Muffler Acoustics - Application to Automotive Exhaust Noise Control

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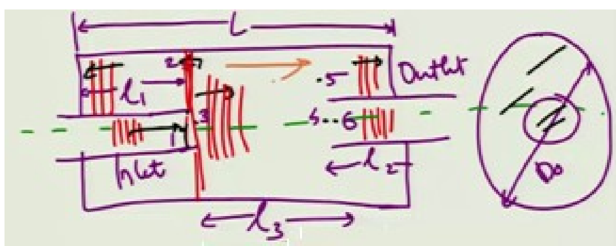
Lecture - 23
Extended - Inlet and Extended - Outlet Muffler Analysis (Continued)

Welcome back. This is lecture 3 of week 5 of our NPTEL course on Muffler Acoustics.

So, as we promised that we are going to analyze, look at the same system extended inlet and extended outlet muffler in which the ports are concentric and the section can be anything; circular section, elliptical section or things like that because we are really assuming a planar wave propagation. So, we are going to look at the transfer matrices between the points 1 and 6. So, the system is now well-known to us. We have discussed it at length in the last lecture, lecture 2.



So, in this lecture 3, our aim would be to basically derive transfer matrices or transfer matrix for the overall extended inlet and extend outlet system and then, use the derived transfer matrix between the inlet and outlet ports to obtain the transmission loss performance based on the planar wave propagation theory. So, what we will do? In this lecture, after deriving the overall transfer matrix, which I can tell you from now is going to be a bit algebraically tedious; unlike the simple expansion chamber because you have two quarter wave resonators, which are you know which are acting together.

And so, the overall transfer matrix would be the terms would be a little bit algebraically tedious to evaluate. So, we will try to take we will try to do as much simplification as possible here and then, go to a symbolic package perhaps maple or something like that.

And for the transmission loss parametric studies, it is very interesting to see how this simple tuning can actually help us to increase the overall performance. For doing that, for you know getting a feel for that, we will probably have to go to MATLAB.

So, probably the later parts of this lecture, we will do some hands on or some practical examples by taking a realistic lens and cross-dimensions of the diameter D and things like that. So, we will do all these things, but before we do that, let us derive the transfer matrix.

So, we will consider the system point 1 is just in on the upstream of the inlet pipe and you know it does not really matter. Just one thing I want to tell you here that the pipe the inlet pipe, inlet and outlet they are of equal diameters. So, there is no difference between the diameters and the uniform across the length that is a simple pipe, cylindrical pipe.

So, the transmission loss of the pipe element itself is 0, because I mean that we can easily see from the expressions, analytical expressions expression for the transmission loss of a simple expansion chamber. If you put m is equal to 1, that is there is no expansion we will actually get t_l is log of 1, that is 0.

So, this reason why we choosing 1 here; just at the interface and 6 here. We could have derived the transfer matrix between this point, somewhere this point and point here. But it does not really matter as far as the if there is nothing upstream or downstream, no muffler element. If you just able to want to derive the transmission loss of between of this element, between this point 1 and 6 will be the same as the transmission loss between this point, let us me call it star and this point.

So, there will be no issues; there will be no issues ok. So, this is what we are going to get, ok. So, we are going to get this overall system. Now, what are the relations that occur between point 1, 3, 2, 4, 5, 6? What exactly goes on? So, basically what goes on is that pressure at 1 is equal to pressure at 3 and is equal to the pressure at 2.

$$\begin{aligned} \tilde{p}_1 &= \tilde{p}_3 = \tilde{p}_2 \\ \tilde{p}_1 &= \tilde{p}_2 = \tilde{p}_3 \end{aligned} \qquad (1)$$

$$\begin{aligned} \tilde{p}_4 &= \tilde{p}_6 = \tilde{p}_5 \\ \tilde{p}_4 &= \tilde{p}_5 = \tilde{p}_6 \end{aligned} \qquad (2)$$

So, this I should have written here at point 2. Why is that? Acoustic pressure at this point is equal to the total planar wave pressure and is equal to the pressure of the section 2. So, this is one of the assumptions of plane wave theory, that the wave comes it is expanding into a larger thing and over the entire cross section, the pressure is equal. So, basically over this cross section, over the cross section, where I am pointing and over this thing and everything is equal. So, it is a same phase, same magnitude, everything is the same.

