Muffler Acoustics - Application to Automotive Exhaust Noise Control

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Lecture - 02
D' Alembert's Solution and 1-D Continuity Equation

Welcome back to the second lecture on the Muffler Acoustics course. So, we left at the following equation,

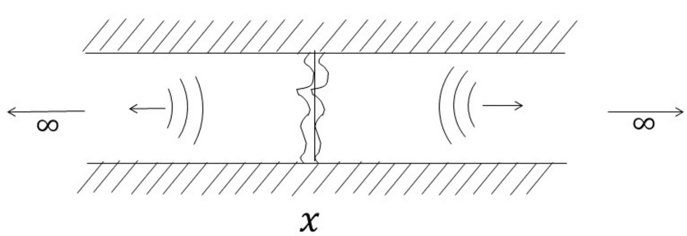
$$\frac{\partial^2 \tilde{p}}{\partial x^2} - \frac{1}{c_0^2}\frac{\partial^2 \tilde{p}}{\partial t^2} = 0$$

One Dimensional Acoustic Wave Propagation Equation. And what we saw was that, $\tilde{p}$ which is your acoustic pressure as a function of space and time is expressed in the following form.

And $$\tilde{p}(x,t) = f(x - c_0^2 t) + g(n + c_{0t})$$

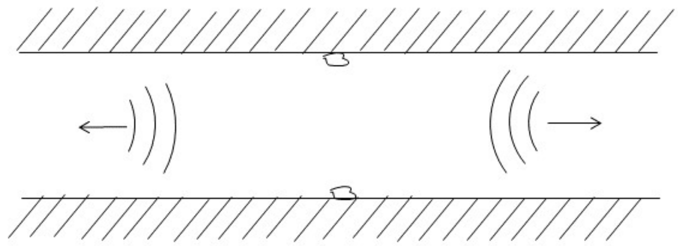
Where *f* and *g* are general functions continuous twice differentiable and they could be like I said sinusoidal, complex exponential, algebraic forms anything which is a continuous function, at least twice differentiable. This term represents the forward propagating wave, this term represents the backward propagating wave or towards the negative x-direction.

So, in the last lecture, we talked about deriving certain solutions for the equations when they are subjective to certain initial conditions. So, let us see, how do we go about doing that.



So, let us say you have a duct, you have a duct and there is something like a pulse, some disturbance; maybe you can consider a membrane, a membrane kind of a thing here inflated membrane or something.

And this goes in the infinity in the positive *x*-direction, now also towards the negative *x*-direction. And say rigid wall duct, only planar waves can propagate; that is wave fronts that propagate along this side and along this side. So, the idea is that when the membrane is completely ruptured, you know let me go to the next slide.



Your duct system is here, and the membrane is ruptured now, which completely you put a pin or something and it produces a noise tuck something like that. And so, it produces a wave, actually two waves are a pair of waves; one that propagates like I said in this direction, and the other one in the negative *x*-direction, ok. So, the membrane is gone, this disturbance which propagates.

The reason that I am discussing about this problem or this situation, because this is again modeled by the wave propagation equation and it has two waves: the one that goes towards the positive *x*-direction, one that goes towards the negative *x*-direction.

And importantly let us say, just at

$$\left.\begin{array}{ll} t = 0, p\,(x,0) = \phi(x)(1) & \\ \dfrac{\partial \tilde{p}}{\partial t} = pt\,(x,0) = 0 & (2) \end{array}\right] INITIAL\ Condition$$

General solution looks like $\phi(x)$. Now, the time wise derivative let us. So, I will call this equation (1), equation (2); these are initial conditions, alright.

So, the idea is that how to incorporate these initial conditions in the final solution. Now, that we know that we have solution given by this equation here; how do we enforce the initial condition given by (1) and (2) into the solution that I show? So, first we will substitute this.

$$f(x) + g\,(x) = \phi\,(x) = p\,(x,0)$$

It, follows by substituting t = 0 in here and what we get is this. So, $p\ (x, 0)$ is this thing.

Now, what about the time derivative? The time derivative if you recall the last lecture, we need to take the derivative respect to the variable; that is here of course it means, with respect to time.

But applying the chain rules, we can go about evaluating the derivative; in any case it works out in the following fashion. When we implement this the second condition in this equation.

$$\frac{\partial \tilde{p}}{\partial x} = f'(x - c_0 t)|_{t=0} + g'(x + c_0 t)|_{t=0} = 0$$

So, what we had done effectively. So, this is 0. Because we are enforcing this condition where I am pointing towards, right.

