Muffler Acoustics - Application to Automotive Exhaust Noise Control

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Lecture – 08,09,10
3-D Acoustic Wave Equation in Rectangular and Circular Waveguides: Derivation, Modal Solution, and Concept of Cut-on Frequency

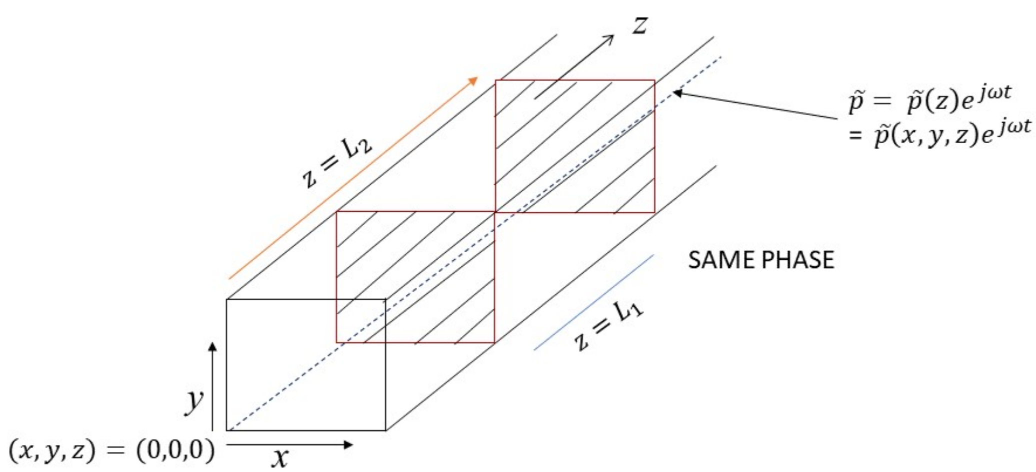
Welcome back to week 2 of this NPTEL course on Muffler Acoustics. So, all this while in the last week which was our introductory week we talked about some basic or simple concepts on wave propagation. You know 1D wave propagation in ducts, some transient solutions and derivation of the 1D acoustic wave equation which is valid in ducts.

So, and then we of course briefly touched upon the effects of mean flow on ducts or on harmonic plane waves that propagate within an infinite duct. We are also I just talked about little bit about boundary conditions, rigid wall conditions and the open-end boundary conditions and all that impedance concepts. Well in real world, however waves are rarely 1 dimensional. Except of course, in ducts which is obviously are focus of this course, very much related to mufflers.

So even within ducts there exist certain frequency based on the cross section of the duct whether it is a rectangular cross section or a circular cross section. There exist an upper bound or upper frequency beyond which wave propagation could no longer be considered planar.

So, 3D effects begin to show up. So, what it means? So, let us consider, you know a duct let me draw a nice photo for you. So here you have this thing, so the idea is that here you have a planar wave. So basically, at any point here, here, here anyway, waves have the same phase. What it means is basically any point you pick on this plane has the same magnitude same value.

So, it is basically a plane surface, now when 3D effects actually one by the way, this is seeing the z-direction. This is saying x, this is a; this is a y-direction we are considering origin to be say somewhere here 0, 0, 0, which is your x, y and z. So, this is a **RECTANGULAR DUCT.**



As I was saying that the only dependence is along the z-direction that is if you consider another plane you know this is let us say this is at z = L_1, this would be at z = L_2. So, you know this is another planar wave front, so the values would be different from what you get from z = L_1.

So, the idea is that the only dependence is now on the z direction. That is if z is changing, then the values will also change. But like I said, there is a; there is a difference between this idealistic or theoretical world and the actual of the realistic environment. In real world beyond a certain frequency, 3D effects will begin to show up. That is to say, the wave propagation or the acoustic say the pressure value.

And of course, we assume time harmonicity. So, the idea is that the waves is no longer propagate or depend only on the z, they will also depend their propagation will also depend on the x and y values. So, what it means is that this surface is no longer be a planar surface.

It would probably have some sort of you know variation across this thing. So, we will probably have a look at some nice mode shape patterns. I draw some nice schematics to drive home the concepts of how the pressure varies across the rectangular cross section or for that matter a circular cross section.

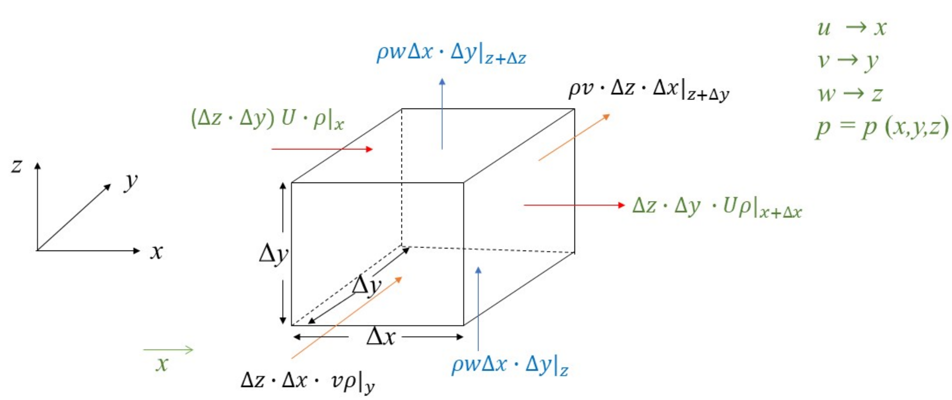
And for that matter, elliptical cross section because mufflers are also elliptical in shape, although it is more complicated. Now, ducts can be subject to rigid wall condition or subjected to lining condition.

So, the purpose of this today's lecture is to present to you all the lecture you know the set of consolidated lectures name 3 to 5, which initially I would begin with derivation of the acoustic wave equation in a Cartesian coordinate system. But involving the other 2 coordinates, x and y and z of course is there.

So that would be your 3D wave equation but in a stationary medium and that will be valid for rectangular ducts. Once we you know implement the appropriate boundary condition whether it is a rigid wall condition or open-end condition. Open-end generally is not used, we use impedance condition but when this set of lectures you stick only to hard walled ducts that is normal velocity is 0.

We will talk about the line ducts later in the probably in the last few weeks or last few lectures of this course, line ducts. So we will worry only about the rigid walled ducts. Now let me just begin to derive the 3D wave equation in the Cartesian coordinates.

And then we can possibly add mean flow term to account to understand all the things pertaining to rectangular duct, concepts of cut on frequency and all that. Before we move onto the circular duct, which is our very important topic that will help us in the later parts of the course.



Cubical Volume Element

Let me draw a system here system something like this kind of a thing. So this is nothing but a **CUBICLE VOLUME ELEMENT OF IN A FLUID** and we see let me use another color and let me actually first also draw the convention. So you know this can be your say the y-direction, this can be your x-direction and this can be your z-direction along the z and y phase. We have the flow that enter here, the one that leaves here.

Note that this is your Δy; this is your Δz. Similarly, this is your Δx, and like this ok. This particular area is Δx into Δy, Δx into Δy. So what we get is your (Δz Δy) area into the density area into the well. We are interested in knowing the amount of fluid that enters this phase per unit time. So if you multiply this by the velocity, say U; the velocity along the x-direction.

So here I must also introduce.

So velocity into the area will give you meter cube per second of the volume flow rate. Now if you want to get the mass flow rate you need to multiply this by the density rho. So all the terms are evaluated at you know at x is equal to let us say it is evaluated at x ok.

Now what do we do after this? We multiply now we also need to worry about the flux that goes from the other phase. So if you are considering this as x. We will do linearization later, so and we will do the linearization part later.

This is the total amount of fluid that enters from the left hand side on the from the you know Δz and Δy phase or z y phase. And amount of fluid that goes on from the opposite side is delta z into ΔU into $U\rho$ at x + Δ x.

Now what happens to the fluid that well comes from the phase z x that is your like this and the one that goes away it is like this. So, this is your Δz into Δx into v along the y direction. It is velocity is ρv and, so this is at y and this is ρv Δz Δx at y + Δy ok. And what about the flux that goes along the or traverse along the z-direction.

