Turbulent Combustion: Theory and Modelling
Prof. Ashoke De
Department of Aerospace Engineering
Indian Institute of Technology - Kanpur

Lecture-35
Turbulence (Contd...)

Welcome back, let us continue the discussion on the turbulence scaling. So, we looked at the wall boundary flows, where we looked at different region like inner region and outer region and overlap region. So, that is from there we will now look at the other pattern like the effect of some roughness and all this thing, that this is where we stopped that you can have an inner layer, log law layer, overlap layer and the outer layer and then if you put them in the Reynolds number plot there would be sub layer,

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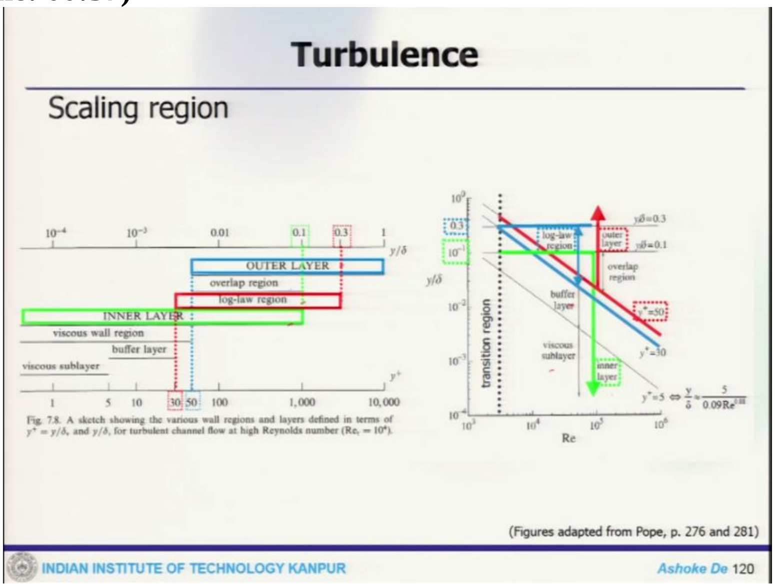

Inner sub layer, buffer layer, and all these finer details one can actually work out and look at. Now we will move to the next set of things that is wall roughness on the turbulent boundary layer. So, what we are interested, this is if you have a boundary layer up there roughness then what is the effect of wall roughness. So, there could be different scaling law for inner layer there would be friction factor.

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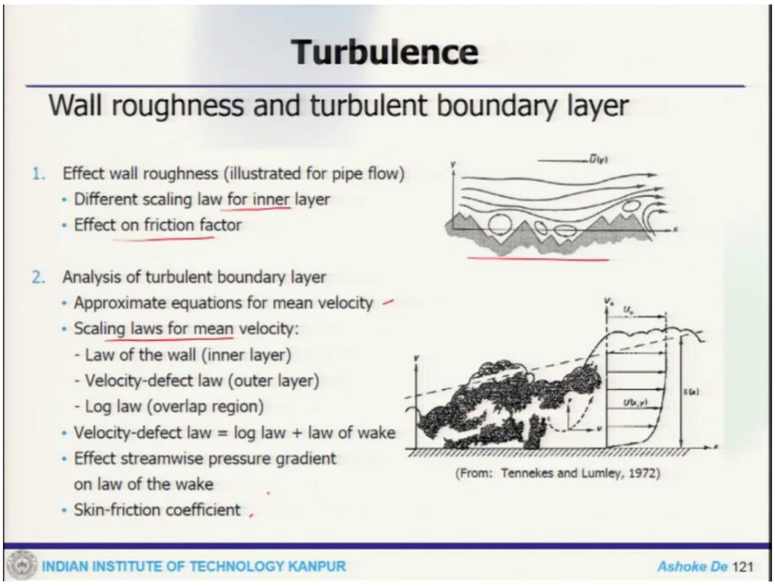

Now, to do the analysis, what one can look at, it can be approximate the equation for the mean velocity then they can have scaling laws, all of them in velocity like law of the wall velocity defect law, log law and then find out the skin friction coefficient and all these things. Now, if you look at the wall roughness, so, what happens in the boundary layer? so, if there is a roughness with characteristics height is then when you scale the velocity profile or velocity gradient, there will be another extra length scale that comes into the picture.

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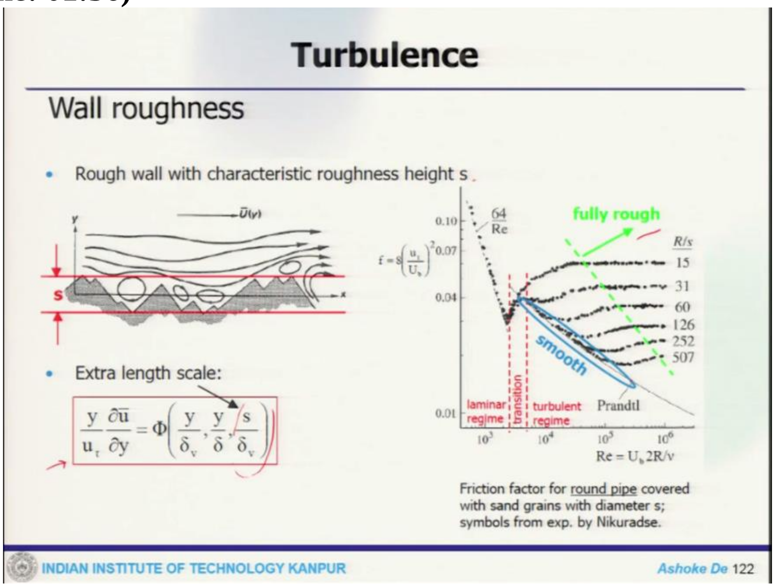

And if someone looks at the plots here, that this is nicely 64/Re plot with the friction factor and this is for smooth pipe and as Re/x increases, so, you can see the effect of roughness. So, from smooth to the fully wrap situation and that is how the roughness is going to. Now, scaling over that outer layer, what would happen outer layer actually unaffected by the wall roughness. So, this is unaffected by wall roughness. So, the larger eddies in outer region scale with delta since is by delta is smaller than one.