So, when we do this, we really mean for that thing. So, we get

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial (x - c_0 t)} \frac{\partial (x - c_0 t)}{\partial t} = -\frac{c_0 \partial}{\partial (x - c_0 t}$$

So, basically going back to this equation, we figure out that.

$$-c_0\, f'(x) + \ c_0 g'(x) = \ 0$$

And here of course, there will be c_0 the students can derive this form, it is very simple, this thing is equal to 0. So, eventually what we get is that, at c naught cancels out. So, we get something like - *f'*(*x)*. So, this is still with respect to that derivative.

$$-f'(x) + g'(x) = 0 \qquad (3)$$

Now, if we integrate equation (3); what we get.

$$-f\ \ (x) + g\ \ (x) = C \qquad (4)$$

some arbitrary constant of integration which is say C.

So, we will call this equation (4) and then go about doing some algebraic manipulations. Now, one thing is there; if you look at this equation, where we got maybe I should have numbered this is (2b) suppose. So, when we combine (2b) and equation (4); what do we get?

$$f(x) + g(x) = \phi$$

$$-f(x) + g(x) = C$$

Now, we just need to add these two things, and simplify to get the following.

$$g(x) = \frac{1}{2}\phi(x) + \frac{C}{2}(5a)$$

$$f(x) = \frac{\phi(x)}{2} - \frac{C}{2}(5b)$$

And then when you subtract say this equation from this one; what we get is the following. So, we have got a function. So, let us call this equation (5a), (5b). So, now, this is right for

$$t = 0$$

So, the total solution or the complete solution at this thing will be what,

$$\tilde{p}(x,0) = \left(\frac{\phi(x)}{2} + \frac{c}{2}\right) + \left(\frac{\phi(x)}{2} - \frac{c}{2}\right)$$

But now we have to find out the solution for a generic time *t*. So, what do we do? We need to translate this thing in terms of $x - c_0t$ and this one will be $x + c_0$. So, for a generic time t, we will get things like this.

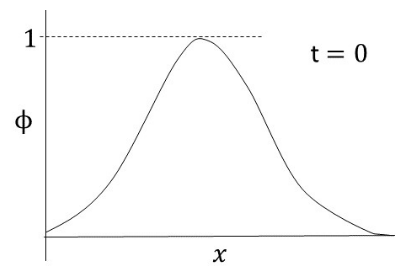
$$\tilde{p}(x,t) = \frac{1}{2}\phi(x - c_0t) + \frac{c}{2}$$

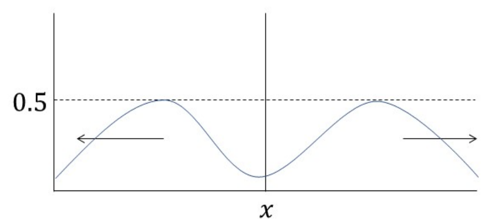
$$+\frac{1}{2}\,\phi\,(x + c_0t) - \frac{c}{2}$$

Now, these two terms will cancel and our complete solution,

$$\tilde{p}(x,t) = \frac{1}{2}\{\phi(x - c_0 t) + \phi\,(x + c_0 t)\} \qquad (6)$$

Let us call this as equation (6). So, then what does it mean? Let us go back to our figure here, when the membrane has completely busted, and two waves are set out in opposite direction. So, let us say based on this condition, what we if we draw the profile x and this is your disturbance ϕ. So, here it is and then based on the solution; if we put $t = 0$, we again get back ϕ.





So, this is self-consistent, but as the time propagates. So, right now we have x, we have ϕ, we have the general Gaussian pulse say at $t = 0$. And so, the wave splits apart. So, it comes down becomes smaller at certain time and then as the time evolves, it starts splitting into two parts; one this pulse propagates in the positive direction, this pulse propagates in negative direction.

So, if this is of amplitude 1 say unity; so, this will be 0.5 that is pretty easy to see from this slide, because you see here the wave actually has split into two parts. And you have basically, we have two waves each of equal amplitude, which is half the amplitude the original disturbance $\phi(x)$ is just that they are propagating in opposite direction.

And the result what happens for free waves that propagate in a duct in opposite directions is basically; under these conditions when you rupture a membrane in a duct and you have no boundary conditions that is nothing is coming back from here, nothing is coming back from here, only you are allowing only unidirectional propagation, no reflections allowed.

Then you basically see the pulse being freely allowed to exit the domain into two equal halves propagating on opposite sides. So, this was for the general case when you have a general waveform, but 0 initial conditions. When we have an initial condition of course, then of course we get what is called a D'Alembert solution.

So, I will probably not go into too much detail in that, rather I will just note down the **initial conditions.**

$$\tilde{p}(x,0) = \phi(x) \qquad (7a)$$

$$\tilde{p}t\ (x,0) = Q(x) \qquad (7b)$$

It is what? Instead of 0, et us say, there is some function Q(x). So, these are our conditions. So, let us name this as, number this has (7a), (7b).

