Smart Materials and Intelligent System Design
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Lecture – 09
Classical Laminated Plate Theory

Good morning students, welcome to the class of Smart Materials and Intelligent System Design. And in this we are in the round 4 in this particular module of composite materials and composite laminates.

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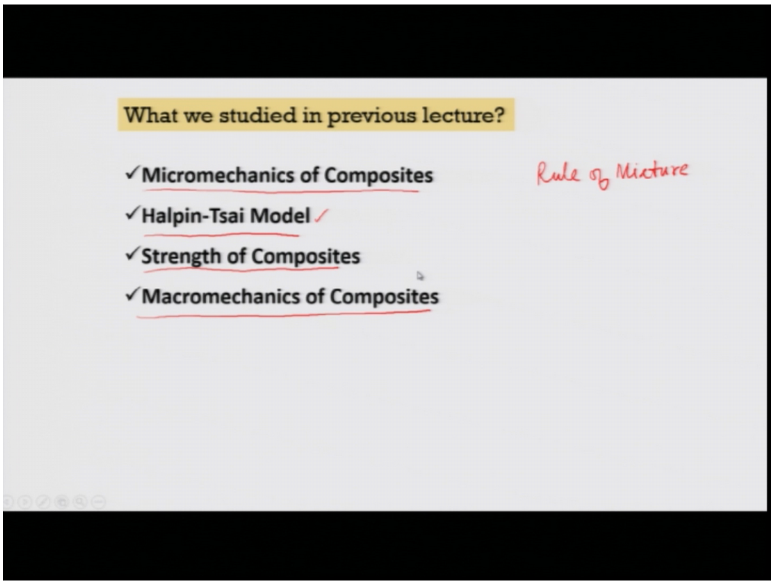

In this particular modules so far, what we have covered? We have covered the micromechanics of composites, we have covered some empirical relationships if you remember that we have talked about the rule of mixture and then we have also shown that in the transfers direction this does not work and there we have use this Halpin-Tsai Model which is any semiempirical you know relationship. And, then we have also talked about that how do you estimate the strength of composites where the key thing is that we assume when the fiber fails the composite fails.

So, as a results we derive the strength, finding out that what is this strain at which this failure is occurring and then getting the strength of the composites. The other key thing there that I discussed if you remember is that, what is the critical volume fraction of fibers that you must keep; not the minimum volume fraction, but the critical volume fraction so that the fiber dominates the response of the system. And, then I have also talked about a little bit just we have introduced that what is called macromechanics of composites. So, from that little bit introduction, we will be expanding the micromechanics part today.

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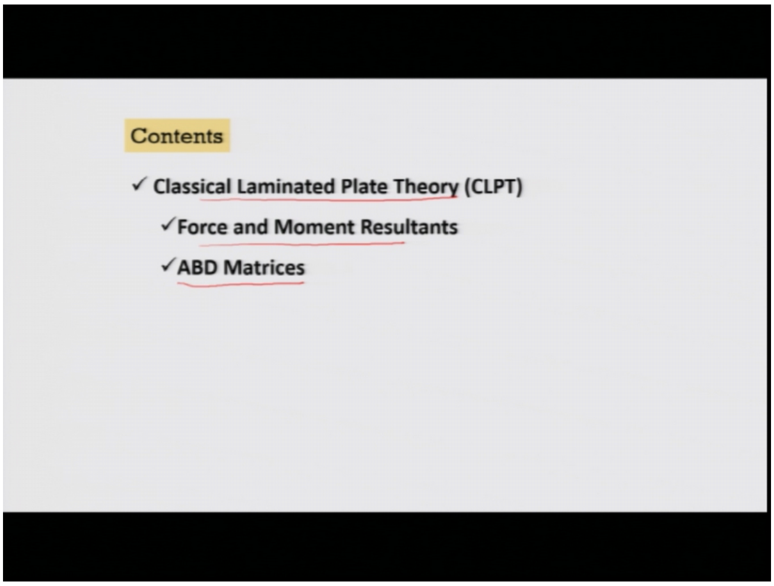

So, what we will talk about is that, classical laminated plate theory. Many of you must have studied the classical plate theory and this is extension of that for composites, which is called classical laminated plate theory. Once we do this, this will help us to develop smart classical smart laminated plate theory ok. So, we have to know first: what is classical laminated plate theory. And then we will talk about force and moment resultants and how to develop something which is very popularly known as ABD matrices.

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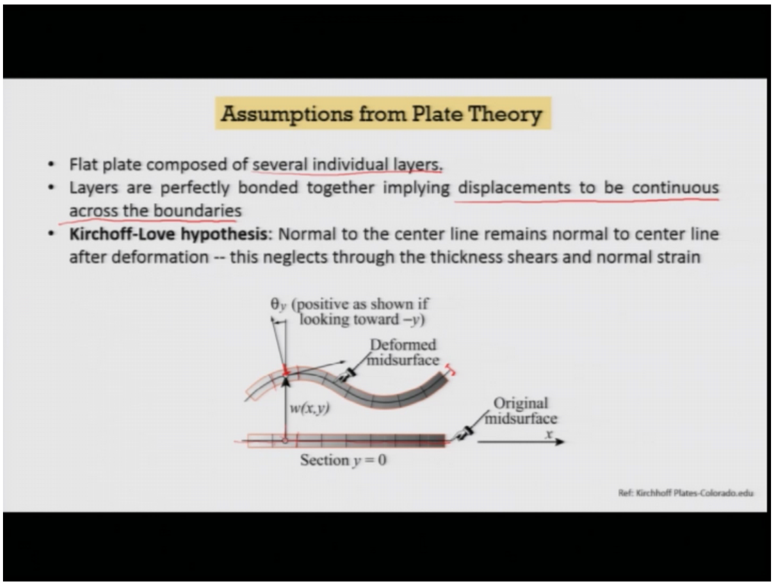

So, the first important point in our classical laminated plate theory is that, what are the basic assumptions in this theory. And the first point was that is that a flat plate in this laminated plate theory will be composed of several individual layers what we call these individual layers? Lamina so, when you have several individual layers lamina is referred as lamini and once you actually join all of them together, then it becomes a laminate. So, from lamina to laminate we will be developing the flat plate that is the first assumption.

