Muffler Acoustics - Application to Automotive Exhaust Noise Control
Prof. Akhilesh Mimani
Department of Mechanical Engineering
Indian Institute of Technology, Kanpur

Lecture - 03
1-D Momentum Equation, Isentropic Equation, Sound Speed

Welcome back to the third lecture for our Muffler Acoustics Course. So, in the last lecture, we were we just derived the continuity equation was usually found in text on fluid mechanics and in acoustics.

$$\frac{\partial \rho}{\partial x} = - \frac{\partial (\rho U)}{\partial x}$$

$$\frac{\partial \rho}{\partial t} + \frac{\partial (\rho U)}{\partial x} = 0 \qquad (1) \qquad => \qquad \rho_t + (\rho U)x = 0$$

$$\frac{\partial \rho}{\partial t} + U \frac{\partial \rho}{\partial x} \ \rho \frac{\partial U}{\partial x} (2)$$

So, we see here the form of the continuity equation is this, and this is one-dimensional of course. And when we expand this thing out, you get something like this – so this entire thing. Now, I want to draw your attention before we begin the momentum equation this term what I have encircled.

$$\frac{D}{Dt} = \frac{\partial}{\partial t}\bigg|_x + \frac{U\partial}{\partial x}$$

$$\frac{D\rho}{Dt} + \rho \frac{\partial U}{\partial x} = 0 \qquad (3)$$

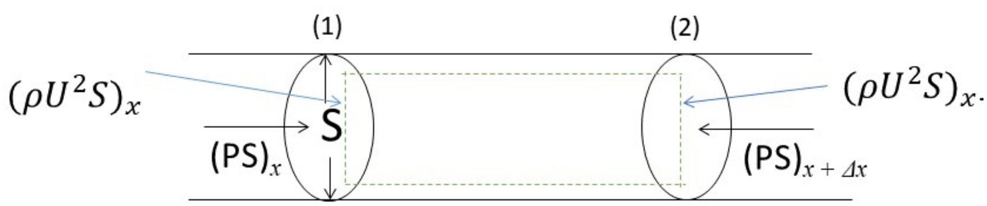
Typically, in fluid mechanics what we have this term is basically also called the total derivative which is given by this operator. So, this comprises of the derivative of variations of a quantity with respect to time by at a fixed location. And here the variation the spatial variation of a quantity at a given time, and U is the velocity in that direction. So, this is called the total derivative also in fluid mechanics.

So, the equation this equation. Equation(2) can also be written in the following form which is rho times rho; so here you are we will call this equation (3). Now, we will probably go to equation to the momentum equation. And to simplify the momentum equation, you will make use of certain forms of the continuity equations – the expanded forms.

Momentum Equation

So, we now get to the momentum equation. We will again consider a cylindrical volume something like this. And let us identify in this cylindrical control volume, what are the forces that act? First is the pressure, force by the fluid pressure. And let us say S is the uniform cross section area.

So, PS at x and the other force that acts in the opposite direction is PS at Sx+Δx. And then there is also the force by the momentum in flux. So, the momentum in flux let us let me write with the different color is



$$(\rho U^2 S) = \boldsymbol{\rho}(SU) - U = \rho Q$$

So, before we proceed further let's understand,

$$= \frac{kg}{S}\frac{m}{s}$$

$$= kg\ ms^{-2}$$

So, and this is like your mass, this is like your volume, volume velocity. So, this entire term is like the mass per second.

$$\overrightarrow{(PS)_x} - \overleftarrow{(PS)_{x+\Delta x}}$$

$$+ \quad \overrightarrow{(\rho U^2 S)_x} - \overrightarrow{(\rho U^2 S)_{x+\Delta x}}$$

$$\frac{\partial(\rho S\Delta x \cdot U)}{\partial t}$$

So, if we balance this, what are we left with?

So, this is like your rho S delta x times velocity so that is your momentum – mass into velocity.

And we need to talk about the time rate of change of momentum, is it not? So, that is how we balance of the forces in the control volume.