So let me use yet another color to simplify things. So, one that enters the volume it is from this from the bottom phase. You know from the bottom phase it enters it from this phase. So basically, by drawing this figure we that we have done is that we have identified in the small control volume cubicle control volume the fluxes that enter from different phases and the fluxes that enter exit from the opposite phase.

So naturally, what the next step is basically to balance out this thing. You know, we are heading towards what is known as.

CONSERVATION OF MASS: we are headed towards that. So what is the difference now?

$$[\Delta y \ \cdot 0z \ \rho v|_x - \Delta y \ \cdot 0z \ \rho v|_{x+\Delta x}$$

this is the difference of the mass; the mass that enters here and the mass that leaves here.

So difference of that will give you the net amount of mass that exit the system. Now if you have a similar thing if you have the mass that goes into the phase here and leave here. So what is that? so that is the difference.

Now we have another thing which is the mass that enters from the bottom and they leaves from the top phase.

$$+\Delta z \cdot \Delta x \;\; \rho_v|_y - \Delta z \cdot \Delta x \;\; \cdot \rho v|_y + \Delta y$$

$$+\Delta_x \cdot \Delta y \;\; \rho_w|_z - \Delta x \cdot \Delta y \;\; \cdot \rho w|_x + \Delta z$$

So Δx, into Δy remember because we are dealing with mass that flux velocity along the z-direction delta z would not be there.

The difference of the flux is that is equal to so this is what have you done basically. We have figured out the net mass inflow net mass inflow through surface of Δv. So this is the net mass influx, the difference is then equal to the net mass that goes away, that is your

$$= \frac{\partial}{\partial t} \left(\rho \Delta x \; \Delta y \cdot \Delta z\right)\Big] \text{ Net mass increase inside } \Delta \text{v}$$

So this is the really the mass that this is the volume delta x into delta y into delta z volume times density kg per meter cube into meter cube that will give you kg right.

So this should be if you take the derivative of that that is basically your kg per meter cube into meter cube. So what we get by this is that we get kg so temporal rate of change, so this is then equal to net mass increase inside Δv. This is the mathematical rendition of the most one of the most fundamental laws that whatever difference of mass we are seeing that comes inside and goes outside.

The sum total of the differences must be equal to the net change of mass within the control volume Δv is nothing but Δx into Δy into Δz this one. So volume times density is the total mass at any instant net mass within that control volume. So time rate of change of that is equal to the net mass in flux through different surfaces of Δv so that is what we get net mass difference.

So this is you know basically what we get. Of course, Δx, Δy and Δz these dimensions are fixed and rho can be changing density. This might change; especially in acoustics we are having a small perturbation thing. So this is what would you produce the acoustic effect. So now let us basically simplify this entire expression. Note that if we divide throughout by Δx, Δy, Δz, what will we get? We will get in the first term

$$\frac{\partial p}{\partial t} = \lim_{\Delta x \to 0} \frac{\rho u|_x - \rho u|_{x+\Delta x}}{\Delta x}$$

$$\lim_{\Delta y \to 0} + \frac{\rho v|_y - \rho v|_{y+\Delta y}}{\Delta y}$$

$$\lim_{\Delta z \to 0} + \frac{\rho w|_z - \rho w|_{z+\Delta z}}{\Delta z}$$

Now, this is clearly you know this is nothing but the partial derivative with respect to the following.

So, let me use another slide and what we will get is basically

$$\frac{\partial \rho}{\partial t} = -\frac{\partial \rho v}{\partial x} - \frac{\partial \rho v}{\partial y} - \frac{\partial \rho v}{\partial z}$$

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho v}{\partial x} + \frac{\partial \rho v}{\partial y} + \frac{\partial \rho v}{\partial z} = 0 \qquad (1a)$$

$$\frac{\partial \rho}{\partial t} + \Delta \cdot (\rho \boldsymbol{U}) = 0 \qquad (1b)$$

So, eventually one form of the continuity equation; so we are dealing with the conservation of mass, so we will get the continuity equation. Or so we will basically get these two equation (1) or same another form let me call this equation (1a) and (1b). So the (1b) is the kind of a more compact form and this is the fully expanded out form.

So u actually let me make it somewhat different from this thing U writing it manually bold, where U is your where U is nothing but

$$U = \begin{Bmatrix} u \\ v \\ w \end{Bmatrix}$$

So you see it is a divergence operator. This guy is a divergence operator, so ρU is a vector, because U is a vector defined like I have done here and divergence operator operates upon this vector to get it get back a scalar product.

So basically, we are adding two quantities. Now we can very quickly put the isentropicity condition here, but I guess will probably wait and wait till the derivation of the momentum equation. So in fact, we will use (1a) form, one the form (1a) into derive the more simplified form of the momentum equation. But before we go ahead, let me just point out to you guys that the same thing can also be obtained through an integral approach.

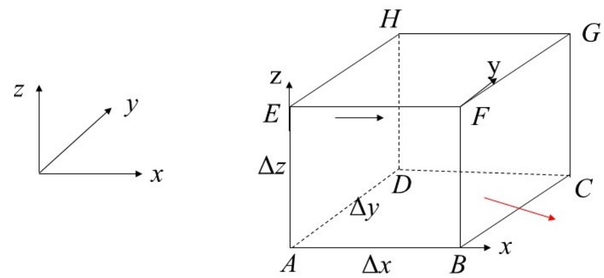
Right now, we took a control volume thing and figured out the partial derivatives and all that. But if we were to do it in a integral derive the integral form of the continuity equation using the same physical principle, the time rate of increase in mass delta *v* is equal to the net mass in flow through surface s.

And get the integral forms of such thing instead of differential forms. We will pretty much arrive at the same thing. So this is a very important concepts I will probably not going to that now. But I guess just leave it here and go to the derivation of the **Momentum Equation**.

So now what we are going to do is that we are heading towards the momentum equation. So this is basically your balance of forces. So let us consider again our cubicle control volume. And let us define a coordinate system which is pretty much the same as was before, so x, y, x along this direction, y along this direction and your z along this direction. So sorry, actually we consider you know the forces.

Let me draw the cube, the unit cube of course, along this direction and you know talk in terms with reference to the phases. So let us say identify the phases. Now we need to first let us be systematic and identify the forces that act along each direction. So here the difference between momentum and continuity equation is that now we are in momentum, we are trying to balance the forces.

So, we will basically end up with not 1, but 3 equations: one for the balance of forces along the either of the directions x direction, y direction and the z direction. Let us consider now the x-direction now. So if we name it as say A, B, C inside vertices D E, this one is F, G, H. So the phase ADHE; the phase ADHE the force that was the pressure that acts along this direction is the or the force that acts along this direction is pressure P.



$$\underline{ADHE} \qquad \underline{BCGF}$$

$$P\Delta y\Delta z|_x \; - \; P\Delta y\Delta z|_{x+\Delta y}$$

Let us work with P or the total pressure, so P into your Δy, Δy is nothing but this one and Δz is this one and Δx is this.

So the difference of the force is equal to what this is the difference. So this is pressure P into the area that will give you force that acts on the phase ADHE and this is the thing that acts on the phase BCGF. So the difference of the two forces will give you the net resultant force and that must be sort of balanced by the momentum.

So on your left phases left and this is left phase, this is the right phase. And you can consider this is the bottom phase, this is the top phase and this is the front phase and the back phase. So we are working with left and right phases as far as x direction is concerned.

So what is the momentum that enters force due to the momentum flux? Because momentum is carried by the moving fluid as it goes through Δv. So the momentum inflow into the control volume that also accounts for the force.

The force due to momentum influx is then basically given by the one that acts through this ADHE phase

$$\rho U \cdot U\Delta y \cdot \Delta z|_x$$

Now area times velocity area times velocity is your meter cube by second.

$$\frac{kg}{m^3}\frac{m}{s} \cdot \frac{m^3}{s}$$

So basically, again like we did in the last week, we basically this has dimensions of force kg meter by second square.

So this is your Q dot kind of a thing, volume flow rate and the volume flow rate and this is the density and the area density, which kind of conveys the mass times this thing.

So in a way, if you group this particular term together ρ times this thing and U is here, so it is like your mass. This will give you your mass, m dot m dot times velocity. So naturally, mu is the momentum and if you see this m dot is with respect to time, so per second will come. So that is why you are getting the units of force was basically this denotes force.