So, whatever pressure value is there just at the interface between the inlet port and the chamber; but right within the chamber, the same pressure we are seeing within the chamber. So, that is there and the same pressure occurs in the interface at the annular cavity thing here.

So, basically what the resulting thing then is p_1 is equal to p_3 is equal to p_2; where, p_1 like I said is the acoustic pressure field in the inlet pipe. But at the interface or the inlet pipe in the chamber, p_3 is the planar wave acoustic pressure field within the chamber circular chamber or elliptical chamber or something like that, but not in the annular region and point 2, $\tilde{p}_2$ that is the acoustic pressure field in the annular cavity, right in the at the interface.

So, this is actually a beautiful analogy between acoustics and electrical circuits. So, we will soon discover that as well and try to draw some electrical equivalent acoustic circuits for this muffler configuration; obviously, by assuming planar wave theory and velocity V_1, the acoustic mass velocity that has to be conserved.

So, whatever pressure field that you are seeing here, V_1 whatever it enters, it gets split into two parts. We assume that based on the plane wave theory that V_1 is equal to V_3 plus V_2. So, basically what happens is that the velocity that enters the chamber here, it is basically split into two parts that is acoustic mass velocity that goes in the cavity here, it gets split here and then here.

So, some part of that is basically being taken away in the chamber and the some other portion of that goes within the annular cavity. So, as a result, what we can do is that we can split V_1 in two parts, V_2 and V_3 acoustic mass velocity that enters the annular cavity region, at the acoustic mass velocity that goes downstream.

And pressure p_4 is equal to pressure again at the outlet, we are I am talking about, the same planar wave propagation fronts are taken. p4 is equal to p_5 is equal to p_6. So, this is equal to p5 that is there; p_4 is equal to p_6 is equal to p_5 and volume velocity or mass velocity again gets split into two parts; V_4 is equal to V_5 plus V_6, V_4 gets split into two parts; V_5 and V_6 that happens. So, now, the question that I am asking is that what do we do with these equations? We eventually, have to worry about the variables with suffixes 1 and 6; is not it? Let us start the process of relating that.

So, p 1 if we write this in the transfer matrix form, we are relating the discontinuity. But now, we also have an extension. So, we have to worry about the impedance of the annular cavity also. So, let me write it down a little bit more cleanly perhaps.

$$-jY_{ann}cotk_0l_1$$

$$\tilde{V}_1 = \frac{\tilde{p}_2}{Z_{ann1}} + \tilde{V}_3$$

$$\tilde{V}_2 + \tilde{V}_3$$

$$\begin{Bmatrix}\tilde{p}_1\\ \tilde{V}_1\end{Bmatrix} = \begin{bmatrix}1 & 0\\ \dfrac{1}{Z_{ann}} & 1\end{bmatrix}\begin{Bmatrix}\tilde{p}_3\\ \tilde{V}_3\end{Bmatrix}$$

$$Z_{ann1} = \frac{\tilde{p}_2}{\tilde{V}_2}$$

Why? Because based on the above transfer matrix, and you know from this thing, impedance of the cavity that we are discussing at the entrance of the cavity or at the beginning of the cavity as we were discussing in the last lecture is given by this expression.

So, V_2 then of course, V_2 is p_2 by Z_{ann1}. So, this is just another way of writing. So, this is actually V_2; this is nothing but V_2 you know and so, that basically gives us the flexibility of writing this like this 1 by Z_{ann1}. So, we get this transfer matrix and another of course, we now, we basically need to worry about the things that happen from 3 to 4.

$$\begin{Bmatrix}\tilde{p}_3\\ \tilde{V}_3\end{Bmatrix} = \left[\begin{array}{c:c}cosk_0l_3 & jY_csink_0l_3\\ \hdashline \dfrac{j}{Y_c}sink_0l_3 & cosk_0l_3\end{array}\right]\begin{Bmatrix}\tilde{p}_4\\ \tilde{V}_4\end{Bmatrix} \qquad (2)$$

So, now let us go sequentially. Now, we will invoke the relation the transfer matrix relation between two points of a uniform pipe. So, what is that? That is p_6 is there, p_3 and 4; I am sorry sorry, not 6, 3 and 4. So, if we go with 4 here and V_4 is here, it is something like this.

So, this is equal to cos of actually one thing that probably I should have mentioned. This is l_1, this l_2. So, let me call this as l_3, the distance right from the interface of the inlet extended inlet and right to the interface of the extended outlet. So, l_3 obviously, that is simple $l_3 = L - (l_1 + l_2)$ know this is important.