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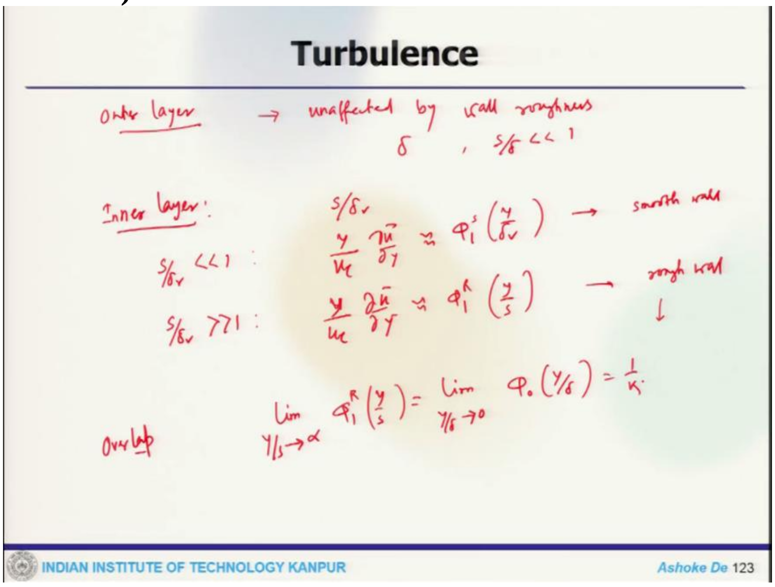

Now the scaling of inner layer depends on inner layer, it depends on S/δ_v. So, if S/δ_v is less than 1, one can write

$$\frac{y}{u_l}\frac{\partial \vec{u}}{\partial y} \approx \phi_1^s\left(\frac{y}{\delta_v}\right)$$

So this is for hydraulically smooth wall or if S/δ_v if it is greater than 1 then

$$\frac{y}{u_l}\frac{\partial \vec{u}}{\partial y} \approx \phi_1^R\left(\frac{y}{S}\right)$$

which is again hydraulically rough wall. Now when you talk about hydraulically rough wall, so here one can think about the friction at wall dominated by the pressure drag instead of viscous drag. So, in this case the viscous effect can be ignored. Secondly, the eddy switch is close to the wall skill with the height of roughness element and the overlap region of rough wall one can write on the limit that

$$\lim_{y/s\to\alpha} \phi_1^R\left(\frac{y}{S}\right) = \lim_{y/\delta\to 0} \phi_s\left(\frac{y}{\delta}\right) = \frac{1}{\kappa}$$

So, if you look at the roughness effect on log law, so, generalization of the log law for wall roughness. So, this is R U + one by Kappa Y by S is by delta V with this factor which is one by Kappa for S by delta V less than 5, if it is greater than equal to 7 it is 8.5.

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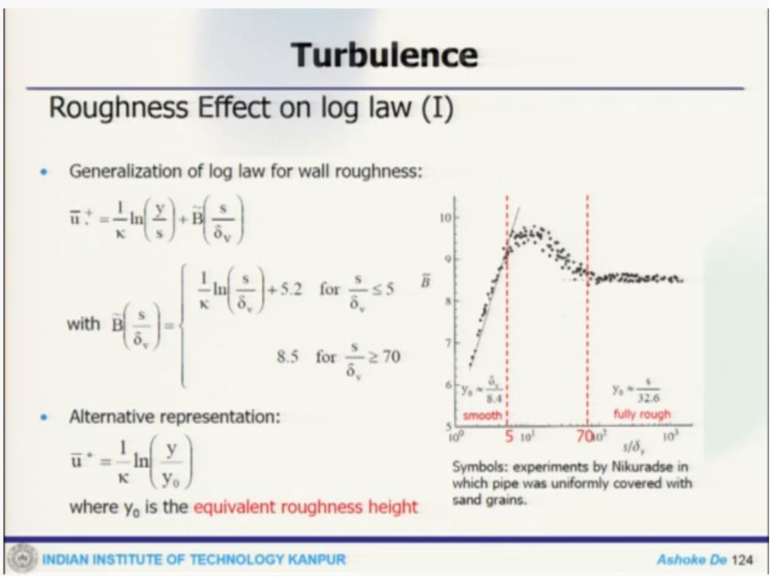

Or in the other way one can represent that u prime for u + is

$$\bar{u}^+ = \frac{1}{\kappa} ln\left(\frac{y}{y_o}\right)$$

where y_o is the equivalent roughness height. Now one can interestingly look at this $\tilde{B}$ versus s/δ_v and that variation and there is an i data base, which is the experimental data one can see, and this is a smooth wall when s/δ_v is less than 5, and this is fully rough beyond 70. And in between, this goes through. So this is from smooth to rough. So, in between the region, slightly mixed, y_o the mix situation one can expect.

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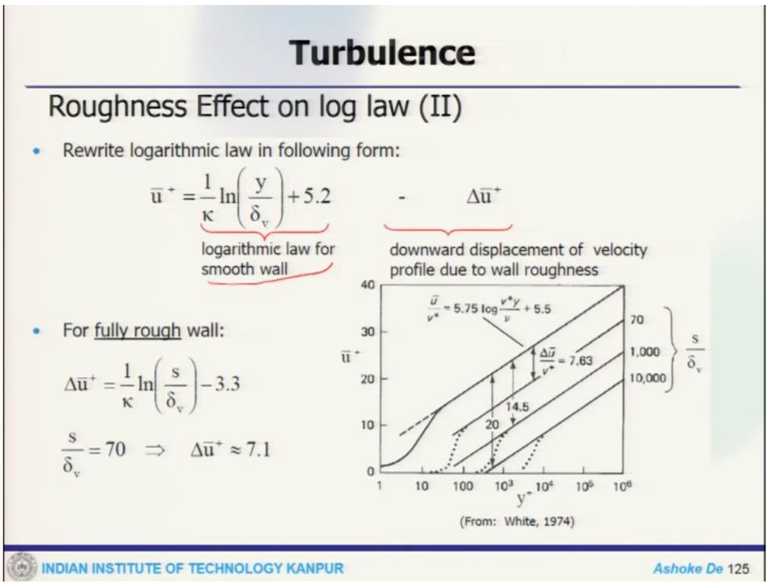