So I would basically club this thing as the following,

$$(\rho U)U\,\Delta y\cdot\Delta z|_x \;-\; (\rho U)\cdot U\,\Delta y\cdot\Delta z|_{x+\Delta x}$$

So we get this kind of a thing now, the one that leaves outside. What do we do now with this particular thing? What we will see is that, basically, these are the forces that act on the phases ADHE. So we need to basically add this term and this term and subtract this term or this term. Because minus sign anyways is already there. So this is the force that acts in the other direction.

$$\rho U\cdot\Delta y\cdot\Delta z|_x - \rho U\cdot\Delta y\cdot\Delta z|_{x+\Delta x}$$

So these are this is the like I said, force carried by the momentum influx and the force acting in the opposite direction due to the momentum out flux and the difference pressure they act on the opposite direction on opposite phases.

$$P\Delta y\cdot\Delta z|_x - P\Delta y\cdot\Delta z|_{x+\Delta x}$$

So this is equal to the net change of momentum inside the control volume.

$$=\frac{\partial}{\partial t}(\rho\Delta x\,\Delta y\,\Delta z\cdot U)$$

$$kg\;\cdot\frac{m}{s}$$

So mass into the velocity along the x direction. Remember u was remember from slide 3; u was the velocity along the x-direction. So this underlined thing is your kg; density times the volume

And this is your meter by second, so kg into meter by second. So basically this gives your μ term like a momentum and time rate of change of momentum is a force. So basically what we do is now we divide throughout by the unit volume that is your Δx,Δy into Δz product of that.

$$\frac{\partial \rho u}{\partial t} = -\frac{\partial p}{\partial x} - \frac{\partial \rho U^2}{\partial x}$$

So after that there is no problem, but there are a few more terms which we did not say did not mention now. But it is true that the momentum is carried along the x-direction momentum flux by these fluxes.

But then there is also your force that along the direction that is carried due to the because some component along the direction by the momentum in flux through each other phases from the front facing and the back phase as well as from the top bottom phase in the top phase. So you get you know basically all those term let us see how we get it.

So, basically the component of the momentum along the x direction due to the force, due to the influx along perpendicular to the page that is through the phases AEFB is basically we will come back to this is the last slide AEFB is given by

$$AEFB \qquad\qquad\qquad\qquad CGHD$$

$$(\rho u)v \cdot \Delta x \cdot \Delta z|_y (\rho u)v \cdot \Delta x \cdot \Delta z|_{y+\Delta y}$$

$$ABCD \qquad\qquad\qquad\qquad EFGH$$

$$(\rho u)w \cdot \Delta x \cdot \Delta y|_z (\rho u)w \cdot \Delta x \cdot \Delta y|_{z+\Delta z}$$

And similarly, along the phase CGHD; CGHD we will be getting the force that. Similarly, the force due to the momentum influx top phases is basically (ρ u).

Well, the idea is how are we getting all these things? Because, you know again, the same thing, the momentum here we are basically the momentum in flux is nothing but it has the dimensions of Q dot, but this is along the your y direction, that is your this direction.

So because your area times the velocity and so if you multiply this by the density, we will get kg or mass along the y direction and that is mass into the velocity will basically give you the momentum along the direction, u being the velocity along that direction. Now, again what we need to do is that idea is fairly simple. We need to add those relevant terms in here. What I will do is that probably get rid of this part just for now and put this thing at the bottom.

$$\rho u \ \cdot u\Delta y \cdot \Delta z|_x - \ \rho u \ \cdot u \ \Delta x\Delta z|_{x+\Delta x}$$

$$P\Delta y \cdot \Delta z|_x - P\Delta y \cdot \Delta z|_{x+\Delta x}$$

$$+\rho u \ \cdot v\Delta x \cdot \Delta z|_y \ - \ \rho u \ \cdot v \ \Delta x\Delta z|_{y+\Delta y}$$

$$= \frac{\partial}{\partial t}(\rho u \ \Delta x\Delta y\Delta z) + \quad \rho u \ \cdot w \ \Delta x \cdot \Delta z|_z - \rho u \ \cdot w \ \Delta x \cdot \Delta y|_{z+\Delta z}$$

$$\frac{\partial \rho v}{\partial t} = -\frac{\partial p}{\partial x} - \frac{\partial \rho U^2}{\partial x} - \frac{\partial \rho U}{\partial y} - \frac{\partial \rho U w}{\partial z}$$

And, you know simplify expressions we will get the following. If you were to simplify this further, or probably what we could do is that

$$(\rho v)_t + (\rho u^2)_x + (\rho uv)_y + \ (\rho uw)_z + Px = 0$$

$$(\rho u^2)_x = \frac{\partial}{\partial x} \ (\rho U^2)$$

Now remember the suffixes signify differentiation with respect to that coordinate. So, when I am writing, you know when I am writing something like this it would mean is the following and similarly for y and z and all that kind of a thing. Now what do we do now?

$$(\rho v)_t + (\rho u^2)_x + (\rho uv)_y + \ (\rho uw)_z + Px = 0$$

We have to basically, you know just like we did for the 1D equation, we will have to cleverly, you know group term. So, let me expand this guy these terms out. So this is like this, and

$$\rho_t U + \rho U_t + \ (\rho v)_x \cdot U$$

$$+\rho v \cdot u + (\rho v)_y \cdot U$$

$$+\rho v \cdot Uy$$

$$+(\rho v)_y \cdot U$$

$$+\rho w \cdot Uz + Px = 0$$

$$+\rho Uz\} \cdot U$$

Now we have to clearly identify term, this of course is plus; this of course is plus. Now if we take you know u common here, u is common here, u is common in this term as well here and as well as here. So you know this is nothing, but we get basically of continuity equation

$$\{\rho_t + (\rho U)_x + (\rho v)_y$$

$$+(\underline{\qquad\qquad}) = 0$$

Entire thing multiplied into u plus all the other terms is equal to 0.

So this clearly this term cancels away why? Because of your continuity equation that, we derived here. Remember I said just a while back that (1a) is the form or probably (1b) whatever it is we will use and we are indeed using that form. So basically the remaining terms what are those then?

$$\rho U_t + \rho U \cdot U_x + \rho v \cdot U_y + \rho w \cdot U_z + \rho_x = 0$$

$$\rho\{U_t + U \cdot \cancel{U_x} + v \cdot \cancel{U_y} + w \cdot \cancel{U_z}\} + \rho_x = 0 \qquad (2a)$$

This is basically a vector equation in the sense that all your u vs they are all vectors.

So perhaps a more compact way of writing this term is then your rho times remember this operator. This is basically your thing that acts upon this, so there this particular term is

$$\rho\left\{\frac{\partial(\cdot)}{\partial t} + \frac{U\partial(\cdot)}{\partial x} + \frac{v\partial(\cdot)}{\partial y} + \frac{W\partial(\cdot)}{\partial z}\right\} + \frac{\partial P}{\partial x} = 0$$

So the compact way of writing the above equation is also basically

$$\rho\frac{\mathrm{D}U}{\mathrm{D}t} + Px = 0 \qquad (2b)$$

So you know either of that is your momentum equation along the x-direction. Similarly, after this little bit of well rigorous thing, we will get the momentum equations along the y-direction as

$$\rho\frac{\mathrm{D}v}{\mathrm{D}t}+Py=0 \qquad (3)$$

$$\rho\frac{\mathrm{D}w}{\mathrm{D}t}+Pz=0 \qquad (4)$$

$$\rho\frac{\Delta w}{\Delta t}+Pz=-\rho g$$

Now actually, this would be 0; but if you consider gravity like I said, it will be minus rho naught times g because this is the body force remember. But we probably ignore it in this course, so we will set these things to 0. So you know basically another thing that we must point out here that we are not actually considering the viscous terms at all. We are considering in viscid equations because, you know we are not considering attenuation due to the viscosity your damping, you know in air.

We are considering only the linear only the Euler equations 0 viscosity in viscid flows. And once we get your momentum equations, then we will probably linearize it to get the linearized Euler equation in case flow is there and then from that we can combine it to get your Helmholtz equation or the wave equation.

Let us see how we go about it. So this is your one equation, this is your another equation. And this is let me call it you know this as (2 a0, (2 b), (3) and (4). Let us not consider the mean flow for a while. Let us work only with stationary flows ok. So we will probably need to have a closer look at this thing. So if you remember last discussions when we begin on the process of linearization.