So, once you fixed up l_1 and l_2, l_3 is obviously, once you determined l_1 and l_2, l_3 automatically is fixed. So, I would simply write this as cos of k_0 times l_3 times. So, this is the T_{11} parameter and this is j into y c. Remember y c was a chamber, c naught by cross section area of the chamber.

And by the way, this the thing that you are seeing here was nothing but minus well, I should I can probably write it here. Z_{ann1} or probably just Z_{ann} cot $k_0 l_1$ because this we saw in the last class. The entire principles of tuning or double tuning the muffler was based on that.

So, because we are trying to cancel out the two troughs in the last lecture; the first trough and the second trough by means of cleverly choosing the extensions, the length of the extensions at the inlet and the outlet. So, it is also called double tuning and these are l_1 and l_2 are general lengths. If they are not equal to l / 2 or l / 4 respectively, then the chamber are then the extended inlet and outlet chamber is not tuned or if you have just a tuning of the inlet and not of the outlet is called singly tuned muffler or single tuned muffler.

So, all this we can actually appreciate when we do some parametric studies shortly. But before that, let us quickly complete this part; So, recall the last week's lecture, when we derive the transfer matrices of the simple tube. So, there was a detailed derivation.

So, probably I guess is a good idea to like this. So, this is what it is. So, we have got let us say let us name this as 1, this relation this is 2, equation 2 and obviously, the relation between the variables at the outlet is the following. It is easy to guess this V6. So, this becomes p 6, this becomes V 6. So, what do we get actually?

$$\begin{Bmatrix} \tilde{p}_4 \\ \tilde{V}_4 \end{Bmatrix} = \begin{bmatrix} 1 & 0 \\ \dfrac{1}{Z_{ann}} & 1 \end{bmatrix} \begin{Bmatrix} \tilde{p}_6 \\ \tilde{V}_6 \end{Bmatrix} \qquad (3)$$

So, this is, this obviously, follows because you get your V_4 is equal to V_6. So, that is what you are going to get and p_6 by Z_{ann2} is nothing but p_5 by Z_{ann} and p by Z_{ann2} is nothing but V_5.

So, we are actually able to retrieve this relation or in other words, this relation the one that is underlined here that forms the basis for the occurrence of this term. So, what happens now is that let me get rid of this part and so, we have got three transfer matrix relation. This is let me call it 2 and this is 3 and here, you get relation 1, transfer matrix relation 1, 2 and 3.

Well, this course is designed in such a manner that the way that I carry out thing is that take some practical examples and introduce certain concepts using them. So, what is the concept that I am going to use now? Because remember we want to relate the points 1 and 6, that is relate the variables at point 1 with those at point 6.

So, now what we need to do is that sequentially multiply the transfer matrices. So, what is that called? That is called Cascading or sequential multiplication of transfer matrices. So, what happens now is that in order to relate point the 1, p_1 V_1 with p_6 V_6, we just need to simply multiply the wave.

Let me reiterate because we will be coming to cases, where the energy propagation, acoustic power propagation does not follow a unique path. There, could be multiple path. So, sequential or cascading of transfer matrices or sequentially multiplying transfer matrices will probably just that thing would not help you.

You need to do something else, we will worry about that much later. But for now, we can just understand that the energy propagates in this direction. So, what do we do now?.

Cascading or sequential multiplication of T matrix or transfer matrices ok, this we will do.

So, you see why is this happening? Why could we do all the algebra that we are trying to do here? Because obviously, unidirectional propagation of acoustic wave will allow you to relate the upstream variables with the downstream.

So, in a way we are trying to eliminate, very cleverly eliminate because you see these are eventually systems of linear equations. We have got 6 unknowns p_1 to p_6; V_1 to V_6 and we have got only 4 variables. So, we can express, obviously the remaining 4 or 4 of them in terms of the remaining 2. So, we could actually end-up solving that using some linear equations, trying to form Ax is equal to V system.