Now, second situation, one can rewrite the logarithm law in the following from,

$$\bar{u}^+ = \frac{1}{\kappa} ln\left(\frac{y}{\delta_v}\right) + 5.2$$

So, this will replace my logarithm law for smooth wall, so we invoke that term, okay. That is my smooth wall effect. And then this is the displacement of the velocity profile you to roughness. So it is essentially taking care of the smooth wall effect plus the variation due to roughness variation. So, for fully rough one this would become

$$\bar{u}^+ = \frac{1}{\kappa} ln\left(\frac{y}{\delta_v}\right) - 3.3$$

that is

$$\frac{s}{\delta_v} = 70$$

So, this will become

$$\Delta\bar{u}^+ \approx 7.1$$

So, this is the same thing, what one can plot for $\bar{u}^+$ by Y^+ and you can see the differences by delta v ratio and how the profile looks like. Now, also for this rough pipe, one can find out the friction factor. From log law, one can estimate the bulk and centreline velocity. So, we have

$$\frac{U_o - U_b}{u_\tau} \approx \frac{3}{2\kappa}$$

So, this one can try out. Now, we can estimate from the law the centreline velocity will be

$$\frac{U_o}{u_\tau} \approx \frac{1}{\kappa} ln\left(\frac{R}{s}\right) + 8.5$$

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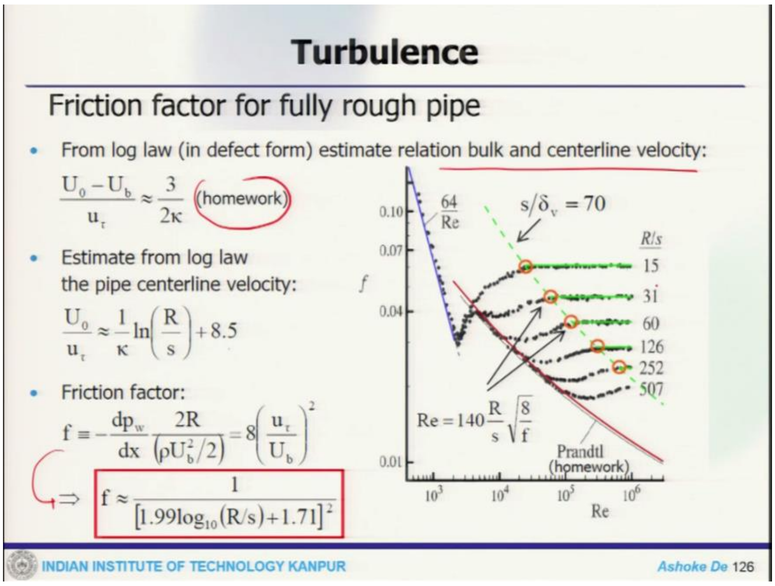

and the friction factor is equivalent to

$$f = -\frac{dp_x}{dx}\frac{2R}{\left(\frac{\rho U_b^2}{2}\right)} = 8\left(\frac{u_\tau}{U_b}\right)^2$$

So, essentially which brings, if f would be of this order

$$f \approx \frac{1}{\left[1.99\log_{10}\left(\frac{R}{s}\right) + 1.71\right]^2}$$

and if you put this thing in the plot, which is a Re, then this is the 64/Re curve and then the other curve one can and this is for different is Y delta v and this is for this Y R/s. Now, if you go to turbulent boundary layer, so this is in flat plate flow becomes turbulent here.

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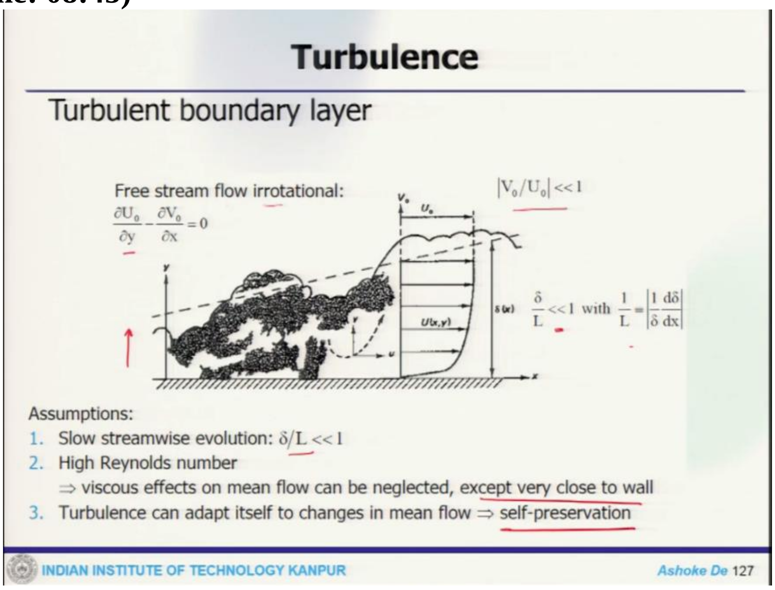

So, this is our boundary layer profile. Here obviously V_o/U_o is quite small that means that the various and industry moves direction and the feast employs the rotational so that will satisfy

$$\frac{\partial U_o}{\partial y} - \frac{\partial V_o}{\partial x} = 0$$

0 and also boundary ethics is δ/L less than 1. So 1/L will be scaled as

$$\frac{1}{L} = \left|\frac{1}{\delta}\frac{d\delta}{dx}\right|$$

Which are kind of associated with that, one is the slowest term was evolution that means δ/L is quite small that means, in this direction that growth is quite faster. Secondly, it is a high level summer case. So, viscous effects on mean flow can be neglected, except very close to wall, were the boundary layer and all these different and it can adapt split to changes in mean flow which is essentially the property of self-preservation. So, now,

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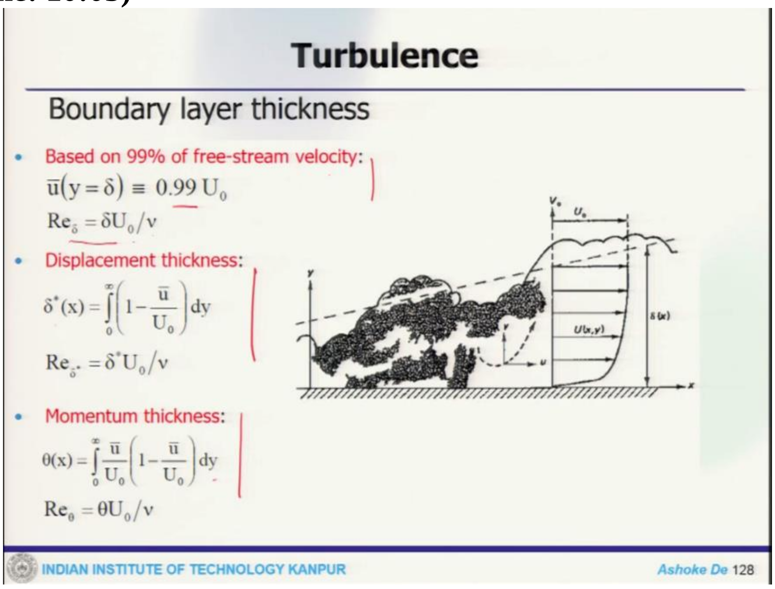