So right now, we have not done any linearization. Just equations Euler equations because in viscid flows is there. What we need to do in the context of acoustics or small signal equation is basically you are start to linearize the terms. So basically we immediately recognize v or uvw they are small quantities, there is no mean flow u naught, or capital U_0, that is no flow along the x direction and this term goes away.

Because when you linearize it you get

$$U = U_0^2 + \widetilde{U}$$

$$V = \tilde{V}$$

$$w = \widetilde{w}$$

Now we will we will assume that, and these two terms you know would also be gone. And obviously, this time also product of two second order quantity. We have seen that derivative does not, you know if I non dimensional arguments we seen in the last lecture that it does not matter, they are also very small quantity.

What essentially, we are left with let me use another slide is basically. And we assume that this particular term this is 0.

$$P = p_0 + \tilde{p} \mid \frac{\partial p_0}{\partial x} = 0$$

That is, there is no gradient of the ambient pressure, ambient pressure is what it is? It is constant. And you know it would be a second order quantity,

$$\frac{\partial \tilde{p}}{\partial t} + \rho_0 \Delta \cdot \overleftarrow{\tilde{U}} = 0$$

Similarly along the y direction these are perhaps the most simplified cases that you will come across, so we get this ok. So this is your set of linearized momentum equations or linearized Euler equations. What about the continuity equation? You know, remember we had a thing, you know this one. So the arguments are pretty much like what we did in the last week for the 1D test case.

In 1D test case, of course this guy was not there, this guy was not there right. But now they are there, so we can easily show that you know rho is again your rho naught plus rho tilde v and w are nothing, but this flow is considered only along the z-direction, or x-direction let us say. When we do all those things we will get the following equation (5).

$$\frac{\partial\tilde{p}}{\partial t}+\rho_0\frac{\partial\tilde{U}}{\partial x}+\rho_0\frac{\partial\tilde{V}}{\partial x}+\rho_0\frac{\partial\tilde{w}}{\partial x}=0 \qquad (5)$$

$$\rho_0\frac{\partial\tilde{U}}{\partial t\partial x}+\frac{\partial\tilde{p}}{\partial x^2}=0 \qquad (6)$$

$$\rho_0\frac{\partial\tilde{v}}{\partial t\partial y}+\frac{\partial\tilde{p}}{\partial y^2}=0 \qquad (7)$$

$$\rho_0\frac{\partial\tilde{w}}{\partial t\partial z}+\frac{\partial\tilde{p}}{\partial z^2}=0 \qquad (8)$$

So this is what is going to be, so we have your you know equations (3 5, 6, 7 and 8; 5 to 8). So these are the equations that constitute equations for getting the wave equation.

Now we will also immediately use isentropicity condition. Now recall from the last weeks lectures,

$$\frac{\tilde{p}}{\tilde{\rho}}=C_0^2\tilde{\rho} = \frac{\tilde{p}}{C_0^2}$$

So what we need to do now? We need to substitute,

$$\frac{1}{C_0^2}\frac{\partial^2\tilde{p}}{\partial t^2}+\rho_0\frac{\partial^2\tilde{U}}{\partial x\partial t}+\rho_0\frac{\partial^2\tilde{v}}{\partial y\partial t}= +\rho_0\frac{\partial^2\tilde{w}}{\partial z\partial t}=0$$

So we get this thing and immediately from this equation we can prove a lot of simplification. You know what we can do? We can differentiate with respect to x. Now similarly, in this guy we can differentiate with respect to time.

So you know what will happen now is that this term, this term, this term would you know from these are the equations, we can just transpose each of the terms underlined here on the right hand side and substitute for the underlined terms here.

$$\frac{1}{C_0^2}\frac{\partial\tilde{p}}{\partial t^2}=\frac{\partial\tilde{p}}{\partial x^2}+\frac{\partial\tilde{p}}{\partial y^2}+\frac{\partial^2\tilde{p}}{\partial z^2}$$

So we will get minus here, so minus can be transposed on the other side to get your this thing, so we will get Laplacian.

$$\nabla^2 \tilde{p} = \frac{1}{C_0^2}\frac{\partial^2 \tilde{p}}{\partial t^2}$$

You are familiar wave equation in full 3D coordinates where,

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial xz^2}$$

but the Laplacian operator in 3D Cartesian coordinate system. Remember, so we have finally, arrived at the 3D wave equation but without a flow. Now we can easily convert this to your wave equation by insisting that,

$$\tilde{p} = \tilde{p}(x, y, z)e^{j\omega t}$$

So, once you do that, we get,

$$(\Delta^2 + k_0^2)\tilde{p} + 0$$

So this is your much sought after Helmholtz equation in 3D coordinate. Let me expand this out completely for you, so we get,

$$\frac{\partial^2 \tilde{p}}{\partial x^2} + \frac{\partial^2 \tilde{p}}{\partial y^2} + \frac{\partial^2 \tilde{p}}{\partial z^2} + k_0^2 \tilde{p} = 0$$

So finally, this is the result that we are waiting for a long time. Now we can actually cleverly guess what will happen before we move on from here, there are lots of things that we could do.

One thing that will come to mind is obviously to go ahead and solve get the variables separate form. If you can analyze from your get back to your engineering mathematics days, your undergraduate days where you should study you know the hyperbolic equations which are separable inseparable coordinate system.

Cartesian system is a coordinate system where solutions can be separated out in xyz coordinates. And then you can combine individual solutions and get the particular modal aspect, modal solution and then expand it out completely in terms of Fourier series this kind of a thing.

So well, that is pretty much what we will do here. So one way is to actually go ahead and do the variable separate from we will do that in this lecture. Now the other thing is of course including mean flow effects while retaining the Cartesian coordinate system. So one way, you know because of lack of time, what I would do in this course in this lecture here as far as 3D things are concerned.

Just include the additional term that would arise when flow u naught is there along the x-direction with the understanding that you guys will also have a look at the lecture on the last week. So you can you know kind of relate to that is relatively straight forward. And then that equation is your Helmholtz equation in Cartesian system with flow along a certain direction.

That is to say the equation that governs wave propagation in a rectangular waveguide, but with the mean flow along the certain along the axis of the duct; then we can go ahead and impose the boundary conditions and all that. And the other thing is, of course change the cross-section so all this while we are worrying about the rectangular cross-section.

What we could do is that get the Cartesian the circular cross section as well, but we will probably sort of worry about that later and then right now just focus on different cases of rigid walled rectangular ducts with or without mean flow. So now let us quickly move ahead and talk about the additional terms that would be present when we consider the effects of mean flow.

$$\rho_0 \nabla \cdot U + \frac{D\tilde{p}}{Dt} = 0$$

$$\rho_0 \left(\frac{\partial^2 \tilde{U}}{\partial x \partial t} + \frac{\partial^2 \tilde{v}}{\partial y \partial t} + \frac{\partial^2 \tilde{w}}{\partial z \partial t} \right)$$

Because we are assuming mean flow along the x-direction, so with that understanding,

$$+ \frac{\partial^2 \tilde{p}}{\partial t^2} + U_0 \cancel{\frac{\partial^2 \tilde{p}}{\partial x \partial t}} = 0$$

$$\frac{D}{Dt}$$

$$= \frac{\partial}{\partial t} + \frac{U_0 \partial}{\partial x}$$

So we will get this kind of a thing ok, so this is the continuity equation with flow along the x direction, but 3D continuity equations. Clearly by setting U_0; if you set this particular term to U_0. We will get basically your get back the equation for without mean flow. Now what is your momentum equation?

Your momentum equation considering you know the mean flow is the expanded-out version,

$$\rho_0 \left(\frac{\partial^2 \widetilde{U}}{\partial t \partial x} + U_0 \frac{\partial^2 \widetilde{U}}{\partial x^2} \right) + \frac{\partial \tilde{p}}{\partial x^2} = 0$$

$$\rho_0 \frac{\partial^2 \tilde{v}}{\partial t \partial y} + \frac{\partial^2 \widetilde{w}}{\partial y^2} = 0$$

$$\rho_0 \frac{\partial^2 \widetilde{w}}{\partial t \partial y} + \frac{\partial^2 \widetilde{w}}{\partial y^2} = 0$$

So we can basically proceed ahead with differentiating with respect to your *z*. Because eventually we would like to get the Laplacian so *y* and this is square and so this we could, you know we can simply transpose out your terms this term here, this time here.

