Muffler Acoustics - Application to Automotive Exhaust Noise Control
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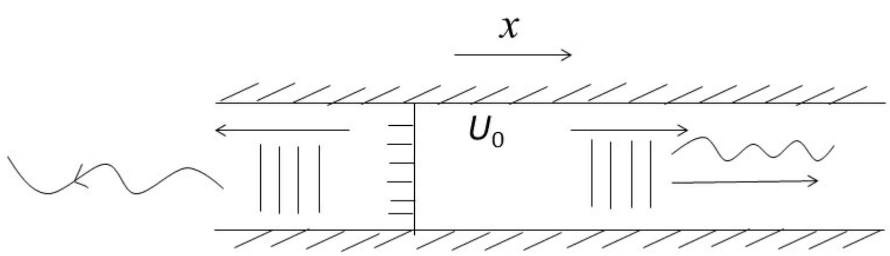
Lecture - 07
1-D Acoustic Wave Equation in Ducts carrying Uniform Mean Flow: Solution

Welcome to the second lecture of week 2; for the Muffler Acoustics course. So before we move ahead, let me bring out the last slide of the last lectures, in which we form finally had the expression for the governing differential equation; for a duct which is carrying a uniform mean inflow along the positive *x* direction.



$$\frac{\partial U_0}{\partial x} = 0$$

$$M_0 = 0.15$$

$$U_0 \ll C_0\,, M_0 = \frac{U_0}{C_0} \ll 1$$

$$\tilde{p}_{xx} - 2\frac{M_0}{C_0}\tilde{p}_{xt} - \frac{\tilde{p}_{tt}}{C_0^2} M_0 \cdot M_0\tilde{p}_{xx} = 0$$

$$(1 - M_0^2)\tilde{p}_{xx} - 2\frac{M_0}{C_0}\tilde{p}_{xt} - \frac{p_{tt}}{C_0^2} = 0 \qquad (10)$$

$$(C_0^2 - U_0^2)\tilde{p}_{xx} - 2U_0\tilde{p}_{xt} - \frac{\tilde{p}_{tt}}{C_0^2} = 0$$

$$(U_0^2 - C_0^2)\tilde{p}_{xx} - 2U_0\tilde{p}_{xt} - \frac{\tilde{p}_{tt}}{C_0^2} = 0$$

$$\tilde{p}(x,t) = \tilde{p}(x)e^{j\omega t} \qquad (11)$$

So going back, this was a system, it was positive x direction mean flow along the positive x direction. And we ended up with equation (10) and (11); in slide number 9 that we have here.

So the first set of equation; equation (10) is obviously in M_0 and all those things are non dimensional form.

This is yet another form, so we will try to derive the solution of equation 10 or equation 11 in this in this lecture, which is going to be a relatively a short lecture. We will just focus on derivation of that, as one emphasized that in the limit.

We will get a stationary wave equation; the equation for a wave propagation in the stationary medium.

So, the solution of that is; very well known, so what is that? Let us; let me write it down.

We will get

$$\tilde{p}(x,t) = Ae^{j(\omega t - k_0 x)} + Be^{j(\omega t + k_0 x)}$$

I am so sorry. And this, of course, is for your equation; Helmholtz equation right. So, this is what it is.

Now, if we insist time harmonicity say in the equation number 10.

$$\frac{d^2\tilde{p}}{dx^2} + k_0^2\tilde{p} = 0$$

$$\frac{d^2\tilde{p}(x,t)}{dt^2} = (j\omega)^2\tilde{p}(x)e^{j\omega t} = -\omega^2\tilde{p}(x)\, e^{j\omega t}$$

I keep making mistakes here I am so sorry. It is tilde; so if we insist that what do we get?

$$(1 + M_0^2)\frac{d\tilde{p}}{dx^2} - 2\frac{j\omega t}{C_0}M_0\frac{d\tilde{p}}{dx} + k_0^2\tilde{p} = 0$$

So if we substitute this particular thing back in equation (10), it is very easy to see; we will obviously get a ordinary differential equation of the form

$$(1 - M_0^2)\tilde{p}_{xx} \ - 2jk_0 M_0 \tilde{p}_x + k_0 \tilde{p} = 0$$

Where p tilde here of course, is just now function of x. Because, like I said, we assume time harmonicity; so this is understood. So I am not writing it again. So, this is the equation that we must solve. So, what do we do now? How to solve this equation (12)? The idea is that just carefully look back in the solution here. So, this is the wave that propagates in the positive direction and this in the negative direction. So, let me also go back to this figure.