Boundary layer thickness based on 99 % of free-stream velocity we can define $\bar{u}$ at y equals to δ is

$$\bar{u}\ (y = \delta) \equiv 0.99\mathrm{U}_o$$

Re_δ would be

$$Re_\delta = \delta\mathrm{U}_o/\nu$$

Similarly, one can find out the displacement thickness which is

$$\delta^*(x) = \int_0^{\infty} \left(1 - \frac{\bar{u}}{U_o}\right) dy$$

and

$$Re_{\delta^*} = \delta^* U_o/\nu$$

and also the moment of thickness which is

$$\theta(x) = \int_0^{\infty} \frac{\bar{u}}{U_o}\left(1 - \frac{\bar{u}}{U_o}\right) dy$$

So, these are the momentum thickness, displacement thickness and free stream velocity these things are very standard and one can define those.

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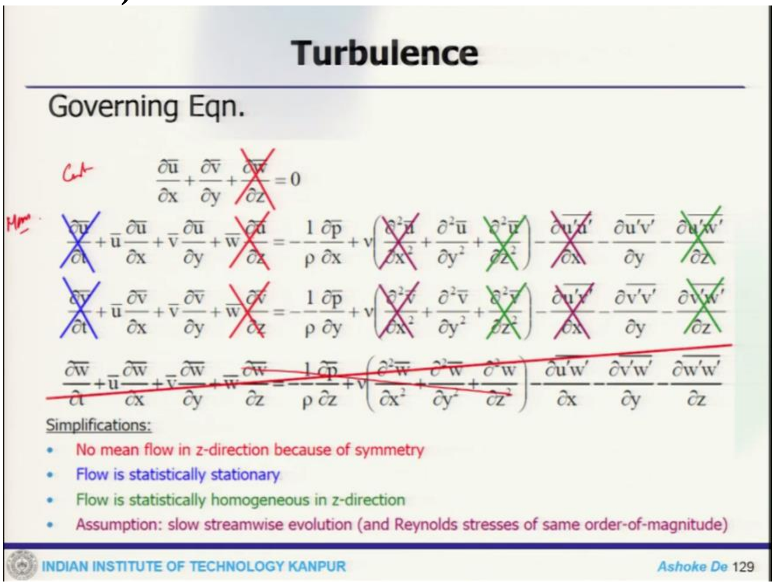

Now, how we simplified the governing equation so, we had continuity equation and momentum equation in 3 dimension in Cartesian coordinate system. So, no meaningful z-direction because of the symmetry. So, these terms goes off, flow statistically stationary. So, the stream wise goes up flow is statistically homogeneous in z-direction. So, get these terms are also going up and this also goes off, the complete gen momentum equation and also streetwise evolution is slow.

We can use the same magnitude analysis for and this term. And now from here we can get our boundary layer equation. So the boundary layer equation we get is that for free-stream region, We have

$$-\frac{1}{\rho}\frac{\partial p_o}{\partial x} \approx u_o \frac{dU_o}{dx}$$

since

$$|V_o/U_o| \ll 1$$

Now we can integrate y momentum equation from y to δ and use zero stress condition at the boundary layer top.

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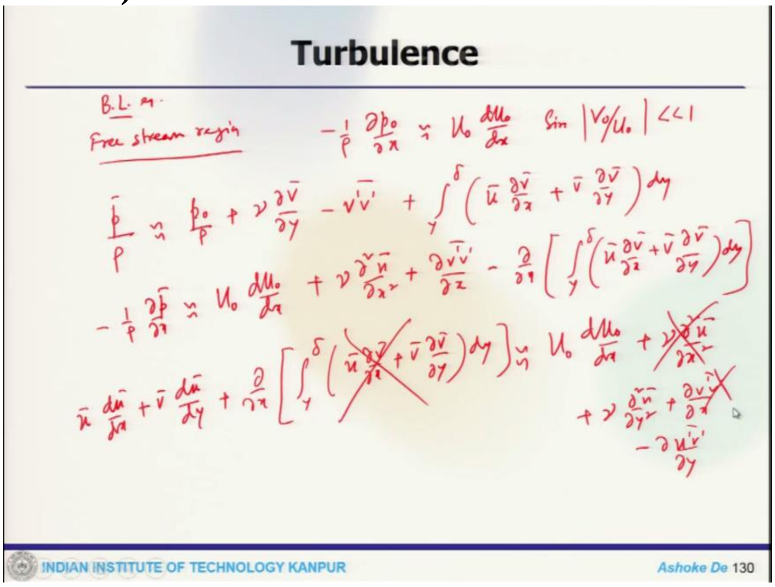

So, that will give us

$$\frac{\bar{p}}{\rho} \approx \frac{p_o}{\rho} + \nu\frac{\partial \bar{v}}{\partial y} - \overline{v'v'} + \int_y^\delta \left(\bar{u}\frac{\partial \bar{v}}{\partial x} + \bar{v}\,\frac{\partial \bar{v}}{\partial y}\right) dy$$

Now we can take the derivative to x and make use of the continuity equation + equation of the free stream. So, that will get us

$$-\frac{1}{\rho}\frac{\partial \bar{p}}{\partial x} \approx u_o\frac{dU_o}{dx} + \nu\frac{\partial^2 \bar{u}}{\partial x^2} + \frac{\partial^2 \overline{v'v'}}{\partial x^2} - \frac{\partial}{\partial y}\left[\int_y^\delta \left(\bar{u}\frac{\partial \bar{v}}{\partial x} + \bar{v}\,\frac{\partial \bar{v}}{\partial y}\right) dy\right]$$

Now we can substitute the pressure gradient into the x momentum equation and what we get from x momentum equation

$$\bar{u}\frac{d\bar{u}}{dx}+\bar{v}\frac{d\bar{v}}{dy}+\frac{\partial}{\partial x}\left[\int_{y}^{\delta}\left(\bar{u}\frac{\partial\bar{v}}{\partial x}+\bar{v}\frac{\partial\bar{v}}{\partial y}\right)dy\right]\approx u_o\frac{dU_o}{dx}+v\frac{\partial^2\bar{u}}{\partial x^2}+\frac{\partial^2 v'\overline{v'}}{\partial x^2}-\frac{\partial u'\overline{v'}}{\partial y}$$

So, there are small terms which is slow evolution of boundary layer or inner stasis of the same of the order. So, these goes off these also goes off and these goes off.