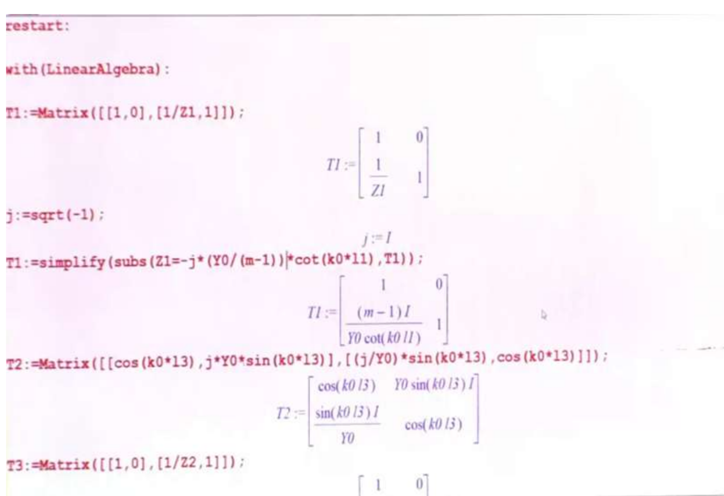
But clever technique is the cascading of transfer matrices which works well in this case and we just trying to relate p_3, V_3 we know. So, we just pre multiply the other matrix with this one and then, post multiply this matrix with this one to relate p_6. So, that is what happens, a little consideration will show you why this is there. So, this is what we get. Let me sort of write it down a bit more clearly;

$$\begin{Bmatrix} \tilde{p}_1 \\ \tilde{V}_1 \end{Bmatrix} = \begin{bmatrix} 1 & 0 \\ \dfrac{1}{Z_{ann1}} & 1 \end{bmatrix} \begin{bmatrix} cosk_0 l_3 & jY_3 sink_0 l_3 \\ \dfrac{j}{Y_3} sink_0 l_3 & cosk_0 l_3 \end{bmatrix}$$

$$Z_{ann1} = -jY_{ann} cotk_0 l_1 \qquad \begin{bmatrix} 1 & 0 \\ \dfrac{1}{Z_{ann1}} & 1 \end{bmatrix} \begin{Bmatrix} \tilde{p}_6 \\ \tilde{V}_6 \end{Bmatrix}$$

$$Z_{ann2} = -jY_{ann} cotk_0 l_2$$

This is going to be a bit tedious. So, what we probably could do is that perhaps use a symbolic computational package named as Maple to further simplify things. And then, it is just a matter of multiplying the matrices which can be very easily accomplished there. Let us get to that.


```
restart:
with(LinearAlgebra):
T1:=Matrix([[1,0],[1/Z1,1]]);
```

$$T1 := \begin{bmatrix} 1 & 0 \\ \dfrac{1}{Z1} & 1 \end{bmatrix}$$

```
j:=sqrt(-1);
```

$$j := I$$

```
T1:=simplify(subs(Z1=-j*(Y0/(m-1))*cot(k0*l1),T1));
```

$$T1 := \begin{bmatrix} 1 & 0 \\ \dfrac{(m-1)\, I}{Y0 \cot(k0\, l1)} & 1 \end{bmatrix}$$

```
T2:=Matrix([[cos(k0*l3),j*Y0*sin(k0*l3)],[(j/Y0)*sin(k0*l3),cos(k0*l3)]]);
```

$$T2 := \begin{bmatrix} \cos(k0\, l3) & Y0 \sin(k0\, l3)\, I \\ \dfrac{\sin(k0\, l3)\, I}{Y0} & \cos(k0\, l3) \end{bmatrix}$$