So here we see the flow is along the positive x-direction, so waves that propagate along the positive x direction they actually they their velocity like I said, like I mentioned at the beginning of my lectures; lecture 1. You are not just doing this, you are doing this, this, this, this, this.

So imagine the particle being somewhere here; where I am pointing. So, it is getting there is a net displacement along the x direction, but it is also doing this; it is also doing this. So, as a result, the waves that propagate along the direction of the flow, so their velocity is increased by a term U_0.

So, you have the equivalent velocity $U_0 + C_0$ the waves that propagate in a direction opposite to the flow; flow is like this.

So, if you propagate like this, you are going against the stream. You know you are fighting the current. So the net velocity will be $C_0 - U_0$ of such a wave. So how will it affect the solution? How will it manifest in terms of wave speed and all that?

So, getting back to the solution that we have here, so we see the spatial part; nothing happens to the temporal part still. We need to worry about the space. That is basically your; you know this term minus k naught x and this one. So this is a term that will get modified.

How will it get modified? So, we will; let me use the space here.

$$k_0^+ = \frac{\omega}{C_0} = \frac{\omega}{C_0 + U_0} = \frac{\omega}{(1 + M_0)C_0}$$

The speed has increased, so it becomes

$$k_0^- = \frac{\omega}{C_0 + U_0} = \frac{\omega}{C_0(1 + M_0)}$$

Similarly, so this you see this beautiful connection this goes somewhere here.

So, this is nothing but the intuition and we see mathematically that intuition, this intuition or this idea will work. So similarly, if we get here, k the waves that propagate in the negative direction will be omega by C_0 - U_0. So I will put a minus sign somewhere here; and this then will become omega by

So, let me write it more cleanly in the ensuing slide, so one of the solutions will

$$\tilde{p} = \tilde{p}(x) \qquad (12)$$

$$\tilde{p}_1(x) = A_1 e^{j\left(\omega t \;\; \frac{k_0 x}{1+M_0}\right)} \qquad (13a)$$

And, what about the waves that propagate in the direction opposite to the flow. So what will happen to that?.

$$\tilde{p}_2(x) = A_2 e^{j\left(\omega t + \frac{k_0 x}{1-M_0}\right)} \qquad (13b)$$

So there will be plus sign somewhere here like this.

So, let me just put these two things; so in order to verify that our idea works; or a solution works. What it will be sufficient to show, demonstrate that the solution given by equation (13 a); that is this one. If we substitute here, the left-hand side should be 0 should identically satisfied.

Let us do it, because once we do it for equation (13a), it is pretty evident and (13b) will also satisfy. So, in that case the complete solution will be the sum total of that; we will probably work around that. But, basically let us do a again a term by term derivation.

So, I will leave aside this x and t and I will basically that what I mean to say is. Let us

$$\tilde{p}_1(x,t) = A_1 e^{j\omega t} + e^{\frac{-jk_0 x}{1+M_0}}$$

$$\frac{d\tilde{p}_1}{dx} = -\frac{-jk_0}{1+M_0} e^{\frac{-jk_0 x}{1+M_0}} (14a)$$

$$\frac{d\tilde{p}_1}{dx^2} = -\left(\frac{-jk_0}{1+M_0}\right)^2 e^{\frac{-jk_0 x}{1+M_0}} = -\frac{-k_0}{(1+M_0)^2} e^{\frac{-jk_0 x}{1+M_0}} (14b)$$

So this is what we are going to get alright. So, let me box it and let me call this as equation (14a) and this as equation (14b),

So, because the complex exponential particular factor that I underline and then what we need to do is basically put this equation as (14a) and b right in (12). So that is all that is required really and we will obviously see that this factor, what I will just underline this thing this actually can be taken common; once we put this thing in equation (12).

Basically the idea is that if we substitute questions 14a and b in the equation (12); the resultant equation would probably be looking something

$$(1-M_0^2)\left(\frac{-k_0^2\tilde{p}_1}{(HM_0)^2}\right)-2jM_0k_0 \times \frac{jk_0}{HM_0}\tilde{p}_1+k_0^2\tilde{p}_1$$

$$=k_0^2\tilde{p}_1\left\{\frac{1-M_0}{1+M_0}-\frac{2M_0}{1+M_0}+\frac{1M_0}{1+M_0}\right\}$$

So, here actually it is more of p_1, where p_1 was your this thing.