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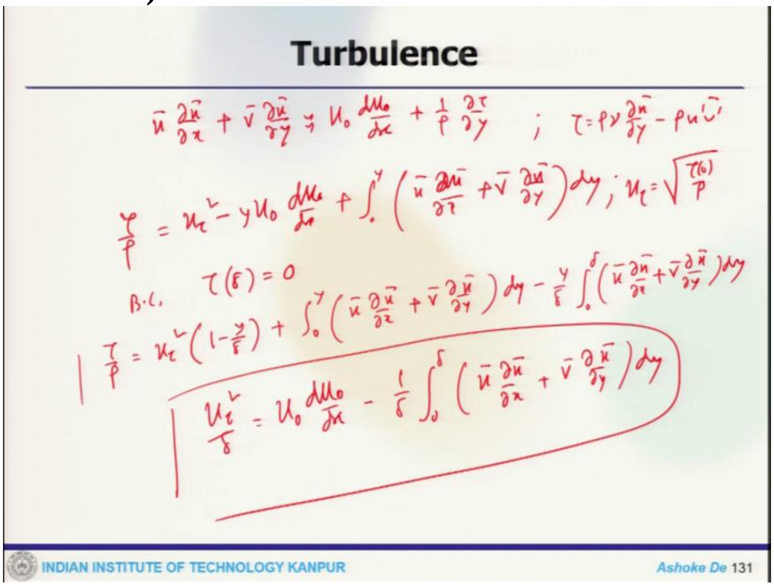

So, from the approximate momentum balance we get

$$\bar{u}\frac{\partial\bar{u}}{\partial x}+\bar{v}\frac{\partial\bar{u}}{\partial y}=\approx u_o\frac{dU_o}{dx}+\frac{1}{\rho}\frac{\partial\tau}{\partial y}$$

$$\tau=\rho v\frac{\partial\bar{u}}{\partial y}-\rho u'\overline{v'}$$

Now we can integrate these equation send from wall to Y. We guess

$$\frac{\tau}{\rho}=u_\tau^2-yu_o\frac{dU_o}{dx}+\int_0^y\left(\bar{u}\frac{\partial\bar{u}}{\partial x}+\bar{v}\frac{\partial\bar{u}}{\partial y}\right)dy;\ u_\tau=\sqrt{\frac{\tau(u)}{P}}$$

Now we use a boundary condition at the boundary layer top.

Where $\tau(\delta)$ is 0. If we put that, what we get is that

$$\frac{\tau}{\rho}=u_\tau^2\left(1-\frac{y}{\delta}\right)+\int_0^y\left(\bar{u}\frac{\partial\bar{u}}{\partial x}+\bar{u}\frac{\partial\bar{u}}{\partial y}\right)dy-\frac{y}{\delta}\int_0^\delta\left(\bar{u}\frac{\partial\bar{u}}{\partial x}+\bar{v}\frac{\partial\bar{u}}{\partial y}\right)dy$$

Where

$$\frac{u_\tau^2}{\delta} = u_o \frac{dU_o}{dx} - \frac{1}{\delta} \int_0^\delta \left(\bar{u} \frac{\partial \bar{u}}{\partial x} + \bar{v} \frac{\partial \bar{u}}{\partial y} \right) dy$$

So, this is what you get, for 0 pressure gradient boundary layer profile if we plot this is how you have this and this is the y by delta.

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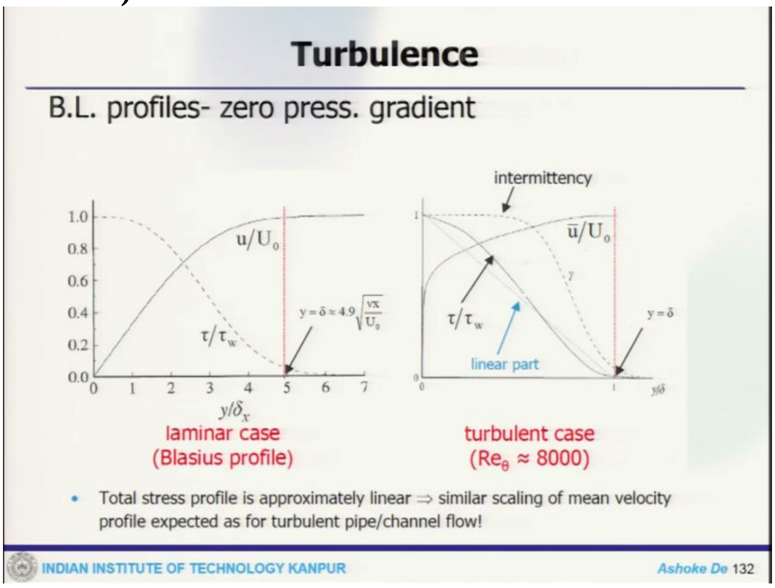

So, this is Blasius profile and turbulent case, this is the variation of the u/δ intermittency τ/τ_w in blue and that is the variation. So, total stress profile is approximately linear. So, one can think about similar scaling of mean velocity profile as for turbulent pipe and now, if we go to the law of the wall, so, there are multiple experiments and simulations have been done to find out or confirm this law of the wall, which is inner layer with y/δ less than point one.