Second point is that layers are perfectly bonded together, implying that the displacements will be continuous remember ok. Displacement and hinds the strain will be continuous across the boundaries. And thirdly there is a you know there is something which is very popularly known as Kirchhoff-Love hypothesis, and that is related to the nature of this deformation assuming that we are talking about thin plates. This you find that such in such cases when the deformation occurs, let us say a bending deformation occurs that is happening here, then the normal to the centre line you know so, this is the normal to the centre line.

So, this is the centerline the geometric midline so, to say normal to that remain straight you know here this section if you focus on this is the section. So, this has undergone bending. So, normal to that remained straight and also it remains normal at this point with respect to the neutral axis. And we would also neglect through the thickness shear and the normal strain ok. So, this direction deformation is actually neglect because it is very thin and through the thickness shear is also neglected in this case. So, this if we keep our assumptions straight then we can build up the theory.

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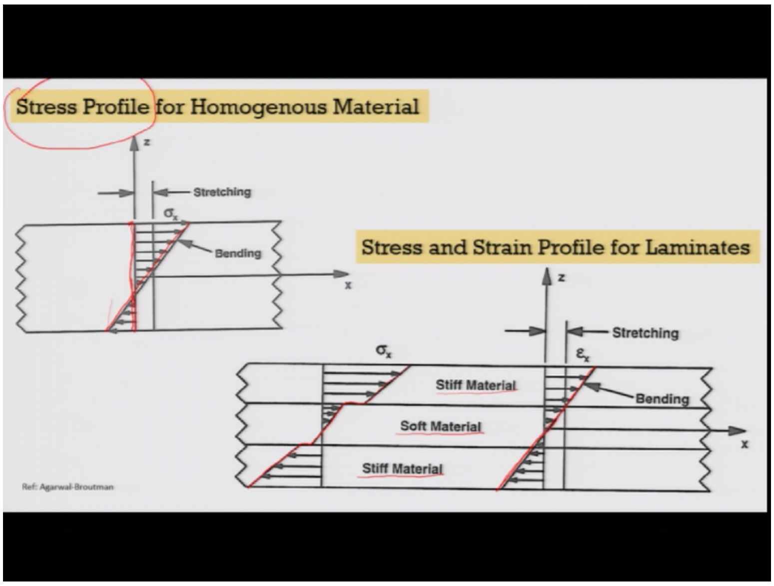

And if you look at the difference between the homogeneous materials and the counter part of it in the classical laminated plate theory, you would see that the difference would be coming in terms of the stress profile. Because if you look at any you know homogeneous material, which is subjected to both bending or and stretching or so, to say sometimes you know asymmetric bending, you would see that the stress profile; however, remains uniform across the cross section ok.

Events I would say you know continuous not exactly reform. If it is stretching it will be uniform, but it is it can be linearly wearing like it is wearing here, but it will it will be a continuous stress profile that we will get. On the other hand if you look at the laminated plate system, then and the same thing is subjected to stretching and bending then let us say it is having 3 parts there is a stiff material there is a soft material there is another stiff material.

So, we had assumed that no matter whatever is the soft stiff etcetera if there is a perfect bonding, then the displacement is continuous and this part also if you look at it that this is the strain profile, which is actually continuous there is no breaking of the strain profile right. And this strain then you multiply it with its modulus of elasticity of each of the layers naturally, in each layer if the modulus of elasticity is different, then will be this kind of breaking of the stress profile ok. So, that is something which is unique of these you know classical laminated plate theory.

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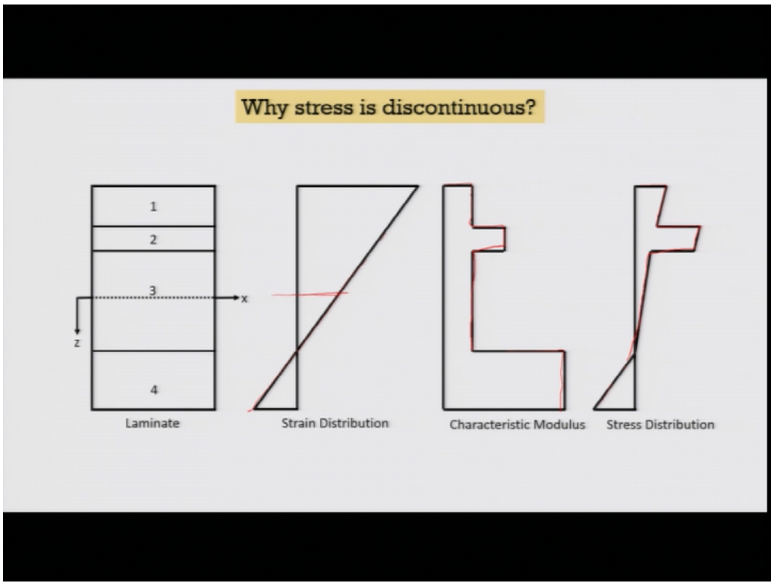

So, why stress is discontinuous? Again this is the strain distribution as you can see there are 2 points to note here that there is always some strain at the geometric midpoint and however, it is continuous as you can see. And if you look at the characteristic modulus let us say this is some kind of a 4 member one, that characteristic modulus could be different at different layers ok.

So, some layers soft some layer stiff etcetera. So, this each layer multiplied by the strain is going to give me the stress. Now since this is different at different layers so, the stress is going to be different, it will be discontinuous across the laminate you know. So, that is why that stress distribution discontinuity is coming up.