And we can basically differentiate both the sides with respect to *x*. So once we do that, the top equation U_0 is mean flow velocity which is constant in space and time. So we will get basically nothing but this thing ok. We probably what we can do is sort of a highlight out this terms and we get this.

We can now again go back to your continuity equation and differentiate with respect to time. So once we do that, we get this, as usual we what we can do is basically replace each of the underlined terms here.

We can replace that with these particular quantities. All these terms, this particular thing as well as the other term $\partial^2 \rho$. All this can be simplified further when we make use of this thing.

For example, this particular term $\partial^2 \widetilde{U}$ all this thing. So this can be eventually be expressed in terms of the other variables and where you have only derivatives with respect to the space. So all this term will be can be translated on the other side.

So we can replace eventually this particular term, then possibly we need to basically worry about eliminating one of these this particular term. So that we can sort of easily do once in fact, this as we know is given by

$$\tilde{\rho} = \frac{\tilde{p}}{C_0^2}$$

So even this term rho tilde can be eliminated and expressed in terms of p. So all we will be left is basically $\partial^2\tilde{u}/\partial^2 x$, which we can again eliminate using the first order equation as we exactly did the week 1 lecture.

So, I am probably not going to do show you those computations. I am just going to write down one final expression that will be valid when you combine the equations that are shown here on this slide on the following slide here into the Helmholtz equation incorporating the convective effects of mean flow which will written in a little compact manner is given by this thing.

$$\left\{\frac{D^2}{Dt^2} - C_0^2\Delta^2\right\}\tilde{p} = 0$$

$$\Delta^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

So now basically now if you expand this out, it is a bit tedious and we can get to it and so D/ Dt this square would mean this operating on some something. What it means is and this then further operates on each of the quantities, operates on this plus D/ Dt of u naught I can take out and this further operates on this thing.

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \frac{U_0\partial}{\partial z}$$

$$\frac{D^2}{Dt} = \frac{D}{Dt}\left(\frac{D()}{Dt}\right)$$

$$= \frac{D}{Dt}\left(\frac{\partial}{\partial t} + \frac{U_0 \partial}{\partial z}\right) = \frac{D\partial()}{Dt\partial t}$$

$$+U_0 \frac{D}{Dt}\frac{\partial()}{\partial z}$$

So if we go to the next slide to simplify the matrix. Let me work out the algebra little quickly.

$$\left(\frac{\partial}{\partial t} + U_0 \frac{\partial}{\partial z}\right)\frac{\partial}{\partial t} + U_0\left(\frac{\partial}{\partial t} + U_0 \frac{\partial}{\partial z}\right)\frac{\partial}{\partial z}$$

$$= \frac{\partial^2}{\partial t^2} + U_0 \frac{\partial^2}{\partial z \partial t} + U_0 \frac{\partial^2}{\partial t \partial z} + U_0^2 \frac{\partial^2}{\partial z^2}$$

$$= \frac{\partial^2}{\partial t^2} + 2U_0 \frac{\partial^2}{\partial z \partial t} + U_0^2 \frac{\partial^2}{\partial z^2} = \frac{D^2}{Dt^2}$$

So this will happen and then what we will get is basically your familiar term which we saw in the first lecture of this week or probably in the second lecture I. So this thing finally, so this is your operator.

Now once we substitute this guy in the above equation, once we substitute this guy in this particular equation, we will get the fully expanded out form.

So that would be

$$= \frac{\partial^2 \tilde{p}}{\partial t^2} + 2U_0 \frac{\partial^2 \tilde{p}}{\partial z \partial t} + U_0^2 \frac{\partial^2 \tilde{p}}{\partial z^2}$$

$$-C_0^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\tilde{p} = 0$$

So we can finally put it in a more formal form. What we could do is? Simplify the algebra to get the following. This will become,

$$= \frac{\partial^2 \tilde{p}}{\partial t^2} + \frac{\partial^2 \tilde{p}}{\partial y^2} + (1 - M_0^2) - \frac{2U_0}{C_0^2}\frac{\partial^2 \tilde{p}}{\partial z^2}\frac{\partial^2 \tilde{p}}{\partial z \partial t}$$

$$-\frac{1}{C_0^2}\frac{\partial^2 \tilde{p}}{\partial t^2} = 0$$

This is exactly what we saw a few lectures back for the 1D case and so this also occurs in your 3D case. So this is the final form and of course, once we insist on sort of time harmonicity let us do that.

$$\tilde{p} = \tilde{p}(x, y, z)e^{j\omega t}$$

Let us assume the harmonic response that,

$$= \frac{\partial^2 \tilde{p}}{\partial x^2} + \frac{\partial^2 \tilde{p}}{\partial y^2} + (1 - M_0^2)\frac{\partial^2}{\partial z^2} - 2jM_0\ k_0 \frac{\partial \tilde{p}}{\partial z}$$

$$+k_0^2 \tilde{p} = 0$$

This is your equation with including the convective effects of mean flow and consider 3D propagation.

So this is really the expanded out version of the more condense version that was presented just a while back. So when clearly we can see when M_0; it reduces back to the original case when there is no flow. Now we have derived this equation without really going into the control volume, at least for this case, but we did it for the case of a zero-mean flow.

Now it basically our job to find out the modal solutions, so the way that we will follow the way the procedure that will follow is basically to find out the modal solutions in terms of M_0 that is for the rectangular waveguide. Infinite rectangular waveguide carrying a mean flow say along the positive z direction; so it will involve the modal solution will involve terms which has M_0 or dependence on the Mach number small Mach number.

And for the more special case of M_0 0 we can get that we can get the case of a rectangular waveguide with zero mean flow. And here of course, we are considering rigid wall waveguide right. So, we learn how to employ the rigid wall boundary conditions, so this is clearly an equation that can be separated in the different in the 3 orthogonal directions x, y, z. This is in the separable coordinate system. So what we really need to do is that assume

$$\tilde{p} = X(x)\, Y(y)\, Z(z)$$

$$-k_0^2 \tilde{p} = \frac{X''(x)}{X} + \frac{Y''(y)}{Y} + \frac{Z''(z)}{Z}(1 - M_0^2) - 2jM_0 k_0 \frac{Z'(z)}{Z(z)} + k_0^2 = 0$$

So you know you can assume this particular guy is k_x^2 and same thing can be done with this thing. This can be minus k_y^2 ok, now there is clearly no flow along this x- and y-directions. But we want basically a solution that is bounded and it is obviously not 0; clearly, if kx is 0; then we get sort of trivial solutions U_0 sort of interested that or things like that or and if variables.

Let us say this is lambda minus lambda and if lambda happens to be well negative here or just if you can just consider this entire guy as lambda; lambda happens to be positive here. Then we get one part is exponentially increasing, other part is exponentially decreasing. So it is unstable, unstable so it is especially going kind of a wave that is not permissible. So what we would do? Basically is that we will assume kx is real number and k_x^2 would be also be real and then -kx were would be always be negative. So once we do once we insist on such kind of a variation for x- and y-directions.

$$X''(x) + k_x^2 \ X(x) \ = 0$$

$$Y''(y) + k_x^2 \ Y(y) \ = 0$$

Leading now remember

$$m^2 + k_x^2 = 0 \qquad \Rightarrow \qquad m = \pm ik_x$$

$$n^2 + k_x^2 = 0 \qquad \Rightarrow \qquad n = \pm ik_x$$

$$X(x) = C_1 e^{-ik_x x} + C_2 e^{-ik_x x}$$

$$X(x) = B_1 \cos k_x x + B_2 \sin k_x x \qquad (1)$$

$$Y(y) = B_3 \ cos \ k_y y + B_4 \ sin \ k_y y \qquad (2)$$

And this similarly would be this kind of a thing.