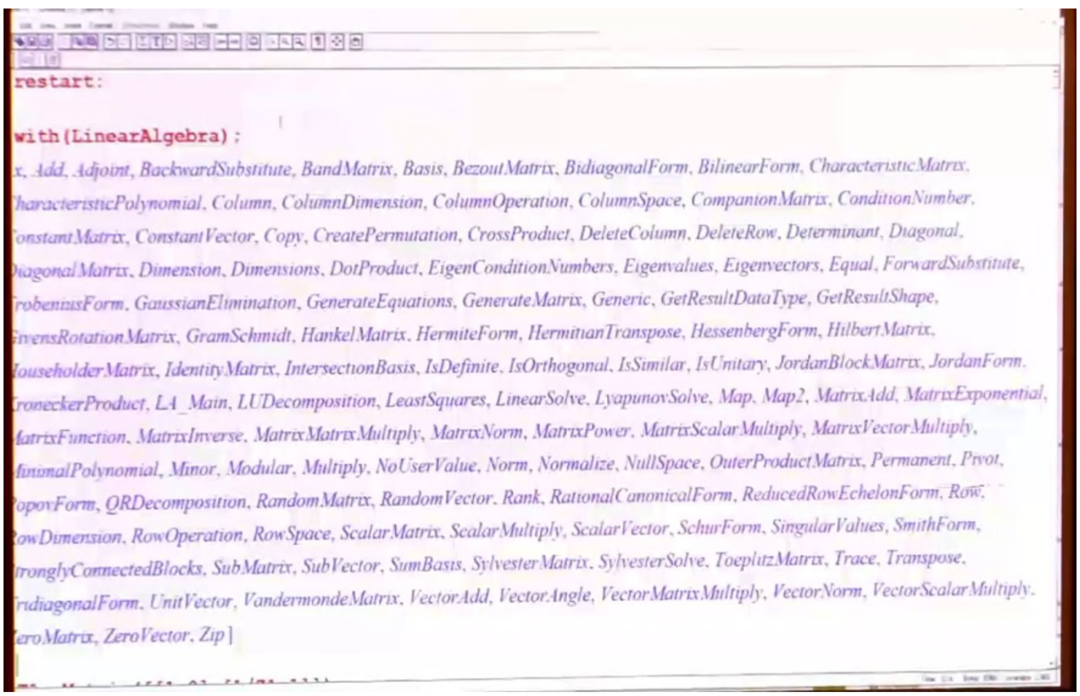
```
T3:=Matrix([[1,0],[1/Z2,1]]);
```

$$\begin{bmatrix} 1 & 0 \end{bmatrix}$$

Let us use maple. So, what we will do now is that we will probably go to symbolic package maple like I was saying. So, maple is used by academicians and also, people who work in industry to work with computer algebra. It really helps in efficiently simplifying algebraic expressions which are otherwise sort of quite tedious. So, what we do is that? We can just go to maple.

And classic window or something like that would open and you can hit this button here, will open new worksheet. So, we have already opened one worksheet in which some commands were written.

Restart means starting from fresh, none of the variables are defined, you have to define it a fresh and linear algebra package is something that will invoke lot of linear algebra commands like matrices and how to multiply matrices and so on. If you and this is the one that is highlighted is a colon. So, if you do not do it and just put a semicolon here.



So, all these things will pop up and in order to suppress that we use a colon. Now, T1 very quickly is the matrix is how you define a matrix; T1 is a variable. This is a symbol to define anything and matrix is something that you write. So, matrix is 1 0 1 / Z1 and 1. So, these are the first row elements; these are the second row elements.

So, Z1 is the impedance. So, the annular cavity. Now, remember, let me go back to the presentation slides, where this encircled boxed expressions are your impedances for the annular cavity.

Similarly, So, this is what you get using the derivations that we have done so far.

Now, entire thing I am clubbing in as Z1 or Z2 as this thing. So, before we move ahead, let me just take the liberty of defining another few symbols which will actually help us in simplifying things. So, inlet and outlet are assumed to be the same here. So, that is the characteristic impedance.

$$Y_0 = \frac{C_0}{S_p}$$

So, let us say that is it is so for both inlet and outlet. Now, for the chamber, what is it? For the chambers Y_C is C_0 by S chamber, keep this aside and for the annular cavity, since annular cavity characteristic impedance of annular cavity 1 and 2 are the same. We get,

$$\frac{S_c}{S_p} = \left(\frac{\rho_0}{d_0}\right)^2 \quad Y_c = \frac{C_0}{S_C} \rightarrow \frac{C_0}{\frac{S_p S_C}{S_p}} = \frac{Y_0}{M}$$

$$= M$$

So, remember, we calling this as M. So, this becomes M - 1, for this thing right and is there any way that we can possibly express the chambers impedance? So, this becomes.