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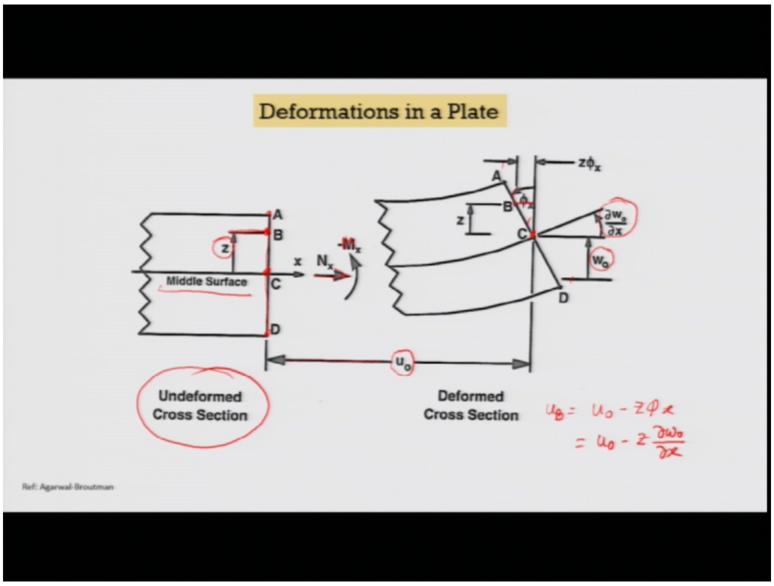

Now, let us look at the deformation in the plate and let us try to get an expression for it. So, this is what is our left side is our undeforrmed section and this is the middle surface of it, we are focusing on a point which is at a distance z on a particular cross sections ok. There are 4 points that we have considered here point A and point D which are at the 2 extremes point C as the geometric peak point and point B which is at a distance z from the middle surface. Now, as I told you that this whole thing is subjected to some kind of a stretching force as well as some type of a bending force there is some sign conventions with the bending because of which you know we have to put the positive and the negative signs accordingly.

So, as long as you follow one regular sign convention likes (Refer Time: 08:30) positive (Refer Time: 08:31) negatives etcetera then you know it is fine ok. So, now, with this sign conventions as this you know plate is subjected to both stretching and bending. So, what will happen is that this midpoint will be undergoing a midpoint stretching right. So, if you if you have checked it in the other these things I have shown you that there will be strain at the midpoint it will not be neutral axis in that sense would not exist at the geometric mid plane so, there will be stretching. And that stretching we have denoted it as u 0 that is the first along the x direction midpoint stretching is u 0.

Now, not only that point C will be also undergoing a rotation not point C, but this line A B C D. So, if you call this to be A prime B prime C D prime C prime because from C to

C prime it has moved. So, you will see that there is a rotation that has taken place here. And this rotation if the you know displacement at this particular point the transfers displacement is denoted as the w 0 then this rotation is considered as dou w 0 dou z.

So, once I know this rotation, then at this particular point B what are the deformations that is working in the system? Along the direction x it is u 0 first and then there is something that we have to deduct here, because at this point it is not u 0 at this point it is u 0 what is that? We have to deduct that is for a small angular deformation it can be written as z phi x, in other words u displacement at B will be u 0 minus z dou w 0 dou x that is the basis of my building up of the constitutive relationship. So, you have to keep this point in our mind.

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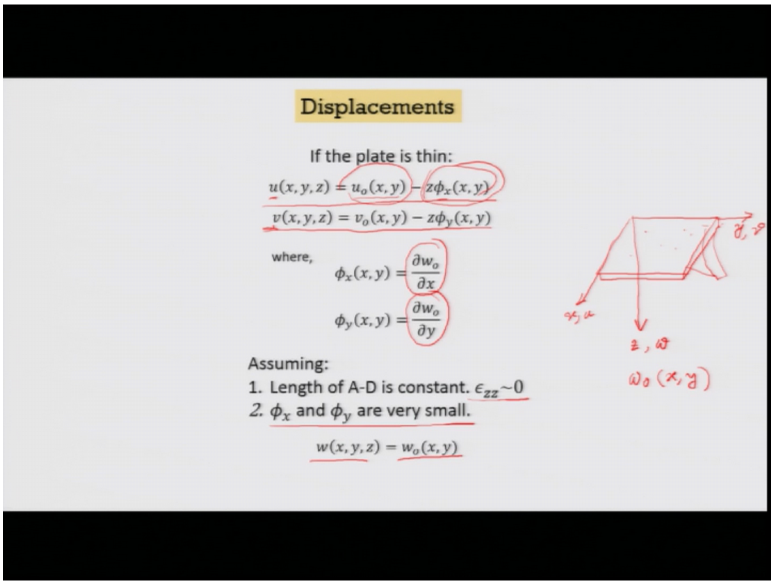

So, now you see we have writing more formally then u and v because this is a plate. So, you will be having both the direction of displacements. So, initially we have shown it how it is going to happen for a beam, but now let us say that this is x direction. So, u is the displacement and this is y direction and v is the displacement and this is z direction say and w is the displacement along z direction. So, u will be mid plane u 0 x y and minus z dou w 0 dou x that is the phi x and that you know as a function of x y you know that is what will be u at any point. Similarly v at any point will be v 0 x y minus z you know dou w 0 dou y in this case.

So, phi x y is a partial differentiation of w with respect to x and phi y x y is the partial differentiation of w 0 with respect to y. So, w 0 itself is varying from point to point right it is not a function of z. We have assume that the you know across the thickness you know it this w 0 is not going to get change, because that is what is the assumption of a Kirchhoff's love plate theory. But w 0 at various points the deformation will be different right. So, if this fellow is subjected to some kind of a bending, if you imagine that there is some bending that is taking place in the system then in a very kind of a you know exaggerated way, then the w 0 will be different at different locations of the plate so that is what is this particular thing would denote.