So now we have to employ the boundary condition. So what does boundary condition mean? Basically, we have the pressure component along the x and y directions vary according to say equations 1 and 2 ok. We know from our Euler equations this is something like this. So we will get your pressure something like this.

$$\rho_0 \frac{\partial U}{\partial t} = -\frac{\partial p}{\partial x} = -\frac{\partial X(x)}{\partial x}$$

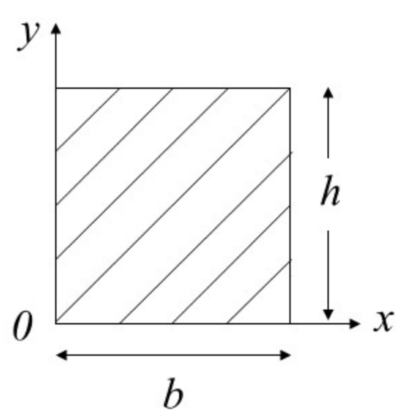
Clearly, this is a partial derivative so the other components is a pressure the y component as well as the z component should not really bother us. So, whatever their forms are we just probably would be interested in the x part. So that would probably we have to take a derivative of your x function. So once you do that, what will we get?

$$= -\{k_x B_1 \sin k_x x \ + B_2\, k_x x \ cos\ k_x x\}$$

Now we need to; we need to simplify this thing further, k x obviously, we have taken out and let me take minus k_x out.

$$+k_x\{B_1 \sin k_x x \ + B_2\, cos\ k_x x \quad \}_{x=0}$$

Now if you remember, our cross section was something like this. You know x y so this is 0, this was say let me call this as breadth is b, height is *h*.



So now, once we simplify the what are the what are the boundary conditions?

At $\quad x = 0$

$$U = 0$$

$$U = 0 \ \rightarrow x = b$$

$$B_2 = 0$$

So now other condition is invoking this particular guy and we will see,

$$B_1 \sin k_x b = 0$$

$$\sin k_x\, b = \sin n\,\pi$$

$$k_x b = n\,\pi \;\; k_x = \frac{n\,\pi}{b},$$

$$n = 0, 1, 2, 3, \ldots.$$

It is not quite trivial in the sense that when you put k x is equal to 0, so n is equal to 0. That means k x the wave number along that direction for the lowest order mode that is 0.

But still you will have wave propagation and we see this is a special case when n is 0; when an n is 0. Now in a similar manner, if you do the same exercise along the y direction, we will eventually end up getting something like this. So this is I made a small mistake I guess. So I think it was something like a this is b because that is what we agreed upon. And this was also b; b and remember our a height of the duct was h.

So we will get,

$$k_y = \frac{m\pi}{h} \;\; m = 0, 1, 2, 3, \ldots.$$

So this is again a special case and we will see why it is special. Now, after having derived the wave numbers k and k x and k y; so these are the transverse wave numbers or the wave numbers along the x and y-directions respectively and they signify higher order modes. So you know once we so let me just write down the X component of the solution.

So remember B2 based on this condition B2 was 0. And similarly, we can also get that B4 is 0. So eventually we are getting solution in terms of cos or cosines not sins for a rigid walled duct. That is, we are here we have rigid wall condition or velocity normal velocity 0 along perpendicular direction.

Soft wall in a hypothetical case or in a lining highly absorbent lining and that will set pressure to 0. So we will get sins in terms of cos wave numbers would be also different. So here we are getting something like, let me if I were to combine the solution. Let me write down something like this so this will be say some constant

$$X(x)\, Y(y) = C_{mn} \cos\frac{n\pi}{b} \cos\frac{m\pi}{h}$$

This was it where m and n ranging from 0, 1, 2, 3, 4. Now we have this kind of a thing. Now the thing is that if we go back to this equation the way the moment we try to we agreed on separating the variables ok. If we put $-kx^2$, $-kyx^2$ here and let us assume that this guy let me use another color.

This is minus kz wave number along the z direction or the propagation. Remember in the 1D case the underline this term and this term were not there. And of course, the other terms were there when you have a mean flow. Now suppose even if suppose you set mean flow to M_0.

So it is clearly evident that for the simple case of no mean flow and 1 dimensional propagation, simplest asset can get; I am sorry kz^2 was equal to k_0^2 or k in or kz is equal to +- k_0 that is wave propagating along the positive z-direction and negative z-directions are respectively ok.

So clearly, we initially we figured out that such waves are called planar wave fronts is not it. So now when you have kx and ky to be 0 they are not trivial solutions. They might be trivial for that equations but when we put kx and ky is to be 0 we will get planar wave fronts. That is, there is no variation along the x and y direction.

The only variation in the pressure values that can happen is the is when the wave propagates from one z-distance say z_1 vary it moves from z_1 to z_2. The variation is only along the upstream or the downstream direction there is no variation along the cross section of the rectangular duct. That is if you have I just drew the cross section of the duct while back. So along the cross section the pressure remains constant for the case kx is equal to ky is equal to 0 or when mn = 0.

So, anyhow what is obviously important is that we have to substitute k_x^2 and k_y^2. And also assume well insist that is what we do that this guy is -k_z^2 we will keep this thing here. And what we will do? So what we will do? We will put this -kz thing and one multiply carry about our regular business of multiplying this.

And remember we are getting a term k naught square here. This is 0 and then there is another term which is your related to damping.

$$-k_x^2 - k_y^2 - k_z^2(1 - M_0^2) + 2jM_0k_0k_z$$

$$\frac{Z''(z)}{Z(z)} = -k_z^2 \qquad + k_0^2 \; = \; 0$$

Now let us consider the case of zero mean flow. Let us consider, $M_0 = 0$. We are introducing for the first time the concept of cut on frequencies or cut off frequency. So when you have such,

$$k_0^2 = k_x^2 \; + \; k_y^2 + k_z^2$$

So what is this cut on frequency? We will soon figure out in different context.

So we can

$$k_z = \pm \sqrt{k_0^2 - \left(k_x^2 + k_y^2\right)}$$

The guy under the bracket that is well, you can consider this like this. So when clearly

$$\text{if} k_0^2 < k_x^2 + k_y^2 \quad \rightarrow \; negative$$

Then the term under the bracket is negative that would mean you have. What does it mean? It means kz is your imaginary is purely imaginary ok. So that would mean remember now k z is multiplied by i times kz that is what you know that is what it will happen.

You know once you insist this kind of a variation, where I am; when I am pointing. I try to put on your kx in ky value and try to solve for try to figure out what is kz in basically, once you figure out your k x k y in terms of the boundary condition and substitute that.

And then try to solve for your then try to solve for your zz dash z dash dash by z or z dash z phi variation along the direction and for that you must come to the step of assuming this kind of a; this kind of variation what I am presenting here and then when you need to solve for kz.

So we will see that if k z is purely imaginary then it is multiply, then the solutions are either spatially growing, or exponentially decaying. So basically, such waves are called with the solutions which decay exponentially in space, such waves are called evanescent waves. In our case, what will happen for a rigid walled waveguide? What will happen is that kz will be equal to this is of course, the case without any mean flow. We get

$$k_z = \pm \sqrt{k_0^2 - \left(\frac{N\pi}{h}\right)^2 - \left(\frac{M\pi}{h}\right)^2}$$

So now we would get this kind of a solution.

$$\frac{2\pi f}{C_0} = \frac{w}{C_0} = k_0 \geq \sqrt{\left(\frac{N\pi}{h}\right)^2 + \left(\frac{M\pi}{h}\right)^2}\ (m,n)$$

$$(0,0)$$

If it is less, then it does not. Now let us revisit the discussion when we

$$M = n = 0$$

So what do we get under such a situation? We will get k naught is always greater than 0. So whatever k naught remember this. So for whatever frequency of excitation, you have for a rigid walled cylindrical I am sorry rectangular waveguide rigid walled rectangular waveguide.

Whatever frequency of excitation is there, the first mode is the plane wave mode, which always propagates. By definition it is called plane wave because there is no variation along the rectangular cross section what we discussed a while back. So basically your 00 mode that is a plane wave mode that is the lowest mode and it will always propagate.

Now the depending upon the geometry, typically we consider b is equal to h. That is a square waveguide that is waveguide with the square cross section so that things become easier to you know analyze all that need not be square. So then we can figure out what is the first mode to get started or to get or to be propagated.