Now, let us simplify. We divide throughout by Δx, and take the limit Δxtending to 0 that is to say we consider a infinitesimal element when we do that we end up and of course, we are considering a cylindrical prismatic element kind of. So, S is uniform, it gets cancels out.

Basically what we left with is and this.

$$-\frac{\partial P}{\partial x} - \frac{\partial \rho U^2}{\partial x} = \frac{\partial \rho(U)}{\partial t}$$

$$\frac{\partial(\rho U)}{\partial t} + \frac{\partial(\rho U^2)}{\partial x} = -\frac{\partial \rho}{\partial x} \,(4)$$

$$\frac{\rho U_t + U\rho_t}{\partial x} \quad (\cdot)_t = \frac{\partial}{\partial t}$$

$$+ \rho U \cdot Ux + U(\rho U)_x \qquad\qquad (\cdot)_x = \frac{\partial}{\partial t}$$

$$= -\frac{\partial p}{\partial x} \qquad\qquad (5)$$

So, basically, we transpose all the terms on the other side to get the following form. I have purposely put this thing in here, where this is capital P different from $\tilde{P}$. This is the total pressure. When we linearize the equations, we will understand what is meant by total pressure. It is thesum of the ambient pressure plus the perturbation pressure $\tilde{P}$. So, we need to wait till the linearization happens. Before that, let us go back to this equation.

And so, we see this kind of a thing ok. Let me name this is number 4. And let me expand out number 4 to get. And by this, I mean this; and by this, I mean this; – that is partial derivative with respect to x and with respect to t, this here can be any quantity. So, here we have Ut + U. ρ_t. And then we need to employ a clever trick, we need to group a term in terms of ρ U. So,

So, now, we have got the expanded form of the momentum equation in the terms of equation (5). Now, let us use these forms either equation form (1) or form (2) to further simplify the momentum equation given by 5. So, you will write down this in a little cleaner manner. Notice that the term U here and U here – this is common. And here rho is common, and rho is common.

So, this is a important, because then we can actually group at

$$\rho\,(U_t + U\cdot U_x) + U(U\rho_t + \rho U)_x \;= -\frac{\partial P}{\partial x}$$

Now, I like to draw your attention here this equation the one that I have encircled is nothing, but the continuity equation written in the form 1.

This is nothing but your ρ_t+$(\rho U)x$= 0, is it not? So, since we do not have any sources the term is homogeneous this entire thing goes away. So, leaving us with the following term Ut + this is what we get. And this again can be simplified further. If you recall the definition of total derivative in terms of here is equal to minus times this, equation (6) or equation (7).

$$\rho(U_t + UU_x) = -\frac{\partial P}{\partial x}\,(6)$$

$$\rho\frac{DU}{DT} = -\frac{\partial P}{\partial x} \qquad (7)$$

So, basically the simplified form of the momentum equation, then is given by equation 6 rho times U_t + U into U_x = - Δ_x p. So, we have derived the momentum equation which we will also find the standard fluid mechanics text and the continuity equation as well.

And now what is remaining is the equation of state which relates basically your pressure and density through some function. So, let us talk in detail. We will keep this equation (6) as well as equation (2) would be of importance to us we will keep this equation aside. So, in the meanwhile, what we will do is that focus on equation of state.

So, here you have your equation of state. So, instead of using conservation equation, we are using the equation of state. And one well-known equation for a perfect gas is

$$P = k\rho T$$

$$R = 287.08\frac{J}{kg}$$

So, this is the perfect gas equation which relates the pressure and temperature. But another form of equation which is often used and particularly useful in acoustics is some generalized relation between the pressure has some function of density and the entropy per unit mass.

$$P = p\ (\rho, S) = p\ (\rho)$$

So, P is a function of ρ – ρ and entropy.