So, the idea is that, if the initial condition that is with respect to the time, derivative with respect to time if that is non-zero, Q(x) in general is nonzero; then what will happen? Is it still, how does the solution gets modified? So, here what we actually get is, I will write down the solution rather than working it out, which is left as a derivation for the student's.

$$\tilde{p}(x,t) = \frac{1}{2}\{\phi(x - c_0 t) + \phi(x + c_0 t)\}$$

And now what is the other term to incorporate this non-zero condition? So, to incorporate this Q(x) condition, that is non-zero initial condition when you take the derivative of the pressure field at temporal derivative of the pressure field.

we get is the following.

$$\tilde{p}(x,t) = \frac{1}{2}\{\phi(x - c_0 t) + \phi(x + c_0 t)\} + \frac{1}{c_0}\int_{x-c_0 t}^{x+c_0 t} Q(y)dy \qquad (8)$$

Maybe you can have a different choice of integrating variable,

So, clearly if Q is 0, you get back the original solution; if Q is nonzero, then what we probably have to do is that integrate find out hopefully a closed sum solution of this particular integral, put in the limits in terms of these variables. The physical significance is that we no longer have even for a stationary medium; the medium is not moving note that. In the duct that we had seen here; there is no flow, it is just an empty duct in which air is filled, a metal duct made of steel brass whatever.

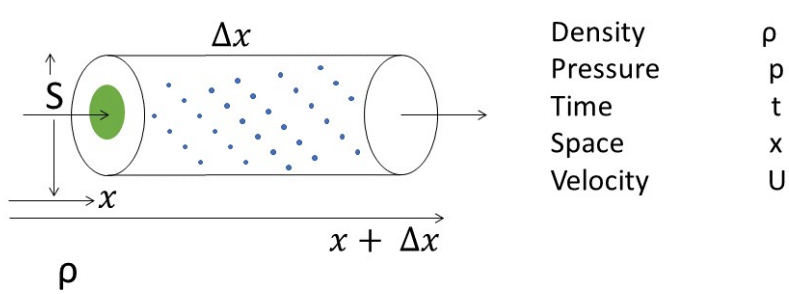
And you just rupturing a membrane. So, if you have a 0 initial conditions with respect to the temporal derivative; the resulting wave that you see here given by this solution let us say (8), you no longer have symmetrical waves, you havesome kind of a bias. So, the waves will no longer have will propagate with equal amplitude in both directions; they might be different depending upon how Q(y) is.

So, students can take up that problem; but the essence of this derivation of this example was that, to show how free waves exist in a duct. And note that there are two solutions, which you must carefully note; because you know this equation is a second order differential equation, second order hyperbolic equation. So, it must have two solutions, two linearly independent solutions:The forward and the backward travelling wave.

We will see in due course, in the next few lectures or so that, when we implement certain boundary conditions, especially when we enforce what is called the time harmonicity; we can get the wave equation become Helmholtz equation in the frequency domain and then we can define your progressive wave, your standing waves, force solution and all those kinds of things.

So, we will take up those topics in the next few lectures; but for now, what is important is to get to the derivation of how we arrived at the second order differential equation, which governs wave propagation that is important.

So, let us begin the derivation of the one-dimensional acoustic wave equation that we had seen in the last lectures, a detailed derivation of that, because this will form the basis of the muffler acoustics basis largely for the 1D analysis of mufflers.



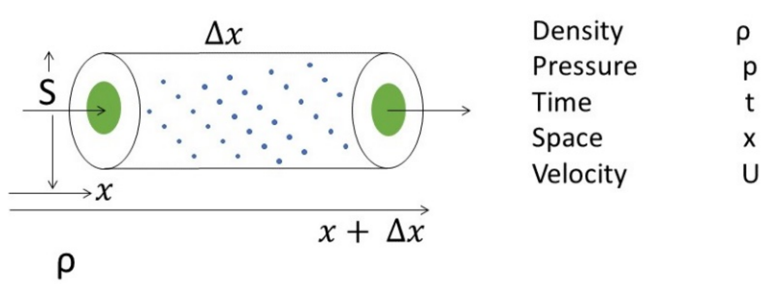
So, let us consider in a fluid element, a small element, let us say a cylindrical element something like this and the mass. Let us say we have certain mass that goes in this phase, which is x; let us say this is x and this is a small element x, Δx. So, this distance is $x+\Delta x$. And the density let us clarify a few things, density of the fluid.

Let us say this is air is ρ and pressure is p. So, this p is different from the acoustic pressure which will name separately, and we will have a slightly different convention in terms of $\tilde{p}$, we will get to that. So, p is the total pressure and time of course is donated by t space, is a 1D space, so it is x.