So now we got this kind of a solution or the expression for the cut on frequency to summarize this particular concept of cut on frequencies. We will have a wave number and n m this particular m n mode which is nothing but let us say C m n is associated. I just had written it y by h you have this.

$$(M,N) = C_{MN} \cos\left(\frac{Mxx}{b}\right) \cos\left(\frac{Nxy}{h}\right)$$

$$k_0 > \sqrt{\left(\frac{Mz}{h}\right)^2 + \left(\frac{Nz}{h}\right)^2}$$

So now this mode will propagate if your k_0 is greater than this thing cut on that shows. In such a case; in such a case, this mn^{th} mode will be cut on or it will no longer be evanescent. If k_0 is less than this value. Then such waves are called evanescent waves they decay specially.

You know you have two solutions e to the power you know something like minus mx or minus m z plus e to the power plus some other constant times e to the power *mz*. The part which is decaying exponentially that is called evanescent parts and so that will not propagate. So that is what is going to happen. What we need to do basically now is derive the full modal solution.

Let us, for simplicity assume that mean flow is not there, that is a stationary medium is there. So what will we get? We need to sort of go back to this guy when the mean flow is not there. Of course, this term goes away and you simply do not have this term. We just have 1 and then we get our regular stuff and that we can substitute in this equation ok. And the solution for this particular equation is simple and we can how to solve this by assuming a certain thing. So we have

$$Z''(z) + k_z^2\, Z(z) \;= 0$$

$$Z(z) = A_1 e^{jk_z z} + A_2 e^{jk_z z}$$

We will have this kind of a thing where kz is given by the expressions that were noted above.

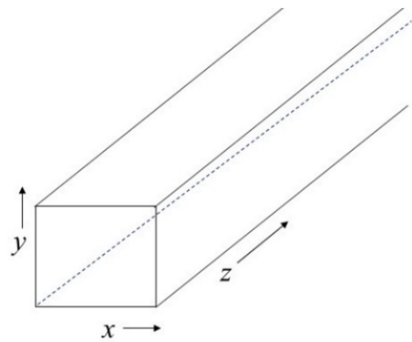
$$p(x, y, z) = X(x)Y(y)Z(z)$$

So under such a situation if we go back to our definition or variable separate form which is your this. And, of course I am not writing the time factor let us assume that it is time harmonic. We are going to get, So under such a situation we will be getting our complete modal solution which will be of the form.

$$= \sum_{M=0}^{\infty} \sum_{N=0}^{\infty} \left(\cos\frac{M\pi x}{b}\right)\left(\cos\frac{N\pi y}{h}\right)$$

$$\left(C_{1,M,N}\, e^{-jk_{z,M,N}} + C_{2,M,N}\, e^{-jk_{z,M,N}}\right)$$

So this is your finally the solution that we get the modal solution, the much sought after solution that we were working so hard to get it is a solution.



That is for rectangular infinite rectangular waveguides that is this direction x is here, y is here. So clearly when you have a pressure variation given by the encircled term.

So that means higher order modes and we have this wave front that propagates along the positive direction, one that propagates along the negative direction. Again derive cut on frequencies in terms of the corresponding wave number. So we just saw that k_0^2 when,

$$k_0^2 \left(\frac{M\pi}{b}\right)^2 + \left(\frac{N\pi}{h}\right)^2$$

$$\frac{z\pi f}{C_0} > \sqrt{(\cdot)^2 + (\cdot)^2}$$

It will be mode will be propagated un attenuated that is the waves will not be cut off will be cut on. So remember we can write this thing as you know 2π f by C_0 and this is greater than this value, is not it?

$$\frac{2\pi}{\lambda} > \pi \sqrt{\left(\frac{M}{b}\right)^2 + \left(\frac{N}{h}\right)^2}$$

$$\lambda < \frac{2}{\sqrt{\left(\frac{M}{b}\right)^2 + \left(\frac{N}{h}\right)^2}} \qquad M \cdot N$$

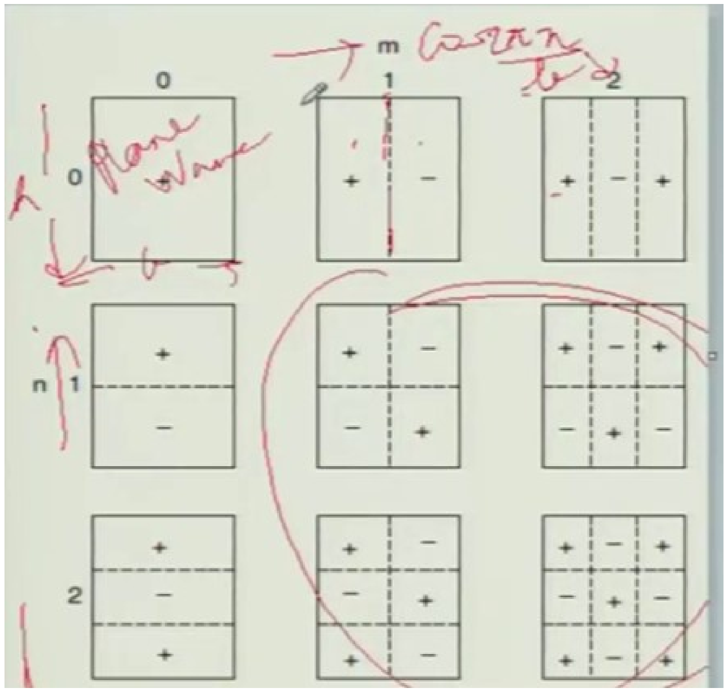
So here we are getting this in the denominator and remember we can take. So this these guys will get cancelled and we will eventually get the following relation.

That means if the excitation wavelength is less than this value then the particular mode m n mode will propagate. This is yet another alternate statement. The reason that I am emphasizing this multiple times is because this is central to our when we will the 3D analysis of mufflers. We will see that although we have infinite solution but no, its not necessary to consider all the infinite terms.

You know had that mean the case it will be practically impossible to do it, so we need to worry only about the first few modes. In particular only those modes will propagate. So at low frequencies only the planar wave is important, nothing else is important. Of course, the first few evanescent modes if we consider that are solution accuracy might improve for some muffler configurations at low frequencies.

But only beyond the cut on frequencies we may beyond the first cut on frequency we may have to consider modes other than the planar wave mode. So the idea behind you know considering this thing is that only the first 2 terms in the modal solution is important.

Its not necessary to consider all the terms ok. So that is of great importance here so that is the case now. Clearly, we can also derive the there is another concept that needs to be done that is the volume velocity concept. For that, we have to understand the velocity along the z directions.



But before that, let me present to you figure which shows you the modal distribution or the mode shapes of a rigid walled rectangular waveguide. So clearly, this is your plane wave mode. This is your plane wave; you know plus means that you know across this rectangular cross section.

As we promise, this is your b and this is your height ok. So across this cross section you have your acoustic quantity to be constant. Now when you go for a modal m, well m is equal to 1; but n is 0. So you get you know the acoustic pressure is the region is divided into 2 halves of which are in opposite phase.

So this is given by your function cos of you know cos of m is 1 here so 1 πx / b. At x is equal to 0 you are getting whatever you are getting and as we go along the duct x is equal to when it is b; then it is minus 1 here. So there will be some transition point at x is equal to b by 2 that is half the distance where this is 0, so this is your nodes.

What is important here is the nodal pattern that is the dotted line you see that is exactly that is the line where exactly the pressure is identically 0. So basically, that line divides the rectangular cross section into 2 half of opposite phases. So, at all times they will be so if you consider the point here and immediate point here they will be exactly opposite in a phase.

As we go for higher solutions if you consider 2 here that is this particular solution and major things from here. So basically, then gradually the phase is shifting from plus to negative and again going back, so you have two nodal lines. So, it divides the rectangular cross section into regions of alternating phase from plus to minus; minus to back to plus.

So, these are called modes along the x-direction. Similarly, you have modes along the y-direction opposite phase along the y-direction. And then these modes all the set in you can arrange this in a matrix. So if you go along the direction, these are modes along the y-direction but there is no variation along the x.

Similarly, here you have variation along the x. But no variation on the y, these modes are all cross modes where there is simultaneous variation of acoustic pressure along the x- and y-direction both. So this obviously comes out very nicely from applying the boundary condition as we saw.

And they are nothing but lots of the trigonometric functions, so this is very classical solution know very well known. So now what we probably can do is have a look at the solution when we have mean flow.