So, when the losses are negligible which is usually the case in acoustics entropy remains constant. So, what we essentially have is an isentropic condition that is the pressure is a function of density alone in which case it becomes P as a function of density alright. So, for most gases, the form that is usually used is the isentropic equation which I am going to talk about now.

$$\frac{p}{p_0} = \left(\frac{\rho}{\rho_0}\right)^{\gamma} \qquad (7)$$

$$P_0 = Static\ pressure$$

where

$$\gamma = \ \ Adiabatic\ gas - constant = 1.4\ for\ dry\ air$$

$$P = Static\ pressure$$

$$\rho_0 = Static\ dencity$$

So, this equation I would say this is equation (7) is particularly useful for gases. For liquids equation (7), isentropic relation may not be applicable a more general approach is required in terms of the condensation factor. We will probably not go into that.

Now, we will probably stick to the isentropic equation which basically relates the pressure and the density at a certain state. The $P_0$and $\rho_0$may be considered as an operating point or static point. And what we are really interested in is that how does small changes in pressure are related to the small changes in density.

So, for now what we are going to do is define a quantity called C^2 which is your nothing but sound speed a very, very important quantity in acoustics that is a quantity which tells you how acoustic disturbances propagate. So, sound speed squared is given by

$$C^2 = \left.\frac{\partial P}{\partial \rho}\right|_{S=constan} = \frac{dP}{d\rho}$$

$$C^2 = \frac{dP}{d\rho} (8)$$

Now, since S is constant, partial entropy we have only one variable rho. So, C^2 is this thing.

Now, another useful simplification will arise when we consider the total pressure as

$$p = \mathrm{P}_0 + \widetilde{\mathrm{P}}$$

but the acoustic pressure if you recall the first couple of lectures that is acoustic disturbances. So, I want to point out here that you know this $\widetilde{\mathrm{P}}$ is basically your small perturbations is something that hits your ear drum.

When you are watching this video, when you, when the sound is being emitted from the loudspeakers of your device – laptops or whatever, so it is basically disturbances are created in the pressure in the medium air between your device and your ear drum.

So, this, this pressure fluctuations are hitting your eardrum the noise that is the disturbances that are created by the device they are hitting your eardrum and that is what is causing the sensation of sound. So, it is basically this $\widetilde{\mathrm{P}}$ is what we are after. So, this is what we are interested in acoustics. And as you can appreciate you know when you do this linearization, we just begin the process of linearization. Obviously, lot of things must be done the momentum and continuity equations we just begin.

So, $\tilde{P}$is your acoustic pressure and that is what we are going to relate to small changes in density which is given by

$$\delta\rho = \tilde{\rho} - \rho_0 \qquad (9)$$

So, basically what is happening is that if you revisit equation number 7 and substitute these relations, let us call these relations as relation number (9). And substitute relations (9) and (7), what we get, what do we get?

$$\frac{p_0 + \tilde{p}}{p_0} = \left(\frac{\delta\rho + \rho_0}{\rho_0}\right)^{\gamma}$$

$$\left(\cancel{1} + \frac{\tilde{p}}{p_0}\right) = \left(1 + \frac{\delta\rho}{\rho_0}\right)^{\gamma}$$

Now, if we apply our binomial theorem or expand this out, you get

$$= \cancel{1} + \frac{\gamma\delta\rho}{\rho_0} + \gamma(\gamma_t)\left(\frac{\delta\rho}{\rho_0}\right)^2 + HOT$$

So, then what happens is this go away this goes away.

And what we are left really

$$\frac{\tilde{p}}{p_0} = \frac{\gamma\delta\rho}{\rho_0} + \cancel{\frac{\gamma(\gamma-1)}{2!}\left(\frac{\delta\rho}{\rho_0}\right)^2} + \cancel{HOT}$$

which we will probably ignore including we will probably ignore these terms and will focus only on the small order terms which

$$\frac{\tilde{p}}{\delta\rho} = \frac{\gamma p_0}{\rho_0}$$

$$\frac{\tilde{p}}{\tilde{\rho}} = \frac{\gamma p_0}{\rho_0} = C^2(10)$$

So, if you recall equation number (8), this was for the generalized sound speed basically not necessarily evaluated at the operating condition given by P_0or ρ_0. What I mean to say is that the relation between $P_0$and ρ_0 is not linear, it follows the isentropic law.