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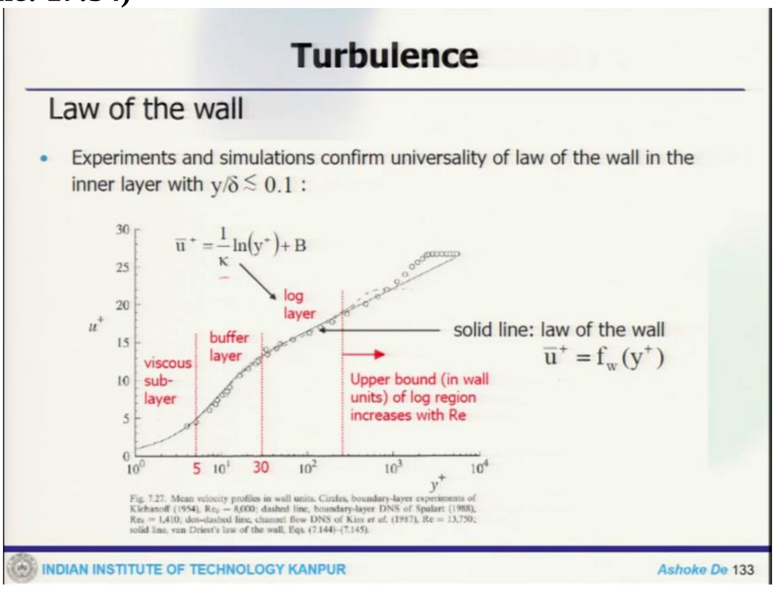

So, this is a plot for u^+ versus y^+. So law of the values u^+ is a function of y^+. So, this particular zone where y^+ less than somewhere 5 it is a viscous sub-layer then 5 to 30 is the buffer layer and 32 somewhere above hundred is log layer, where it follows this log profile and this is the upper bound in one limits of log region with increase in x. Now, if we look at different layers now, inner layer, what do have mean velocity profile, total shear stress τ/ρ is

$$\frac{\tau}{\rho} = \nu\frac{\partial \bar{u}}{\partial y} - u'\overline{v'}$$

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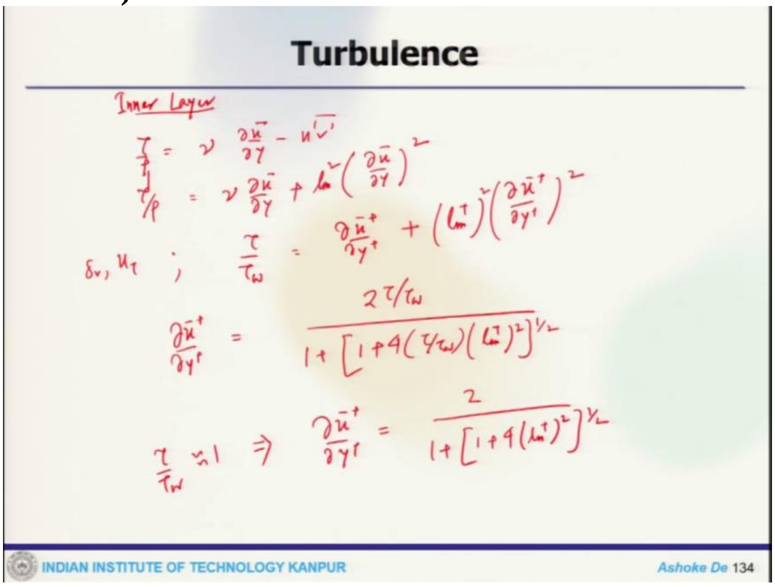

Now, we can use prandtl mixing hypothesis to get

$$\frac{\tau}{\rho} = \nu\frac{\partial \bar{u}}{\partial y} + l_m^2\left(\frac{\partial \bar{u}}{\partial y}\right)^2$$

Now we can normalize this one by δ_v and u_τ so that we get

$$\frac{\tau}{\tau_w} = \frac{\partial \bar{u}^+}{\partial y^+} + (l_m^+)\left(\frac{\partial \bar{u}^+}{\partial y^+}\right)^2$$

Now, once we solve for the velocity gradient, what we get this

$$\frac{\partial \bar{u}^+}{\partial y^+} = \frac{\frac{2\tau}{\tau_w}}{1 + \left[1 + 4\left(\frac{\tau}{\tau_w}\right)(l_m^2)^2\right]^{1/2}}$$

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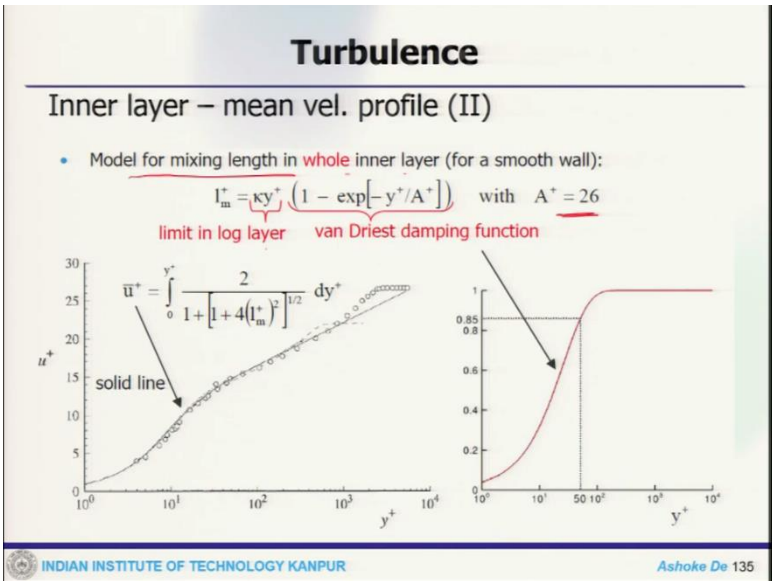

The inner layer $\frac{\tau}{\tau_w}$ is order of 1 which gets us back that it should be

$$\frac{\partial \bar{u}^+}{\partial y^+} = \frac{2}{1 + [1 + 4(l_m^+)^2]^{1/2}}$$

So, if we look at the inner layer mean velocity profile so model for mixing length in whole inner layer for smooth wall. This is

$$l_m^+ = ky^+[1 - \exp\left(-\frac{y^+}{A^+}\right)]$$

So this is the Van Driest damping function and that follows like this, which is y^+ and u^+ y^+ follows this line, where A^+ turns out to be 26. So there are evidences also with the measurements and simulations for this kind of profile, which are.

