level, I can find the thickness at any other level of the same involute tooth profile.

For continuous transmission of rotation from one gear to another, it is obvious that is imperative that a pair of teeth must remain in engagement, before at least until the next pair of teeth comes into engagement. It should not happen that, one pair of teeth has lost its engagement and the next pair of teeth has not come in engagement, because then there will be no transmission. So, to maintain continuous transmission of rotation from one gear to another, it is imperative that a pair of teeth must continue to remain engaged, at least until the next pair of teeth has come into engagement. This phenomenon is studied in terms of a geometrical quantity which we call contact ratio.

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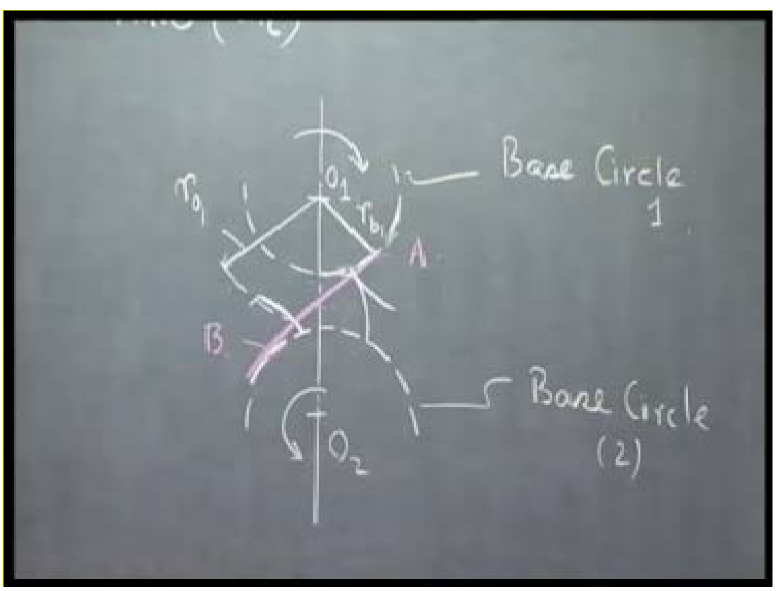

Let me now define the contact ratio and try to get the expression of this contact ratio. So we study contact ratio for involute tooth profile, we use the symbol M_c, indicate a contact ratio. Let us say, this is one base circle and the centre of this gear is at O_1 and the centre of the other gear is at O_2. This is the other base circle 2 and base circle 1. If we remember that, it is the common tangent to this pair of base circles. If I draw the common tangent to this pair of base circles, that defines the line of action. This is A and this other point of tangent is B. So we know the contact between a pair of gear teeth will always lie on this line AB.

Let us talk of one tooth of this gear. Suppose, this gear is rotating this way and to the teeth action, where teeth engagement it is rotating this gear in distance. If we consider a tooth on this gear, the first it comes into engagement at its outer circle or the addendum circle and because the contact point must lie on this line, when the addendum circle comes here in contact with this line AB that is, where the contact starts on this face of the gear tooth. This is one gear tooth.

Now, how long this gear starts rotating this way? How long this tooth will remain in contact with the tooth on the other gear? Gear number 1, so long the outer circle of this gear is in contact. This is the addendum circle of the gear 1. Let me call it radius r_{o1}, that is outer radius of gear 1 and this is r_{b1}. The contact will as this tooth rotates, this tooth comes here and when the tooth on this gear which is pushing it. Let us say here, if I draw this line. This tooth comes and rotates, and when this outer circle leaves this line, the contact is lost with this particular gear.

So if draw to this tooth here, this is where the contact **(incomplete video).**