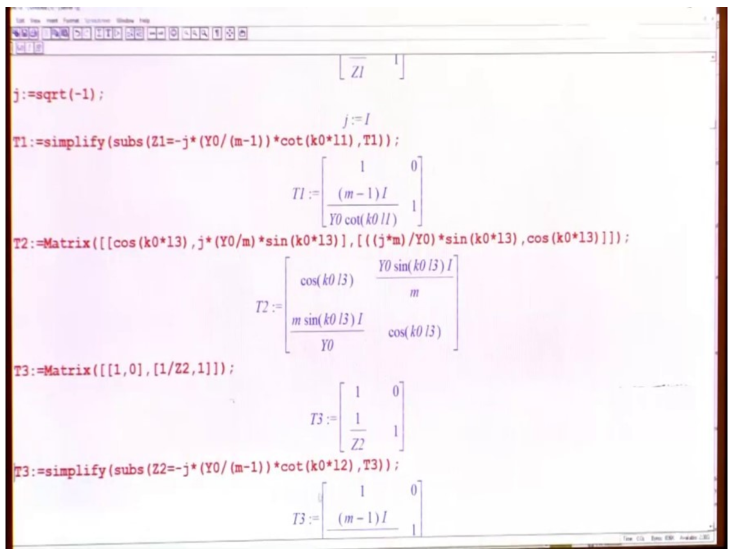
$$Y_{ann} = \frac{C_0}{S_C - S_p} \rightarrow \frac{C_0}{S_p\left(\frac{C_0}{S_p} - 1\right)}$$

$$Y_{ann} = \frac{Y_0}{m - 1}$$

So, Y naught divided by M will be your characteristic impedance of the chamber. So, we can possibly substitute all these things here. So, let me just invokes these thing. Now, let us go back to our maple and let us begin to simplify things. Now, j is square root of minus 1 imaginary number and T 1, now we substitute Z 1 is equal to minus j into y of annular cavity.

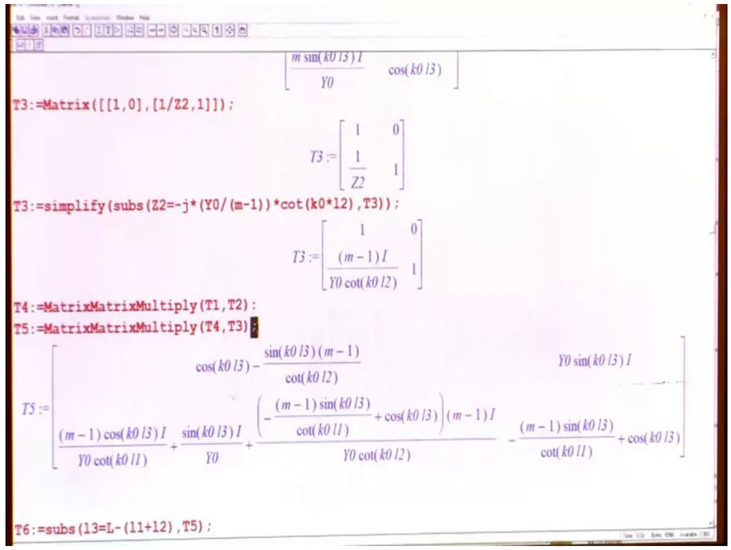
So, that we just saw it is Y naught divided by M minus 1 into cot of k_0 l 1 and we substitute this in T1. Well, the idea of this brief digression is not just to not to teach you, not to introduce here a tutorial in maple; but possibly we will it is a more like an ad hoc approach or you know introducing some symbols as we need them.

So, what we do is basically let me just begin again with linear algebra package and this is square root minus 1 and once we simplify, we get this. Now, this is the matrix for this T2 is the matrix for the part which is of uniform cross section. So, here we put l_3; remember l_3 was capital L chamber length minus l_1 plus l_2. So, that is what we get cos of k naught now.

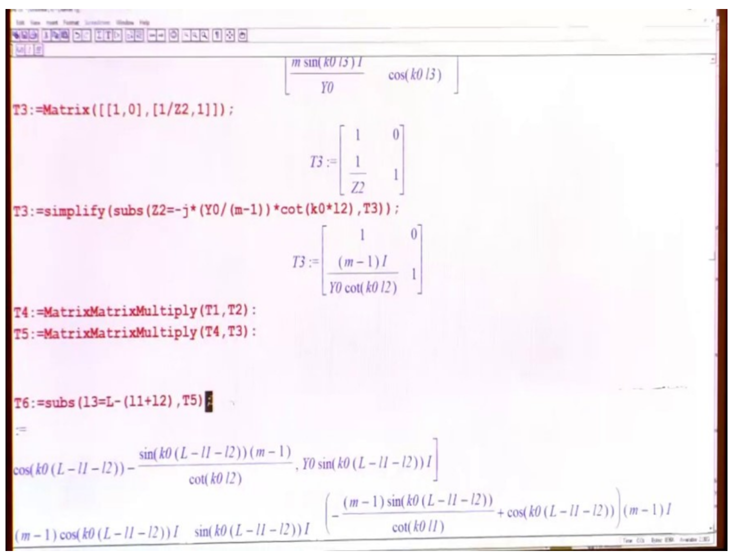


Here, there is a small change here it is m. Let me write it as m and m would probably go into the numerator. So, we get this sort of an expression, where m is nothing but the expansion ratio. So, m is nothing but Sc by Sp and it is the square of the ratio of the diameters of the chamber to that of the port.

