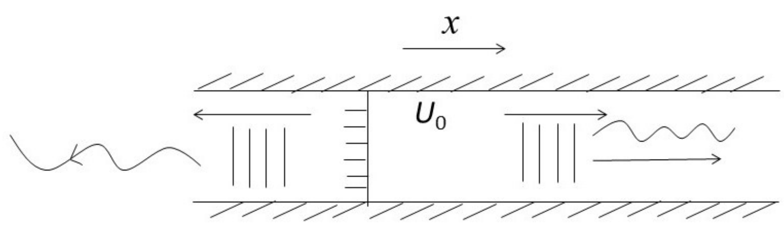
Muffler Acoustics - Application to Automotive Exhaust Noise Control
Prof. Akhilesh Mimani
Department of Mechanical Engineering
Indian Institute of Technology, Kanpur

Lecture - 06
1-D Acoustics Wave Equation in Ducts Carrying Uniform Mean Flow: Derivation

Welcome to week 2 of the course on Muffler Acoustics. The first lecture of this week we will basically resume our discussion from where we left in the last weeks lecture. In this today's lecture, we are going to focus on planar waves that propagate in ducts which carry mean flow that is 1D wave propagation ducts, but now including the convective effects of mean flow in the ducts.



As you can see from this figure, you have a duct and say U_0 is the uniform flow velocity. In ducts it is never actually uniform flow, it is actually more like a plug flow something like this, but we approximate that as a uniform mean flow and what we interested particularly is that

$$U_0 \ll C_0$$

That is the Mach number

$$M_0 = \frac{U_0}{C_0} \ll 1$$

Typically, we consider an automobiles; we really exceed 0.1, 0.15 perhaps at the max; 0.15 Mach number. So, we will be happy will be contended with designing systems or analyzing wave propagation with maybe maximum of Mach number 0.15 or something like that. So, we going to develop the basic theory underpinning wave propagation, planar waves with mean flow of x; so, that is going to be the focus of this lecture.

So, let us develop the maths; if you recall the last week's lectures, we first talked about the continuity equation ok. And so what was that?

$$\rho_t + (\rho U)x = 0 \qquad (1)$$

So, analysis remains the same other than the fact that rho now

$$\rho = \rho_0 + \tilde{\rho}$$

$$p = P0 + \tilde{p}$$

$$U = U_0 + \widetilde{U}$$

Particularly, if we substitute this and this in here; what we get?

$$(\rho_0 + \rho)_t + \{((\rho_0 + \rho)((U_0 + U)\}_x = 0$$

So, now the analysis is pretty much the same let us number this as equation 1 and one substituting all this and simplifying expanding out; I am sorry, we end up with this thing. So, obviously, like in the previous class we

$$\frac{\partial \rho_0}{\partial t} = 0$$

So, this then simplify is to

$$()_t = \frac{\partial}{\partial t}$$

$$()_x = \frac{\partial}{\partial t}$$

So, and then term by term we need to expand out; the thing is that are;

$$\widetilde{\rho_t} + (\rho_0 + U_0 + \rho_0\widetilde{U} + \tilde{\rho}U_0 + \tilde{\rho}\widetilde{U})_X = 0$$

But probably just wait for a while and now we are assuming the flow is uniform along the axis of the duct that is along this is the x direction; I am sorry forgot to mention.

So,

$$\frac{\partial U_0}{\partial x} = 0$$

flow is fully developed; actually it is a uniform flow, so it is by definition its uniform across space and we also get; so that would; density; obviously, constant with space.

So, here you get something like this and then rho tilde and U x and U naught plus rho U x is equal to 0. So, that is what we going to get; obviously, this term then goes away and what we are really left with this term is also a secondary term; this is a second order term, while this is a zeroth order term; this is a second order term because rho tilde and U tilde small, their products are also smaller.

So, we will probably ignore this term and ultimately retain only the following terms;

$$\tilde{\rho}_t + \rho_0 \widetilde{U}_x + \tilde{\rho}_x U_0 = 0 \qquad (2a)$$

So, this is what we are going to get.

Let me write down this equation in a cleaner manner perhaps on the next slide which is

Now as a check of self-consistency; it is important that we should be able to get perturbed continuity equation pertaining to 0 mean flow from equation (2). How do we get that? If you simply substitute if you put U_0= 0 that is a stationary medium, what would you get? You will get that is your old familiar equation for the continuity one.

$$\tilde{\rho}_t + \rho_0 \widetilde{U}_x = 0 \qquad (2b)$$

So, let us call this (2b) and this is (2a); now this is if you have this condition, but in the more general case when you have a non-zero uniform mean flow, equation (2a) will apply.