But for small changes in acoustic pressure that is basically small signal equation strictly only for small signal equation, this relation $\tilde{P}$by $\tilde{\rho}$small disturbance is the sound speed or disturbance is propagate at constant rate and that is given by gamma times P_0 by ρ_0, and it is denoted by C_0^2.

Suppose if the signals are no longer small as it happens in the amplitude or disturbance is no longer smaller as it typically happens in non-linear acoustics, the sound speed may not be constant, it depends on the complete process. But usually for small signal equations we can consider P_0 as equivalent to atmospheric pressure, at that point and ρ_0 to be the ambient density.

So, once we substitute those values, let us see what

$$C_0^2 = \frac{1.4 \times (1.013 \times 10^5)}{1.2}$$

$$C_0 = \sqrt{(1.1818 \times 10^5)}$$

$$= 343.778\, m/s$$

$$C_0 \simeq 344\, \frac{m}{s} 20^o = 7$$

So, this of course, is that standard temperature in pressure which is 20 degrees temperature – normal temperature and pressure.

$$C_0 = \sqrt{\gamma RT}$$

$$C_0 \propto \sqrt{T}$$

For colder regions or where the temperature does drop, sound speed will propagate at a slower rate as compared to the cases where temperature is high.

So, if the value that you saw here 344 meter per second, this was at 20 degrees based on the 1.29 thing. Similarly, when you have a colder thing the density of the air is bit higher, and then you will have typically a lower value of sound speed. So, basically this is the relation then.

And now what we can do using our aim then is to basically use these relations equation (10) substitute that in the previous equations for the simplified equations for the continuity momentum, and further linearize it based on the conditions that we set about that is basically your equation (9).

So, what we will do is basically first we will write down this as

$$\tilde{p} = C_0^2 \tilde{\rho} \tag{11}$$

And we will put that equation (11) in equation number (6) as well as equation (2) and go about simplifying things. Let us begin with the continuity equation and see what we are getting. Let us first linearize it. So, I need to write down this equation, equation (2) which is,

$$\frac{\partial \rho}{\partial t} + \rho \frac{\partial U}{\partial x} + U \frac{\partial p}{\partial x} = 0$$

So, before we begin the linearization, an important thing to be mentioned here is that,

$$\tilde{\rho}\, \rho - \rho_0$$

that is your small changes in the acoustic density which is nothing but your total density minus the ambient density. This value rho tilde is much, much smaller than the ambient density that is to say,

$$\frac{\tilde{\rho}}{\rho_0} \ll 1$$

$$\tilde{\rho} \ll \rho_0$$

Similarly, we will also find well actually this, this one more thing this condition leads you to the condition,

$$\tilde{P} \ll (\rho_0\, C_0).\, C_0$$

So, this particular quantity is very important. It is called the characteristic impedance. We will talk about that after some time. But before that there is another restriction that is the particle velocity should be $V \ll C_{0.}$

So, basically you have this, you have this and this thing. So, the acoustic density perturbation $\tilde{\rho}$ should be much less than the ambient density or the acoustic pressure should be much less than the characteristic impedance times the sound speed as well as the particle velocity should be much smaller than the acoustic sound speed or the sound speed just generally disturbance speed at which the disturbance propagates.

Also important thing to be noted here is that the characteristic impedance $\tilde{\rho}C_0$ is significantly different for water as compared to air. So, generally people tend to believe that that small disturbance is small acoustic disturbances in the pressure should be much much less than the ambient pressure. But that might be true for air, but for water it is very different.

What I mean to say is that the condition

$$\frac{\tilde{p}}{\rho_0} \ll 1 \qquad 10^5 \ \times 1.013$$

While it is obvious, but the more fundamental condition,

$$\tilde{p} \ll (\rho_0 C_0) C_0$$

So, if you put in the values for air, you will see both the equations will pretty much give you the same things, but for water will be very different, for water it will be very different.

$$413 \ \ 343 \ \simeq 10^5 \ \times 1.4$$

So, they pretty much give the same restrictions for air, but for water it is very large. Anyhow, we will come back to our discussion on in air, and we will begin the linearization process in the next lecture. Thanks.