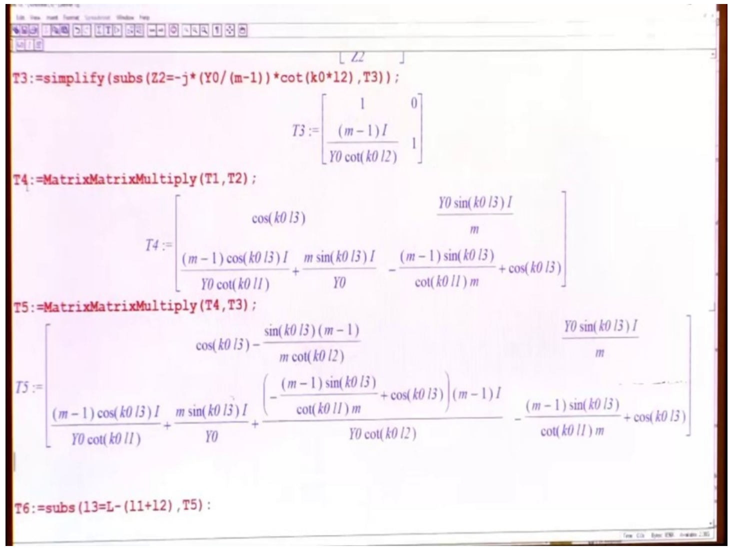
So, m that is why it comes in the denominator and here, it comes in the numerator. So, you get this expression for T2 and then, T3 will become this particular thing Z2. So, now, similarly, if we substitute an expression for Z2, what we get is this particular thing.



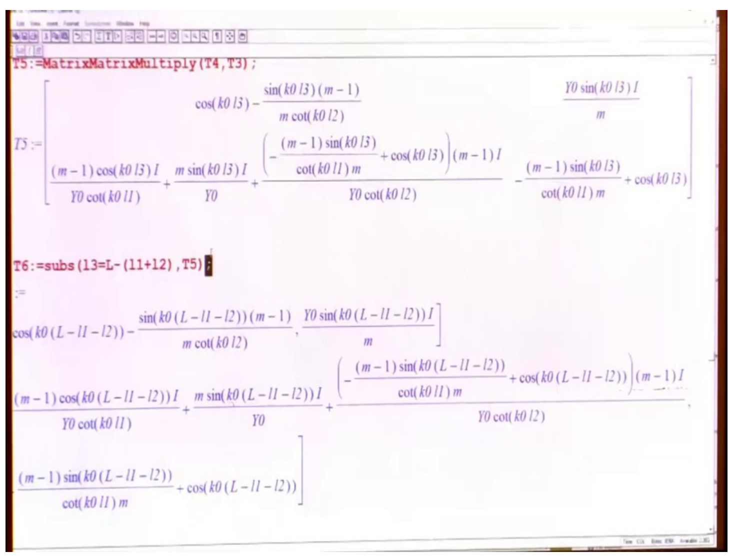
T3 matrix is this and we once we substitute this, we will get the this particular expression. For now, just let me suppress all the outputs and show you all the things one at a time.



Because that way it is probably easy to comprehend things. Now, T1, T2, T3 matrices we have got just like we got in our derivation, the long hand derivation that we did. Now, T4 matrix how is it obtained by multiplying T1 into T2. So, matrix, matrix multiply so that you get T1 into T2. Once we that is basically you multiply, the sudden area discontinuity matrix, where you have a protrusion inside the cavity extended inlet with that of the section with uniform cross section.



So, once we do that, we get T4 and then, once we multiply T4 with T3, what do we get? So, T4 matrix is the combination of this and this is what we get here. So, now, T 5 matrix is the final transmission loss matrix that we obtain. Now, if the things can become even more complicated, if we were to substitute *l*3 is equal to L minus *l*1 plus *l*2. So, I have just suppressed the output.