Now, let us focus our discussion on the momentum equation.

So, we recall again here that

$$\rho(U_t + UU_x) + P_x = 0$$

This we saw from a last week's course, last week's lectures.

Now, again we pretty much do the same thing that is to say we have the same perturbation quantities;

So, now is a time where we substitute the pressure one in this one; so when we do that we would eventually end up with this thing. So, I am dropping out the thing like U; I am already simplifying this. So, I am directly.

$$(\rho_0 + \tilde{\rho})(\tilde{U}_t + (U_0 + \tilde{U})(U_0 + \tilde{U})_x) + \tilde{P}_x = 0$$

$$\frac{\partial U_0}{\partial t} = 0$$

$$\frac{\partial p_0}{\partial x} = 0$$

with understanding that ambient pressure, atmospheric pressure does not change with a space its constant. Now, what do we do after this? We should probably simplify the terms and once we do that, we are going to get

$$\rho_0\tilde{U}_t + \tilde{\rho}\tilde{U}_t + (\rho_0 + \tilde{\rho})\ (U_0 + \tilde{U})(U_0 + \tilde{U})_x + \tilde{p}_x = 0$$

$$\rho_0\tilde{U}_t + \{(\rho_0 U_0 + \rho_0 U^2 + \rho^2 U_0 + \rho^2 U^2)(U_0 + U)_x\} + \tilde{p}_x = 0$$

Now, we have this term it can the simplification can get a little tedious. So, well in order to keep the interest going; I would assume certain simplifications, you can work it out on your own. So, I will underline the second order terms which you; by now I should notice that this is definitely a second order term.

So, I will probably not carry forward this thing in the next step;

$$\rho_0\tilde{U}_t\ \ + \rho_0 U_0\tilde{U}_x + \rho_0\tilde{U}\tilde{U}_x\ + \tilde{\rho}U_0\tilde{U}_x + \tilde{\rho}\tilde{U}U_x +\ \tilde{p}_x = 0$$

$$\rho_0 U_t^2 + \rho_0 U_0 U_x^2 + \tilde{\rho}U_0\tilde{U}_x +\ \tilde{p}_x = 0$$

So, the whole thing needs differentiation I am; so sorry; here you have your this thing not the whole thing well.

So, well, we get this and this term of course, would not survive; this one U_0 because once you differentiate with respect to x, it is; this is not there. So, eventually you will end up with only $\tilde{U}_x$ and I would get this thing here.

Now, what we get

$$\rho_0 \widetilde{U}_t + \rho_0 U_0 \widetilde{U}_x + \tilde{p}_x = 0$$

So, this clearly a second order term coming up for us;

So, this term would also drop out, eventually leaving us with only the following which may also be written in a more sustained manner; the following form. Look eventually, whatever we do as a check of self consistency; it should boil down to the state when you do not have mean flow. So, this let me name this is equation 3 and clearly when U is 0, if U is 0; you get back your familiar momentum equation that is

$$\rho_0\left(\widetilde{U}_t + U_0 \widetilde{U}_x\right) + \tilde{p}_x = 0 \qquad (3)$$

$$if\ U_0 = 0$$

So, that is your Euler equation; linearized Euler equation, momentum equation along the *x* direction. So, we have equation (3) and we have our equation (2a); is not it? That is your continuity equation with mean flow and equation (2a) and your equation (3) is your momentum equation with mean flow. So, the idea then is to combine equation (2a) and (3).

So, let me write down both these equations,

$$\tilde{\rho}_t + \rho_0 \widetilde{U}_x + \tilde{\rho}_x U_0 = 0$$

Here you are rho; rho tilde x and your other equation that we just derived right now

$$\rho_0(\widetilde{U}_t + U_0 \widetilde{U}_x) + \tilde{p}_x = 0$$

So, other than these two equations; it is important to also recall that we have the isentropicity relation; is coming from the equation of state for a lossless fluid which is what we are really considering.

$$\frac{\tilde{p}}{\tilde{\rho}} = C_0^2 = \quad \frac{\tilde{p}}{C_0^2} = \tilde{\rho} \qquad (4)$$

So, we have this; what we need to do; is that use this equation 4 in the above equations. So, once we put this particular relation isentropicity relation in here, what do we get?

We get,

$$\frac{\tilde{\rho}_t}{C_0^2} + \rho_0 \widetilde{U}_x + \frac{U_0}{C_0^2} \tilde{p}_x = 0$$

Alright, this is what we are going to get, but of course, we need to simplify this thing further.