And we have assumed as I told you that epsilon zz is 0 and phi x and phi y are very small, that is why is z phi x is happening otherwise you have to really take the angle and you have to take the tangent of the angle and get that deformation. And w x y z is nothing, but w 0 xy that is at the midpoint whatever is the deformation along the z direction, that is what we will con considered to be the deformation along the z direction across the cross section of the plate.

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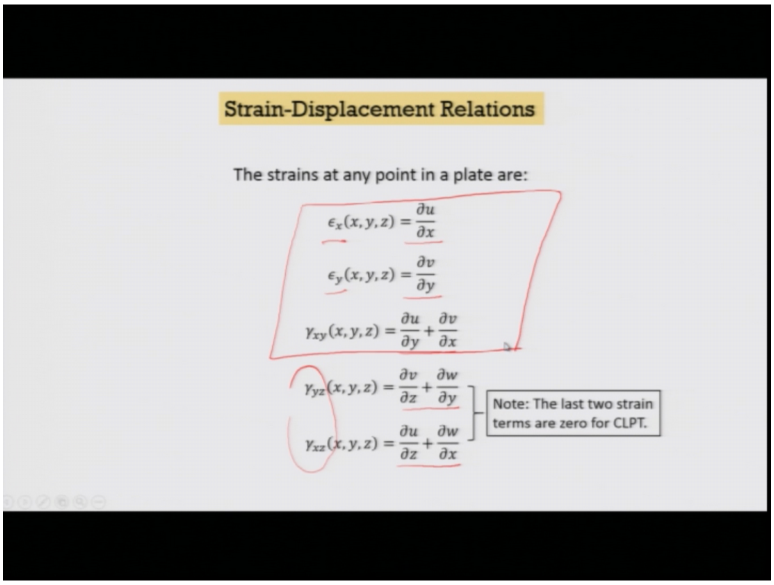

Now, then once you know the deflections your, you know you understand the deflections most critical thing is done. Now, you have to apply the definition like for the strain epsilon x what is it? It is dou y dou x and what is epsilon y? It is dou v dou y. Gamma x y is dou dou y dou v dou x gamma y z dou v dou z dou w dou y and gamma x z is dou v dou z dou w dou x. Now these 2 terms you know dou v dou z plus dou w dou y and dou u dou z plus dou w dou x. So, gamma y z and gamma x z these 2 terms are considered to be 0 for the CLPT; so, we are left with only 3 terms now epsilon x epsilon y and gamma xy and let us try to actually derive this terms now.

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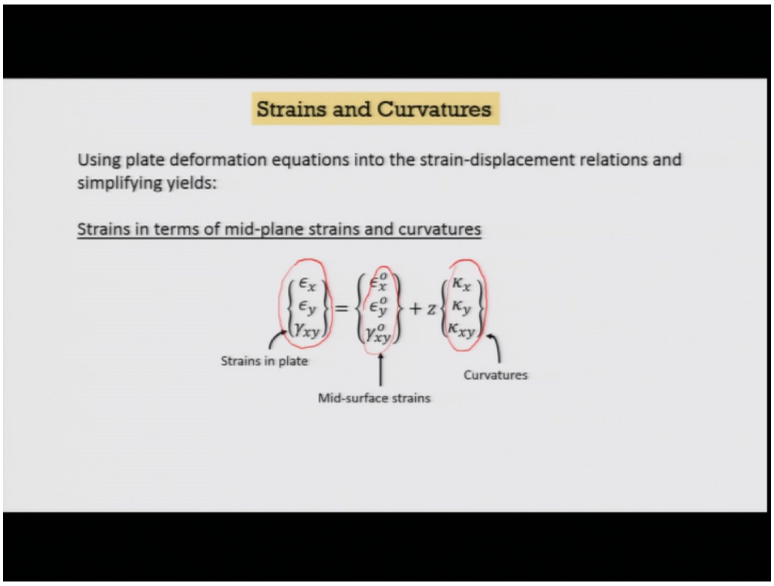

So, if I do that what you will find is that, there will be one part of it which will be if you look at it that in the displacement expression itself you have 2 parts right one part is related to the mid plane and another part is related to this angular you know motion of that particular layer. So, he here also in terms of the strain, you are going to get one part which is related to the mid plane surface strains epsilon x 0 y 0 gamma x y 0 and another part which is related to the curvature. So, all these 2 parts together is going to give me the strain in at the plate at a particular point of coordinator x y on the plate you get epsilon x epsilon y and gamma x y.

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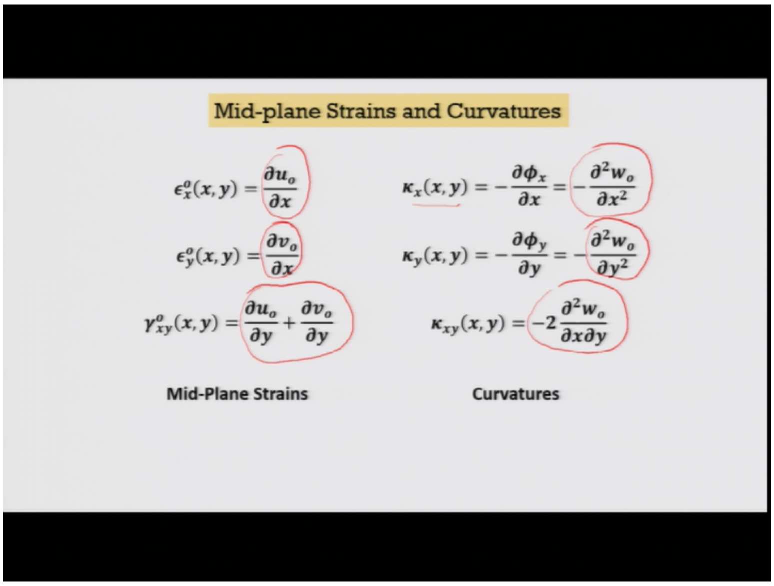

So, with this you know thing the what will be epsilon x 0 it is dou y u 0 dou x. What is the kappa x? If you evaluate it would become minus dou 2 w 0 dou x 2. Similarly epsilon y 0 dou v 0 dou x and the kappa y which is curvature in the y direction that is minus dou phi dou y by definition and that would become minus dou 2 w 0 dou y t and what is the shear strain gamma x y 0? That is dou u 0 dou y plus dou u 0 dou y and will there be a curvature? Yes there is the curvature which is minus 2 dou 2 w 0 dou x dou y. So, this are the you know definitions of the strains and the curvatures in the system.