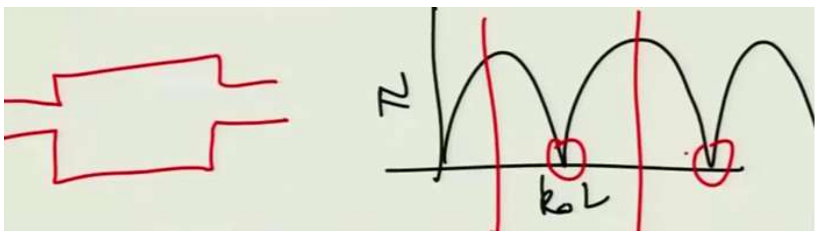


But if I do not, I get this. So, you know it is just a matter of substituting this expression. So, I rather suppress the output here and you know just focus on this particular matrix here. And with understanding that *l*3 is equal to L minus *l*1 plus *l*2. This looks to be a little bit quite tedious expression.

Because right now, you know when we derive the transmission loss expression for a simple expansion chamber, there was really only two variables, actually 3 or probably yeah 3 or 2 depending upon your conventions m, that is the area expansion ratio which is capital D_0 by small d_0 and L the length of the chamber. So, everything else was expressed in terms of m and L. There were actually only 2 variables, if you go back to our slide somewhere here; m and L.

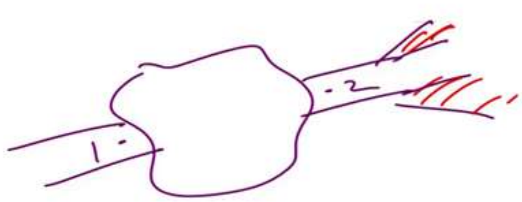
$$= 1 + sin^2 k_0 L \left(M - \frac{1}{M}\right)^2 \quad M = \left(\frac{D_0}{d_0}\right)^2$$

$$TL\ 10\ log_{10} \left\{1 + \frac{1}{4}\left(M - \frac{1}{M}\right)^2 sin^2 k_0 L\right\}$$



And frequency of course, is there. But in the maple expression that I just presented now, we saw as many as you know m is there, expansion ratio, then here you have three other variables; the chamber length capital L, extension length l 1 and l 2 and the l 3 is fixed of course once you select the other two and area expansion ratio m and Y_0.

So, we have a bunch of variables to play around with and this looks to be a little bit complicated expression. So, the best practice then is to derive the transmission loss expression for a general element.



Suppose you have a general element and you have inlet point here, outlet point here. You know the relation,

$$\begin{Bmatrix} \tilde{p}_1 \\ \tilde{V}_1 \end{Bmatrix} = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix} \begin{Bmatrix} \tilde{p}_2 \\ \tilde{V}_2 \end{Bmatrix}$$

once we have this relation, we can possibly obtain the transmission loss by assuming the anechoic termination, here for a generalized element.

This is the general this could be any element. There could be perforates, linings and it need not just be a simple expansion chamber. So, we have a generalized transfer matrix parameters, also known as the four-pole parameters.

$$[T] \quad \rightarrow \quad \text{Four} \; - \; \text{pole parameters}$$

[T] parameters are also called Four - pole parameters. So, what we intend to do now, is that derive a generalized expression for the transmission loss for a general element in terms of the four-pole or T matrix parameters T_{11}, T_{12}, T_{21}, T_{22} for a stationary medium.

And once we know the inlet and outlet diameters and then, based on these generalized expression, what we are going to do is that get a feel of some idea, where you can expect some peaks, resonance peaks or where transmission loss is will exhibit a resonance peak at animation peak there in the spectrum or where it can express a trough.

And once, we know the generalized expression which should be straight forward to derive, we can use the transfer matrix for an extended inlet and outlet element and plug in all the parameters and obtain or the rather complicated or rather tedious expression for transmission loss and then, possibly draw some conclusions.

We can actually explain using the tree matrix parameters, why is that by choosing appropriate lengths we that l by 2 or l by 4, we should be we are able to cancel out the troughs and all that sort of a thing.

We should also be able to explain basically using the transfer matrix parameters, I mean basically using this thing we can carry out some parametric studies in MATLAB that I am going to present in the next lecture. So, using certain parameters and how do we choose certain things. So, till that time, we probably have to stop and I will see you in the next lecture.

Thanks a lot.