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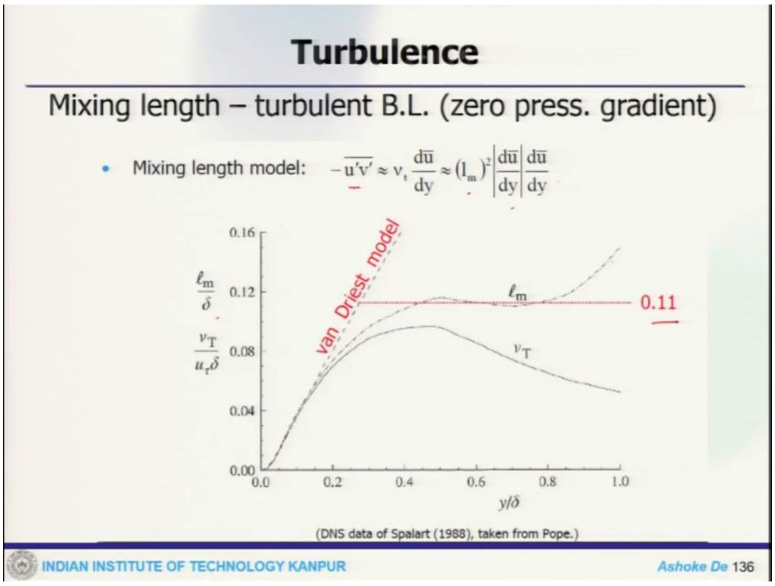

Now, if you look at the mixing length boundary layer, which is a zero pressure gradient case, then the mixing length model gives you that is inner stress the order of ν_T and velocity gradient which one can write l_m^2 and magnitude of the velocity gradient to the velocity gradient. And if you plot those things, this is how l_m is varying. This is how ν_T is varying with y/δ. And this is the van driest model and for 0.11. So, one can recreate these things very nicely.

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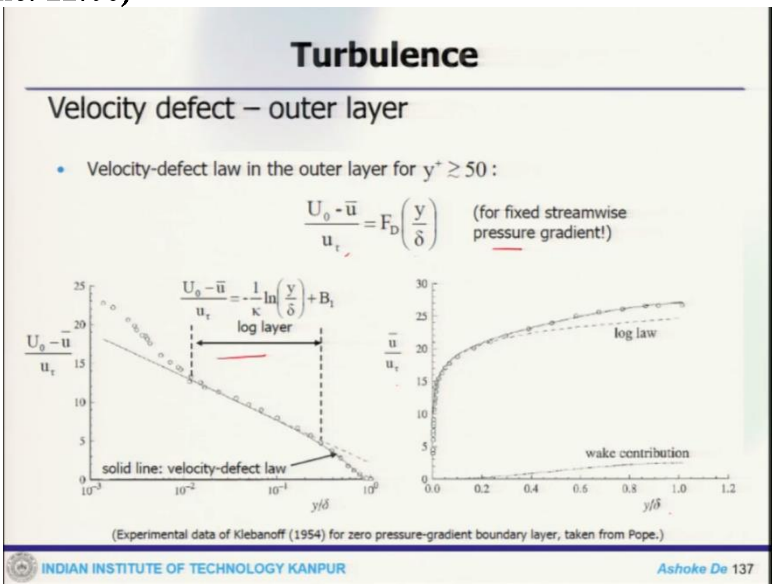

Now, there will be velocity defected the outer layer also. Now, when y * > 50. We can have an velocity defect which is

$$\frac{U_o - \bar{u}}{u_\tau} = F_D\left(\frac{y}{\delta}\right)$$

So, this is for n fixed stream is pressure gradient. And you can see when we plot these velocity defect with y/δ, this is the line it follows and this is the log layer. And if you only look at the normalized velocity profile mean velocity profile y/δ, this is how it follows and there is a contribution comes from the wake.

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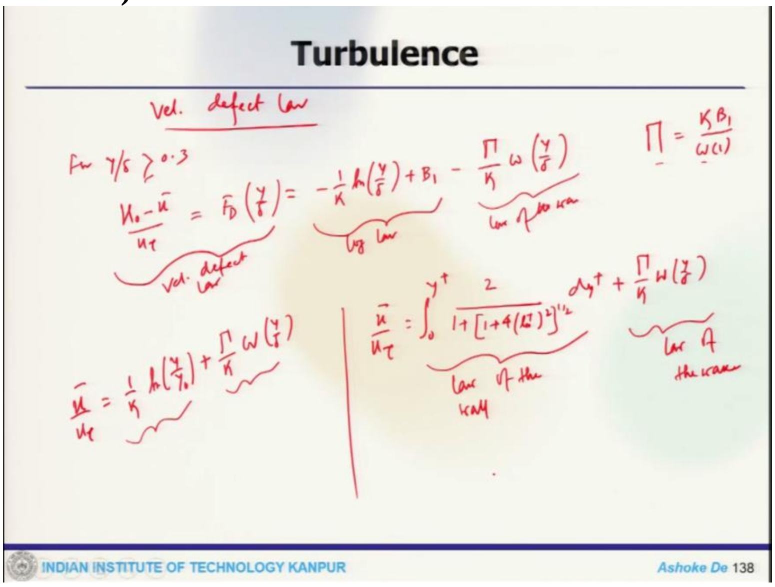

Now, this is velocity defect law, we can find out now let us say for y/δ greater than 0.3 the log law deviates from the velocity defect law. The difference between 2 laws is the law of the quick. So, what one can write is that

$$\frac{U_o - \bar{u}}{u_\tau} = F_D\left(\frac{y}{\delta}\right) = -\frac{1}{\kappa} l_m \left(\frac{y}{\delta}\right)$$

So, this is our log law outer scaling. So, this is velocity defect law. This is log law and this is the law of the wake where this is

$$\Pi = \frac{\kappa B_1}{\omega(1)}$$

here ω is the wake function and this we extend parameter or calls parameter. So, you can have an alternative formulation which looks like

$$\frac{\bar{u}}{u_\tau} = \frac{1}{\kappa} l_m \left(\frac{y}{y_o}\right) + \frac{\Pi}{\kappa} \omega \left(\frac{y}{\delta}\right)$$

U So, this is again log law of the inner scaling and this is the log the wake.