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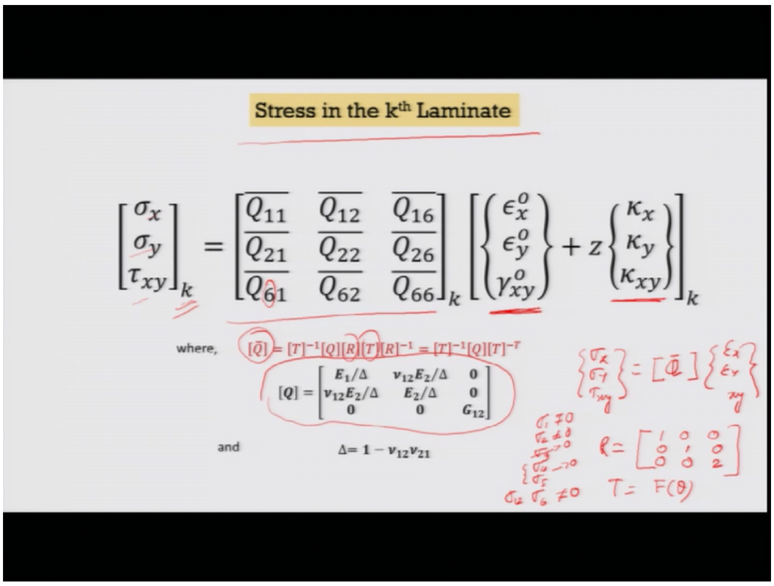

With that we can once you know that you can actually find out the stress in the laminates. So, at any particular kth laminate ok, whatever be the angle of that laminate you can actually get the stress strain relationship constitutive relationship in this manner; that at the kth layer the 3 stresses we are considering sigma x sigma y dou x y they are if you remember that I already told you that sigma x, sigma y and tau x y they can be written in terms of the global coordinates system as Q bar epsilon x epsilon y and gamma and gamma x y.

Now, that Q bar what is written here in terms of Q 11 bar Q 12 bar, 16 bar, 21 bar, 22 bar 26 bar 61 bar 62 bar and 66 bar. So, this 6 here is simply because if you consider that how we have considered all the stress terms. So, it was like sigma 1 sigma 2 and then sigma 3 is actually neglected in this case that is how the accounting is happening then sigma 4 sigma 5 and the final stress these 3 are the shear stress terms in which this 2 are also 0. So, the only term that is remaining is related to sigma 1 2 and that is not equal to 0, these 2 are not equal to 0. So, that is why sigma 1 and sigma 2 and sigma 6 are coming and this Q 6 is rim you know it always reminds us that that is what is the origin of the 6.

And here so, you have the strain which can be split into 2 parts, the mid plane strain and the curvature part as we have already explained. Now, this Q bar you also know that this Q bar can be derived from the Q Q is the you know. So, to say elastic constant matrix in the principal directions along the direction of the fiber, and then you know if you know the transformation matrix, which you depends on the ply angle.

So, if the ply angle is known to me I not E R is very simple structure if you remember R was simply 1 0 0, 0 1 0 and 0 0 2 that was what was the R. So, R is known to me and T is a function of T is a matrix which is a function of theta ply angle. So, that also I had earlier given you we should be able to find out the Q bar. So, once you know the Q bar, the constitutive relationship or the stress in the kth laminate will be known to us so, that is the way you know to get it ok.

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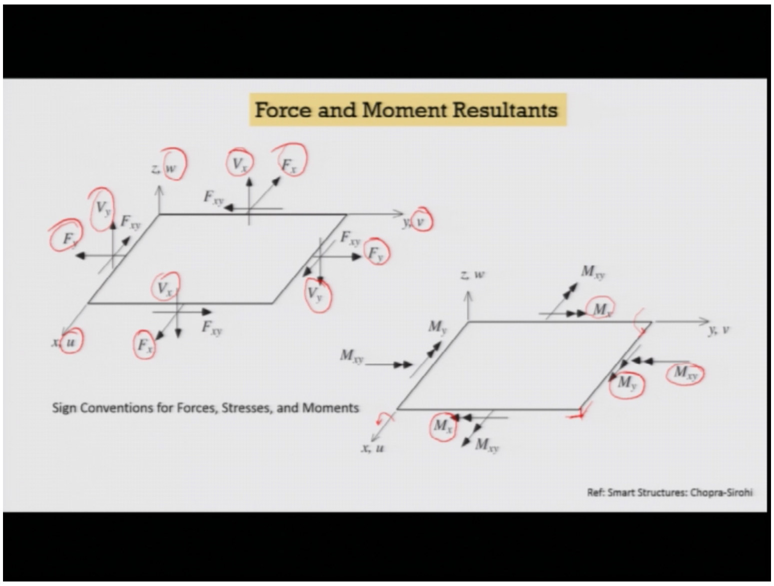

Now, what are the force and the moment resultants that would come for a plate. So, if you look at the you know consider this particular plate with xyz sign conventions, then x direction displacement is u, y direction displacement is v, z direction displacement is w ok. So, with that first of all we have the stretching along the x direction that is given by F x positive expansion ok. So, they are also F x and similarly y direction expansion positive towards a positive y direction that is F y.