So that is because this factor keeps on happening, so basically the point is to take p_1 common and actually also k_0^2 will happen. So, I will take,

So, actually there will be a minus sign somewhere here because, you are taking we have minus so we carefully look.

Minus will be here, I will take this thing common; so taking

$$=k_0^2\tilde{p}_1\left\{\frac{-1+M_0-2M_0+1+M_0}{1+M_0}\right\}$$

So little bit of algebra would basically, show that this is or the final case scenario is; so you can multiply that and this of course, will go away. The cancellations are pretty evident and eventually, it

$$=k_0^2\tilde{p}_1\times 0=0=RHS$$

of your equation (12).

So, eventually what we saw that we substitute this in equation (12). This indeed is a solution and so I actually leave it as an exercise for the students to show that equation (13b) is also solution of equation (1), but this then represents the wave that travels at the speed little smaller or little lesser than the sound speed; because of the convective having some mean flow is propagating in opposite direction and this is also in solution.

So the complete solution then any guesses, how will it look?

$$\tilde{p}\ \ (x,t) = \tilde{p}_1(x,t) + \tilde{p}_2(x,t)$$

$$= \left(A_1 e^{\frac{-jk_0 x}{1-M_0}} + A_2 e^{\frac{jk_0 x}{1-M_0}}\right) e^{j\omega t} \quad (15a)\text{s}$$

So, eventually we are going to get things like here. Basically, this is what it is the solution of the wave equation in a duct which carries uniform mean flow.

If t increases x must also increase, but it increase at a faster rate that the spatial wave number; spatial wave number

$$k_0^+ = \frac{k_0}{1-M_0} k_0^- = \frac{k_0}{1-M_0}$$

some non zero small quantity probably less than 0.15 or something for duct acoustics for automobile mufflers. So its anyways much more than your this thing which is k naught which is this thing. So this is something like your k plus and this is a k minus this is the complete solution then. What about the particle velocity? That is also something that we need to figure out.

$$\tilde{\rho}_t + \rho_0 \widetilde{U}_x + \tilde{\rho}_x U_0 = 0 \qquad (2a)$$

$$if\ \ U_0 = 0$$

$$\tilde{\rho}_t + \rho_0 \widetilde{U}_x = 0 \qquad (2b)$$

$$\frac{1}{C_0^2}\tilde{p}_t + \frac{\tilde{p}_x}{C_0^2} U_0 = -\rho_0 \widetilde{U}_x$$

$$-\frac{1}{\rho_0 C_0^2}\{\tilde{p}_{tx} + \tilde{p}_{xx} U_0\} = \widetilde{U}_{xx} \qquad (9)$$

So the particle velocity, I leave it as an exercise to show that probably we can use this one where you can use your Euler equation; momentum equation to derive the particle velocity, but the simple idea would be to show that particle velocity can be related to acoustic pressure like this.

It is the same; there is nothing different; but it will be something.

Basically, this equation is let us number this as 15,

$$\widetilde{U}(x,t) = \frac{1}{\rho_0 C_0}\left(A_1 e^{-jk_0^+ x} + A_2 e^{jk_0^- x}\right)e^{j\omega t} \quad (15b)$$

So, these are the solutions of acoustic propagation, 1 dimensional planar wave incorporating the convective effects of mean flow. So, I guess we will stop in this lecture here and worry about the 3D effects in the next set of lectures.

So, in our next lecture we will focus on for the first time we are going to consider the very general case of a rectangular duct in which the waves can propagate in both x well x, y and z direction or probably you. Since, we are considering x here *xyz*; we can reorient the coordinate system and probably when take the *z* along the axis of the duct; and *x* and y are the transverse dimension.

So,*xy* and *z* is something like perpendicular to this plane *xyz*. So, the full 3 D solution for the acoustic pressure field will be derived.

And of course, this will be first; we are going to talk about development of the 3 D Helmholtz equation, without mean flow. And incorporating mean flow along certain direction. But, then most of the mufflers they are not rectangular for a certain reason. Because the breakout noise is much more for a flatter section like, a rectangular section. Circular sections are much more rigidity.

So, that is why we use circular mufflers. So, muffler circular chambers, so the idea is to basically get the Helmholtz equation for circular cylindrical ducts also. And derive that for the simple case and then derive the general solutions; and then probably, that is going to be a focus for this week at least. And talk about concepts of cut on frequencies, resonance frequencies and so on. This these concepts will be useful in our later analysis.

Thanks a lot.