So, for complete smooth wall, velocity expression in boundary layer would be

$$\frac{\bar{u}}{u_\tau} = \int_0^{y^+} \frac{2}{1 + [1 + 4(l_m^+)^2]^{1/2}} dy^+ + \frac{\Pi}{\kappa}\omega\left(\frac{y}{\delta}\right)$$

So, this gives you law of the wall with band lyst model this is law of the wake. So, this is law of the wake and this is law of the wall. So, then within the boundary layer you get complete profile. Now, you can find out the wake function.

So, the law of the wake is universal function of y/δ which is consistent with velocity defect law. And w(y/δ) would be

$$w\left(\frac{y}{\delta}\right) \approx 2\sin^2\left(\frac{\pi}{2}\frac{y}{\delta}\right)$$

Now, we can define the wake strength parameter. So it depends on essentially depends on streamers pressure radiant.

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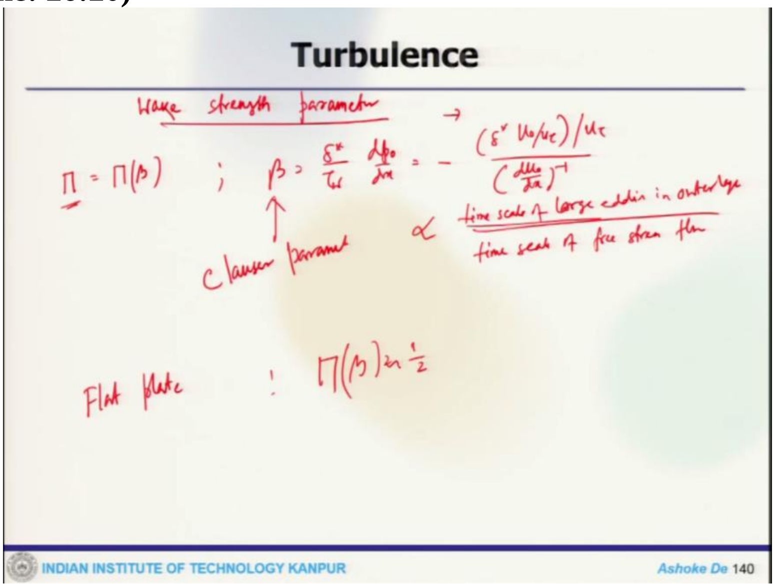

So you can write

$$\Pi = \Pi(\beta);\ \beta = \frac{\delta^*}{\tau_w}\frac{dp_o}{dx} = -\frac{\left(\frac{\delta^* U_o}{u_\tau}\right)/u_\tau}{\left(\frac{dU_o}{dx}\right)^{-1}}$$

which you can think about the time scale how large it is in outer layer this is time scale of free stream flow. So, this is known as clauses parameter. Now, one important thing to note here that turbulent boundary layers are self-preserving. When this pi is constant in x since in this case the large eddies is in the outer layer can adapt to changes in the free stream flow.

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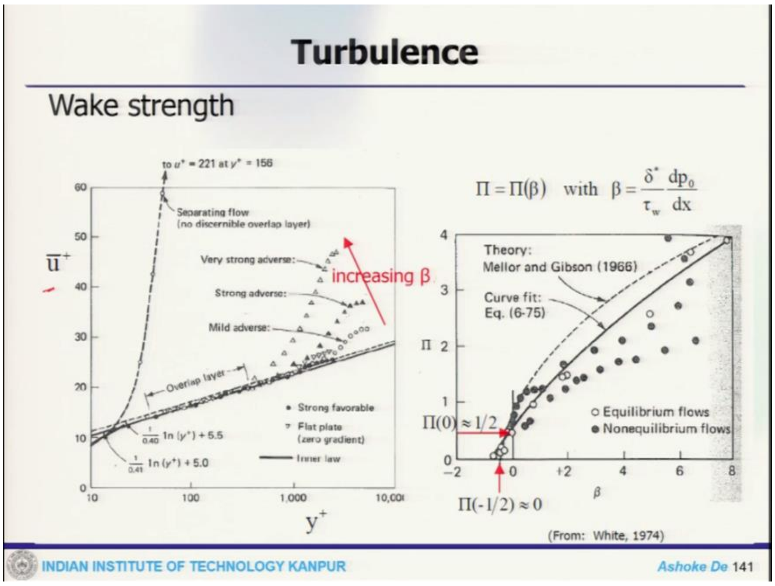

Now, if you look at for flat plate which is a 0% radian in steamers direction this will be almost hub. So, again one can put these things and look at the wake strength. This is your U + masters y + and this is increasing beta mild average strong average very diverse. So, depending on this and this is the separating flow. So, you can actually define this and this is how pi vs between of varies.

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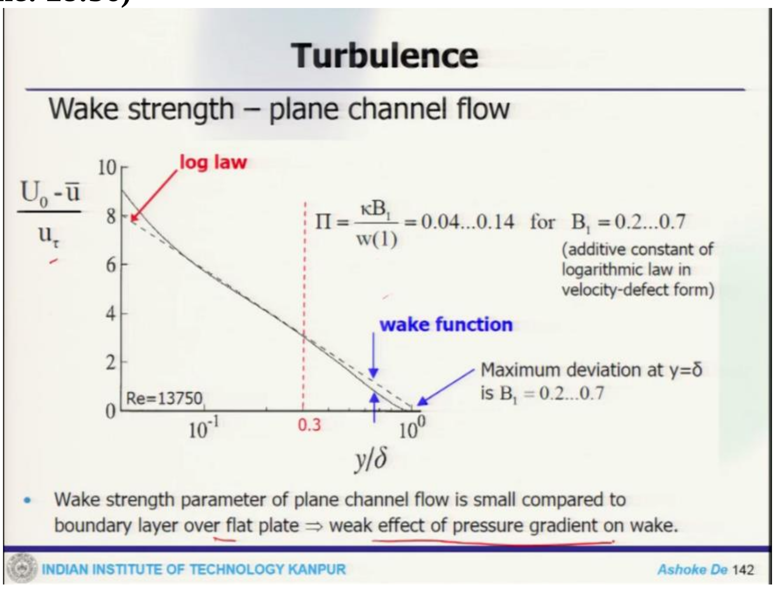

So, you can actually define this wake strength and if you look at for plain channel flow, this is the velocity defect with y/δ and this is our maximum deviation at y/δ and this is the plot for pi which is a different values. So, extend parameter of plain channel small compared to boundary layer or flat plate. So weak effect of pressure gradient on wake. So we'll stop here today and continue this discussion in the next lecture.