So, that is: what is the direction you know kind of convention that we have to keep in mind and for the shear force, what we will always say? Left up and right down. So, this is V y positive and this side is the V y positive. So, that is the y direction shear force a similar shear force you know would be coming here for the V x. So, this is in plane forces and the shear forces. What about the moment well? So, the moment once again is coming up here. So, you know first of all about the x axis the moment if you look at it, then this is actually M x about the x axis it is coming up and then you know this M x must be having its counterpart on the other face so, that is the M x part.

So, about the x axis and similarly about the y axis you will be having the rotation about the y axis itself. So, that will be denoted by the M y ok. So, this is also another bending and that bending is happening about the y axis, and then you also have torsion M x y is nothing, but torsion ok. So, the torsion is happening across this particular cross section of the system ok. So, it is a twisting that is happening in the system that is denoted by the M x y. So, these are the force and the moment resultants that we have to consider in this particular kind of a system.

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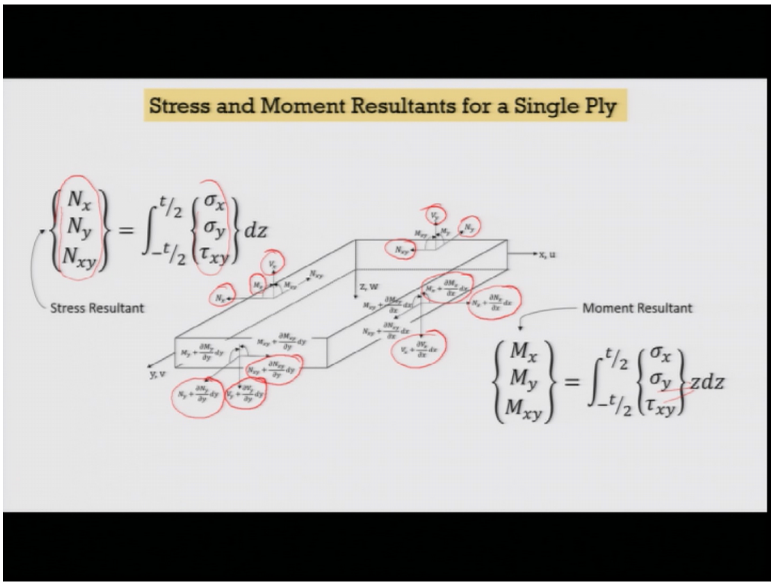

So, now based on this how do we get the stress and the moment resultants, well stress says stress resultant; so, to say N x y N x N y and N x y. So, these are like x direction net force y direction and then the shear so, that is you know obtained by integrating each one of these stress terms ok. And for the moment it has to be you know sigma x z dz sigma y z dz tau x y z z dz where z denotes the distance of the point with respect to the neutral axis and also to say that geometric we will call it as a geometric mid axis.

Now another thing that we have to keep in our mind is that, this if the plate in between is subjected to some loading then the moments are going to change from one face to the other ok. So, that is denoted here if you look at this particular picture, we would see that let us start from the stresses for example ok. The stress resultants if you look at it then you would see that if the stress resultant here is a N y at this point it would become N y plus dou N y dou y d dy once it is subjected to some loading ok. Similarly in the x direction if it is a N x here then it will be N x plus dou N x dou x dx here. Similarly for the shear force if it is V x at this point then it will be V x plus dou V x dou x dx. If it is V y at this section then it will be Vy plus dou V y dy at this section.

And then the shear stress resultants if it is N x y here then it will be N x y plus dou N x y dou y dy here. If you look at the moment part just I will show one of the terms if it is M x here then at the other face it will be M x plus dou M x plus dou M x dau x dxok. So, thus you know the stresses the stress resultants the both like the planar stress resultants are the moments that is the way we have to you know define them the for this particular system.

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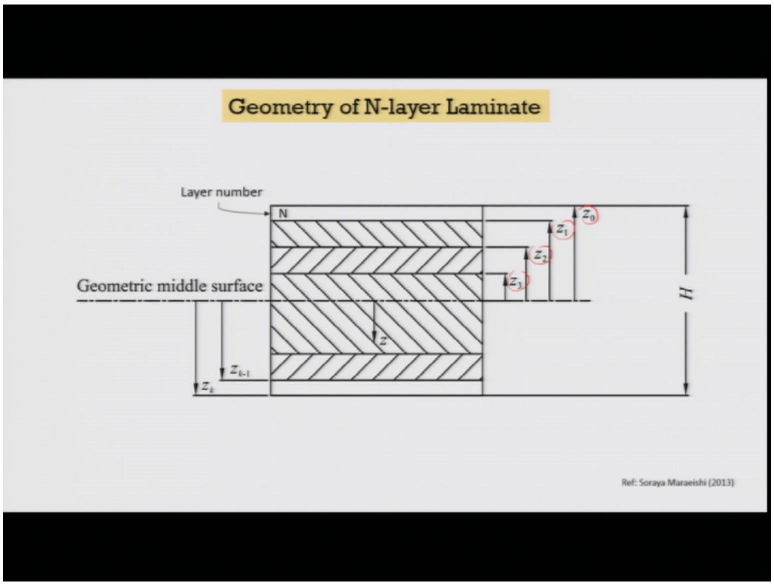

Now, let us consider the geometry of N layer laminate where each layer is. So, this is Z positive direction. So, if you consider that in from this system actually this things will be all negative Z 0 Z 1 Z 2 Z 3 each one of the layer distance from the geometric middle surface and the downward direction it will be positive.