So, we multiply all throughout by

$$\tilde{p}_t + U_0 \tilde{p}_x + \rho_0 C_0^2 U_x = 0 \qquad (5)$$

this is what we are going to get. Let us name this as equation number 5 and what about this one? Momentum equation, you also have this. So, we have successfully eliminated the acoustic density variable rho tilde and the equations that we have got, we have got this particular equation; this one.

And we have also got this equation; momentum equation with mean flow and acoustic quantities $\tilde{p}$ and $\widetilde{U}$ and the continuity equation in which we have only the acoustic pressure variable.

Our job is to combine equation (5), as well as this equation; how do we go about doing so? So, what we will probably do is that we will differentiate this equation mark star which is the same as the previous one with respect to *x*; we will differentiate this equation with respect to *x*.So, what will we get? We will get

$$\rho_0 \widetilde{U}_{tx} + \rho_0 U_0 \widetilde{U}_{xx} + \tilde{p}_{xx} = 0 \qquad (6)$$

Everything else is the same; it says that you keep on putting U_0; and you retain, you get back the stationary medium thing. So, this is we will keep this aside for the time being and worry about the continuity equation.

So, for this equation we will basically differentiate with respect to the time t. So, we get

$$\frac{1}{C_0^2} \tilde{p}_{tt} + \frac{U_0}{C_0^2} \tilde{p}_{xt} + \rho_0 C_0^2 \widetilde{U}_{xt} = 0 \qquad (7)$$

So, we have got this; equation (6) and equation (7). Now what? How do we go about processing these equations further? That is the job. So, what I suggest is that let us open the bracket in this one and let us multiply throughout by rho naught.

So, here you are and we probably what we could do is that you could subtract these two. So, this is; obviously, $\widetilde{U}$ and I forgot to remove the thing here x here; what do we do now? So, what we could do is probably divide equation (7) throughout by C_0^2 and here you also have C_0^2, this will go away and you will just have a rho naught ok.

And then we clearly see that if you subtract (7) from (6) or vice versa, this particular term that I have; I will just encircle this one; $\widetilde{U}_{xt}$, whatever it is that can just cancel, or we can sort of eliminate that term.

So, I will say subtracting (7) from (6); we get. What do we get?

We get.

$$\rho_0 U_0 \widetilde{U}_{xx} + \tilde{p}_{xx} - \frac{1}{C_0^2}\tilde{p}_{tt} - \frac{U_0}{C_0^2}\tilde{p}_{xt} = 0$$

So, we nearly there we almost there, apart from a few things that we further need to do and what is that? So, now this thing should be clear; we only have a term this one; the first term which involves this U term here where acoustic particle velocity I am sorry should be written as U tilde which needs to be eliminated.

So, how do we do that? Actually, before we do that, let me write this in a more and in a form which is we are probably a little bit more comfortable.

And you further get,

$$\tilde{p}_{xx} - M_0 \frac{1}{C_0^2}\tilde{p}_{xt} - \frac{1}{C_0^2}\tilde{p}_{tt} \ \rho_0 U_0 \widetilde{U}_{xx} = 0 \qquad (8)$$

Now, I guess this is probably equation number; let us let me call it equation (8) and we need to eliminate.

$$\tilde{p}_{xx} - \frac{M_0}{C_0^2}\tilde{p}_{xt} - \frac{1}{C_0^2}\tilde{p}_{tt} - \rho_0 U_0 \frac{(\tilde{\rho}_0 + U_0 \widetilde{U}_{xx})}{\rho_0 {C_0}^2} = 0$$

Substituting equation (9), in equation (8), we get

If we go back to our continuity equation, that is this particular equation (2a); we can probably

$$\frac{1}{C_0^2}\tilde{p}_t + \frac{\tilde{p}_x}{C_0^2}U_0 = -\rho_0\tilde{U}_x$$

So, this is what we get. And so the end result is that we will be getting things like this. So, I am just calling equation (9) because we have got equation (8) out there. Let us substitute equation (9) in the last equation 8, we just need to differentiate again with respect to x.

$$-\frac{1}{\rho_0 C_0^2}\{\tilde{p}_{tx} + \tilde{p}_{xx}U_0\} = -U_x^2 \qquad (9)$$

So, I would probably do it somewhere here itself. So, let us directly substitute equation (9), in equation 8; substituting equation (9), in equation 8;

we get,

$$\rho_0 U_0 \tilde{U}_{xx} + \tilde{p}_{xx} - \frac{1}{C_0^2}\tilde{p}_{tt} - \frac{U_0}{C_0^2}\tilde{p}_{xt} = 0$$

$$\tilde{p}_{xx} - M_0\frac{1}{C_0^2}\tilde{p}_{xt} - \frac{1}{C_0^2}\tilde{p}_{tt}\ \rho_0 U_0 \tilde{U}_{xx} = 0 \qquad (8)$$