And other than that, we have velocity, velocity of the fluid particles that is U; let us say this is U. So, one thing before we begin with the influx and out flux things that I am going to talk about; we are assuming here that the air particles here which are there everywhere, you know it is a; it is a continuum.

Although they comprised of molecules; but we are considering the entire fluid element to be a part of a continuum. So, that is why all these equations that we are going to present now will apply. So, the flux that enters this phase is given by ρ and let us say S is the cross-sectional area. So, what is the total mass of the fluid that enters?

So, before I begin this derivation, I must mention here that, when we derive the wave equation; we first need to get to the continuity equation and then the momentum equation. And in addition to that, we have equation for state isentropic equation. So, all these equations they will be derived from the first principles and then linearized; because we are doing linear acoustics in this course and then combined suitably and finally, we will arrive at that.



Continuity equation.

So, the detailed derivation begins from what is meant, what is known as the continuity equation. So, what does the continuity equation do; is an expression for the conservation of mass, so whatever mass goes inside the phase say 1 and whatever mass that comes out or the phase 2 and what is the relation between the mass that comes goes into the system from phase 1, the one that goes out the phase 2.

What happens to the changes within this volume; that is within the entire volume, what are the temporal changes? So, we are going to see that the detailed derivation in this slide. So, I need to probably go back and forth.

So, the mass that enters this phase 1, then is given by

$$(\rho\, SU)_x - (\rho\, SV)_{x+\Delta x} = \rho SU$$

with the notation x and one that leaves the phase 2 is minus. Because we are trying to understand how much mass is left in the system at time. So, if you have this amount of mass that enters within the system the flux and the one that goes out of the system. So, this is the net mass that is left inside the system, right. And what is that? There must be some time change of mass, is not it? Here you have density and surface area and then you have velocity.

So, the mass that enters the phase 1 is given by$(\rho\, SU)|_x$, and the one that leaves the phase at 2 is also.

$$(\rho\, SU)|_x - (\rho\, SU)|_{x+\Delta x} = (\rho S \Delta x)_t$$

$$Inflex \qquad outflex$$

And so, this is the total, this is the mass, which is influx, this is out flux. And what is that net, what happens to the difference? So, this is related to the temporal change within this volume; that is the orangish part that I have shown the entire cylindrical volume.

So, what is the volume of that? S into Δx, and ρ; if you multiply that by the density ρ, that will give you the mass. So, temporal derivative of that is balance to the net influx and out flux. So, this note that these changes are dynamic, they are happening both in space and time; there is constant influx of the mass from one side, constant out flux and the difference of that will give you the temporal change of mass within that element.

So, this is how we arrive at this one. So, now, let us simplify this equation and s is of course, we are assuming here a constant or a uniform cross section area. And so, what we do essentially

$$\frac{(\rho U)_x - (\rho U)_{x+\Delta x}}{\Delta x} = \frac{\partial \rho}{\partial t}$$

we are doing temporal derivative.

So, in the limit that Δx is tending to 0, infinitesimal element; that is this element is getting smaller, Δx is smaller. So, what do we get? What we get? Let me write this in this manner, this can be rewritten.

$$\frac{\partial \rho}{\partial t} = -\left\{\frac{(\rho U)_x - (\rho V)_{x+\Delta x}}{\Delta x}\right\}$$

And then, so this is your continuity equation; it basically relates the changes in density within the control volume, which is this one, and with the velocity of the fluid and with time and space.

$$\frac{\partial p}{\partial t} = -\frac{\partial(\rho U)}{\partial x}$$

$$\frac{\partial p}{\partial t} + \frac{\partial(\rho U)}{\partial x} = 0 \qquad (1)$$

So, it basically tells you that, Δt of ρ; that is temporal derivative of the density plus the spatial derivative of the quantity (ρU) that is 0. Here of course, I must mention that in the control volume here, we are assuming that there is no sources; there is nothing that is pumping fluid, so that is why you have put this thing 0. So, that is how you basically got this thing; otherwise suppose if there was a source, you would have put a term Q or something like that.

With the same units as this one, which is your kg per second; but of course, we do not have that, so I strike of this part. And what we will probably go back to this one. So, before we end this lecture, I like to simplify equation (1) in the following form, which we will see will be useful in some other context.

$$\frac{\partial p}{\partial t} + U\frac{\partial p}{\partial x} + \rho\frac{\partial U}{\partial x} \qquad (2)$$

So, this equation (1), dimensional equation for continuity equation which is really an expression for conservation of mass; you also seen lot of fluid mechanic's text. So, we will stop here and then take on the momentum equation in the next lecture. Thanks.