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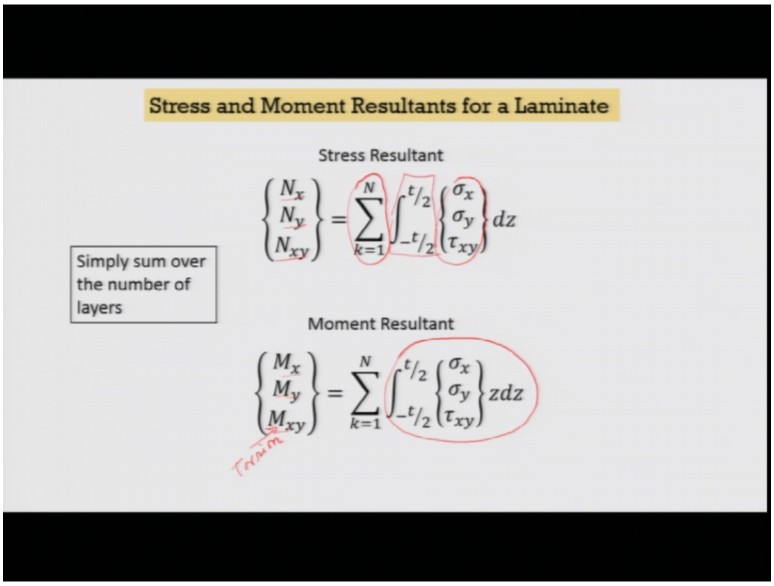

With this Z values you know you can actually now carry out the integration as discrete integrations that is the advantage of it that you find out the sigma x sigma y tau x y as an integration over dz only for a particular layer; so, this minus t by 2 to t by 2 where t is the thickness of each layer. So, only you do it you know for each one of these layers and then you actually sum it up if you have N number of layers you sum it up for N number of layers, that is going to give us the stress resultants N x N y and N x y.

Similarly, if you consider you know the M x M y M xy that is too bending and the torsion this one is the torsion. In this case also you carry out first the integration in each one of the lamina layer and then sum it up and get the you know get it for all the layers. So, that is you know that is something that we are assuming for and that is valid as long as we are in the domain of slender plates where the link to thickness ratio is at least greater than 100 or so.

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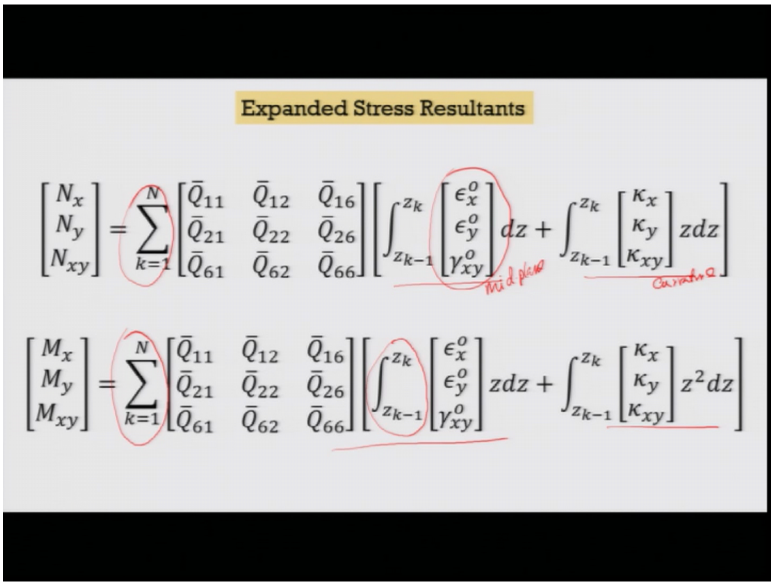

Now, if we keep all this things in mind and our earlier expressions then now in terms of the summation and integration I can get the stress resultants N x N y N x y and I can break them into 2 parts one is the mid plane part and other is the curvature part ok. So, N x N y N x y has these you know mid plane this is the mid plane part mid plane strain component and this the curvature related part.

And same thing we are going to do for the M x M y M x y also the mid plane related part and curvature related part. And integrations we are carrying out from Z k minus 1 to Z k and then we are summing it up for k equals to 1 to n ok. That is the strategy we are doing and in order to derive N x N y N x y and M x M y M x y in terms of the strain of the system the in terms of the mid plane strain and in terms of the curvature of the system.

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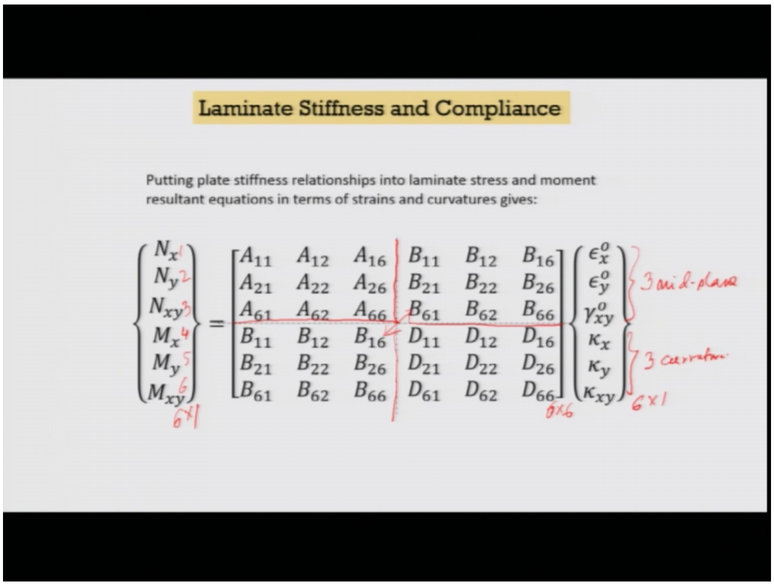

So, that strategy if we keep in our mind then you will see that the laminate stiffness and the compliance will be coming up in this manner that you know if the ply sequence is absolutely (Refer Time: 26:28) symmetry, you are going to get a fully populated matrix ok. How many terms we have here? 1 2 3 4 5 6 so this is the 6 by 1 vector right and here also we have these 3 mid plane strain 3 mid plane and we have 3 curvatures here right 3 curvatures.