Substituting equation (9), in equation (8), we get

$$\tilde{p}_{xx} - \frac{M_0}{C_0^2}\tilde{p}_{xt} - \frac{1}{C_0^2}\tilde{p}_{tt} - \rho_0 U_0\frac{(\tilde{\rho}_0 + U_0\tilde{U}_{xx})}{\rho_0 {C_0}^2} = 0$$

$$p_{xx} - \frac{M_0}{C_0^2}\tilde{p}_{xt} - \frac{\tilde{p}_{tt}}{C_0^2} - \frac{M_0}{C_0}(\tilde{p}_{tx} + U_0 + \tilde{p}_{xx}) = 0 \qquad (9)$$

So, we can clearly see this term here the one that I have underlined in orange and similarly here; so these terms are something that will add up. So, you will

So, we get, $\tilde{p}_{xx} - 2\frac{M_0}{C_0^2}\tilde{p}_{xt} - \frac{\tilde{p}_{tt}}{C_0^2} - M_0 \cdot M_0\tilde{p}_{xx} = 0$

and then you also have this term; let us see you know how things shape up.

so you will get

$$(1 - M_0^2)\tilde{p}_{xx} - 2\frac{M_0}{C_0}\tilde{p}_{xt} - \frac{\tilde{p}_{tt}}{C_0^2} = 0$$

This has pressure differentiated twice with respect to the spatial variable x and this is dimensionless, but you have your pressure; you have your second. So, basically pressure differential with respect to x and then you also have your plus; you will have somewhere you have your second, but here in the denominator you have meter per second. So, this thing will go away; second.

So, you are still good here because dimensionality wise it is fine and you are here you have your p t t. So, your meter per second square; second square, second square; I will cancel here.

$$\frac{(\tilde{p}_x)}{S}\frac{\mathrm{S}}{m}$$

So, I guess we are fine; we are doing good in terms of maintaining the dimensionality of each term and yes.

$$\tilde{p}_{xx} - 2\frac{M_0}{C_0}\tilde{p}_{xt} - \frac{\tilde{p}_{tt}}{C_0^2}M_0 \cdot M_0\tilde{p}_{xx} = 0$$

$$(1 - M_0^2)\tilde{p}_{xx} - 2\frac{M_0}{C_0}\tilde{p}_{xt} - \frac{p_{tt}}{C_0^2} = 0 \qquad (10)$$

And another way of course, is to write this equation in terms of your velocity; absolute velocity rather than non dimensional velocity.

$$(C_0^2 - U_0^2)\tilde{p}_{xx} - 2U_0\tilde{p}_{xt} - \frac{\tilde{p}_{tt}}{C_0^2} = 0$$

And if we multiply throughout by minus sign; so let us see what we get. You get

$$(U_0^2 - C_0^2)\,\tilde{p}_{xx} - 2U_0\tilde{p}_{xt} - \frac{\tilde{p}_{tt}}{C_0^2} = 0 \qquad (11)$$

So, I guess we are there, what we promise to derive, but before we end we probably; it is a good idea to talk about check self consistency things.

If you put $M_0 = 0$ in the equation number (10); it is just $\tilde{p}_{xx}$-$\tilde{p}_{tt}$. So, that is your wave equation; the one that was covered in the last week; so that is your wave equation second order hyperbolic wave equation; for a stationary medium.

And if you insist time harmonicity, what are we going to get? You are going to get the Helmholtz equation; one dimensional Helmholtz equation. Similarly, it is very easy to see if you put U naught is 0 here; that is in this term. So, this term will; obviously, go away; so will this and you will just end up with your same thing; this should not be here ok.

So, if you put U naught 0 here; we just get this term and we will just have this; so it is pretty much the same thing. So, from the purely from the point of self consistency; it is doing good. So, equations (10) and (11); then are basically equations for the one dimensional wave propagation in ducts carrying uniform mean flow.

To derive the solution of this equation; we are going to do some algebraic manipulations that will begin the next lecture with the solution of one-dimensional wave equation including convective mean flow of x; that is a solution equation (10) or (11) equivalently. And then talk about each term; whether it is the waves that propagates in the forward direction and the wave that goes in the negative x direction. So, yes; so I guess we will stop here.

Thanks for attending and stay tuned for the next lecture.