So, we are also we have 6 by 1. So that means, here I am expecting a 6 by 6 matrix of 36 terms. Now that 6 by 6 matrix of 36 terms is subdivided into 4 parts flat this is one part part a and there is a part b part and you can see that these 2 parts are actually symmetric and then there is a part d. So, this is what it would look like with respect to the you know stiffness laminate stiffness and compliance.

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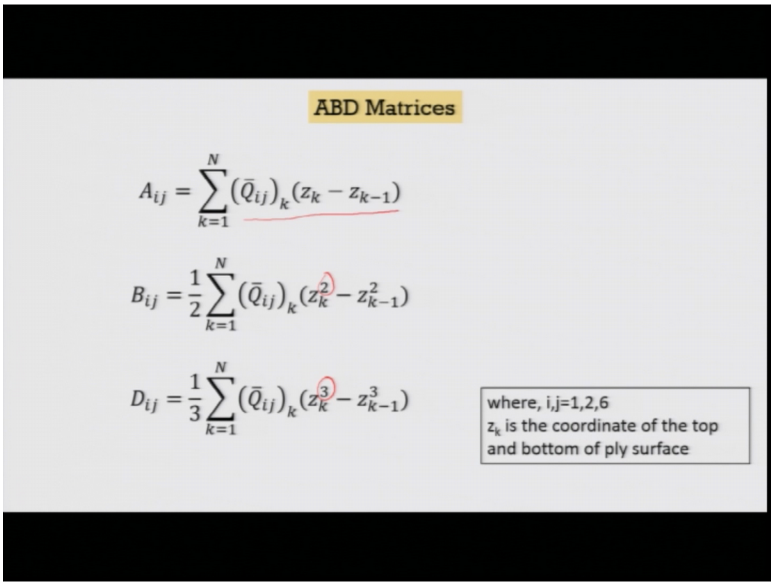

Now, by definition you can get each one of these terms like A terms are all like you know summation of Q i j over Z k minus Z k minus 1, for B terms it becomes square for D terms it become cube that it becomes square for B terms that is very significant we will see to it you know in one of the lectures that the B terms essentially varies for certain layout of the composite.

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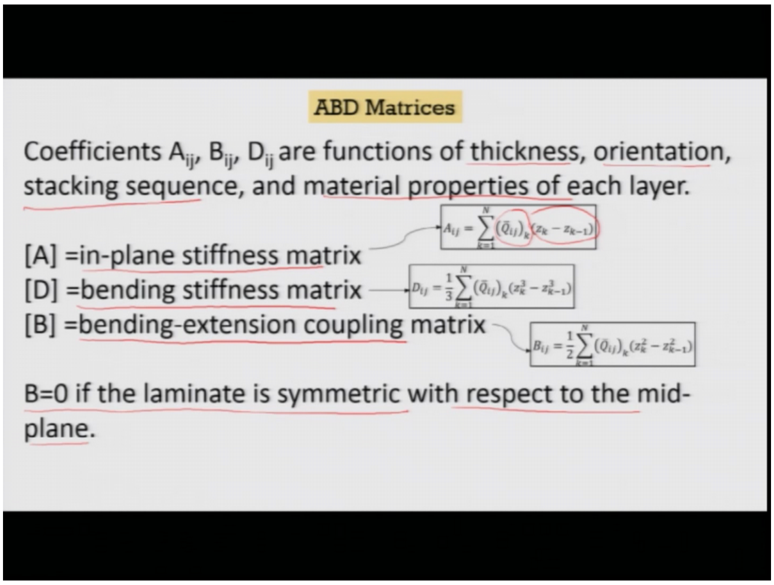

So, now, if you look at the ABD matrices here with the you know from which is fun which will be functions of thickness orientation and the stacking sequence and the material properties of course, of each layer then A i j for example, this is where the material property comes into picture thickness comes into picture and because of the stacking sequence, you will accordingly get this summations and A is known as the in plane stiffness matrix.

D is known as the bending stiffness matrix and B is known as the bending extension coupling matrix. This B will be equal to 0 if the laminate symmetry with respect to the mid plane of the laminate ok. So, this is what is the description of the ABD matrix that we have to keep in our mind based on which we have to develop the further deformation theory.

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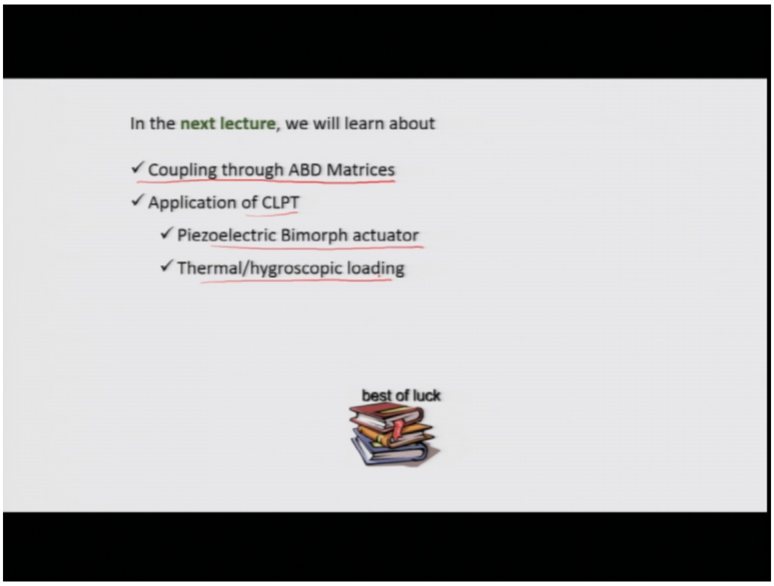

So, this is where we will put an end, in the next lecture we will learn about the coupling through ABD matrices will also talk about some application of CLPT in piezoelectric bimorph actuator the thermal hygroscopic loading.

Thank you.