But before that, it is good to also talk about your velocity along the z direction, which is this is important when we basically introduce get our self a little familiar with 3 dimensional approaches like your piston driven approach or things like that.

So from the Euler equations we can easily show I am not going into the details that u z n the particle velocity for the m nth mode along the z direction is given by this variation. And when we simplify stuff we will get your this kind of a thing.

$$U_{z,M,N} = -\frac{\frac{\partial p}{\partial z}}{j\omega\, \rho_0}$$

$$= \frac{k_{z,M,N}}{k_0}\{C_{1,M,N}e^{-j(-)} + C_{2,M,N}e^{-j(-)}\}$$

$$\cos(-)\cos(-)e^{j\omega t}$$

Now what is important is that along a given if you fix z is equal to some value and if we now want to find out the volume velocity or mass velocity. So that is nothing but if we integrate as we saw from our last few first couple of lectures and also the last two lecture of week 1.

We defined mass velocity as rho times u cross section area into u. In this case it is u m n but then you know this s this is this for the case when u was constant. But what is do we do for the case when your u z m n is varying? So obviously we need to integrate over the cross section where ds is the rectangular cross section as we discuss.

$$V = \rho_0 \int U_{z,M,N}\, dx\, dy$$

So, since it is going from,

$$\left(\int_0^b \cos\frac{M\pi x}{b}\, dx\right)\left(\int_0^h \cos\frac{N\pi x}{h}\, dy\right)$$

It is It is fairly straight forward to do the integration and verify for ourselves. But in case it is

$$= 0 \quad M \neq 0, \quad N \neq 0$$

so basically, we are going to get

$$bh \quad M = N = 0$$

We are just going to get that when what is this really means that is this if either is important. Then basically the contribution in the mass velocity by the higher order modes m n mode is identically 0. That is high order modes if you integrate the velocity contribution by them over the cross-section area that is going to be 0. And this you will see we can show that eventually in our discussions in the later weeks of our course that it is probably one of reasons why a plane wave is a very good approximation.

That is when we assume at a certain area discontinuity if you are probably going to see in the week 5 and week 6 we are going to see. We are going to since that volume mass velocity is constant across the certain need at discontinuity. So all these things will be sort of useful at least for a cylindrical rectangular waveguide and same discussion can also happen for a valid is valid for a cylindrical waveguide.

Now let us quickly write down the solution for the case when you have a nonzero mean flow M_0 is there and then we will sort of end our discussion for the rectangular waveguide.

$$k^{\pm}_{z,M,N} = \frac{\mp M_0 k_0 + \sqrt{k_0^2 - (1-M_0^2)\left\{\left(\frac{M\pi}{b}\right)^2 + \left(\frac{N\pi}{h}\right)^2\right\}}}{1-M_0^2}$$

So clearly, when M_0 is 0 we can get back our familiar expression. We can see this term will not be there. This term will be obviously 0 and this will be just be unity, so you will get back original stuff. Now if that is not the case let us analyze. So what basically it does that mean flow when you have a mean flow M_0 this will the affect on mean flow is to basically lower the cut on frequency.

$$k_0 > \sqrt{1-M_0^2}\sqrt{()^2 + ()^2}$$

When this is the case, then m nth mode will propagate in the presence of a mean flow. So we can sort of do the algebra and figure out then when,

$$\lambda > \frac{2\sqrt{\left(\frac{M}{b}\right)^2 + \left(\frac{N}{h}\right)^2}}{\sqrt{1-M_0^2}}$$

And frequency can also be figured out using that and this solution then can be put in the modal solution thing to get as the final thing. But basically, the modal solution then will sort of look like will be very similar to your case when you have zero mean flow.

And so, here

$$p(x,y,z,t)$$

$$= \sum_{M\pi}^{\infty}\sum_{N\pi}^{\infty} \cos\frac{M\pi x}{b}\cos\frac{N\pi y}{h}$$

$$\left(C_{1,M,N}\, e^{-jk^{+}_{z,M,N}} + C_{2,M,N}\, e^{jk^{-}_{z,M,N}}\right)$$

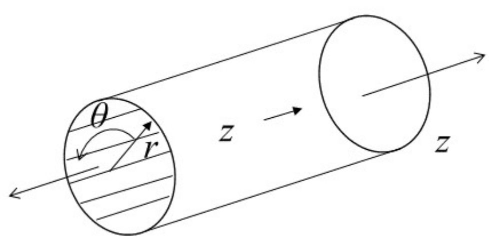
we are going to get. This is what it is and then how we figured out the solution of k z I leave it as an exercise to the students here. And shortly we are also now going to now remember these lectures are combined the lectures 3 to 5 in the week 2 are combined and we are going to have a massive thing.

So shortly now we are going to have the discussion on cylindrical waveguides so for that we need to understand you know it is involved. It is fairly more mathematically involved and we need to what we probably do is that directly get on with the conversion of Laplacian into the cylindrical polar coordinates.

And, then get to the variable separation form and probably not get into the derivation of the Bessel function which is invariably comes up when you deal with circular cylindrical geometries in almost all physical problems.

And then our solutions will proceed, and things will become very interesting because then most of our mufflers are circular in shapes in cross section. So it is important to understand that of course there are some variations also.

Let us start the circular or circular cylindrical waveguide. So what do we need to do now? So **Circular Cylindrical Waveguides**,



$$x = r\cos\theta$$

$$y = r\sin\theta$$

so it is basically, instead of a rectangular cross section we have a circular cross section. So what is the; what is the motivation of doing this? So because most of the silencer is the mufflers they are or of circular cross section like I mentioned while back.

So this is your muffler cross section alright and this is the direction that is extending ok. Now the thing is we need to first introduce some transformation. Remember Laplacian in the regular Cartesian you know system was given by this thing. Now, in order to get to the case of a circular cylindrical waveguides. What we need to do?

We have this kind of a thing. We introduce this transformation z is the z is this direction which is the same and this is your r direction and this is your θ direction. So x and y are related to r and θ by the relations that are presented here on the top.

$$\Delta^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

So using this using our Jacobian matrices and you know basically using conventional rules of partial derivatives that is if you want to do this.

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial}{\partial \theta}\frac{\partial \theta}{\partial x} \quad = \frac{\partial}{\partial r}\cos$$

This can be written something

$$= \frac{\partial r}{\partial x} \; 2r = 2x$$

we can figure out from that relation. Basically its nothing $r^2 = x^2 + y^2$.

In a similar manner; in a similar manner, what we are going to get here,

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial r}\frac{\partial r}{\partial x} \; + \; \frac{\partial}{\partial \theta}\frac{\partial \theta}{\partial x}$$

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial r}\cos + \frac{\partial}{\partial \theta} sin\theta$$

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial r}\sin - \frac{\partial}{\partial \theta}\cos\theta$$

So this is what we are going to get. And if we do the same thing for the y variable we are probably going to get here in probably there will be a minus sign here and here we will probably have your sin theta. This will probably be your cosθ or something like that not very sure of the symbols here regarding the signs.

But you can verify the thing is that we would like to figure out what is this and what is this in terms of the partial derivatives of with respect to r and with respect to theta. Of course, there will be there might be terms which multiply there may be terms involving r or theta that multiply to the partial derivative like this. But after some algebra we can you know radially show that your Laplacian the cylindrical polar coordinates.

Basically your this will become so this is your Laplacian in cylindrical polar coordinates. So if you have to get rid of this term, we will just have polar coordinates or circular polar coordinates. But we have our 3D case, so we will have to retain z term as well. Now what do we; what do we do with this thing?

$$\frac{\partial}{\partial x^2} \; and \; \frac{\partial^2}{\partial y^2}$$

$$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

$$\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2}$$

So, basically the idea is we remember our Helmholtz equation; our Helmholtz equation was nothing but

$$(\Delta^2 + k_0^2)\tilde{p} = 0$$

$$\frac{\partial^2 \tilde{p}}{\partial r^2} + \frac{1}{r}\frac{\partial \tilde{p}}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \tilde{p}}{\partial \theta^2} + \frac{\partial^2 \tilde{p}}{\partial z^2} + k_0^2 \tilde{p} = 0$$

So this is your wave equation in cylindrical polar coordinates. Obviously, there is a lot of work to be done before you arrive here. So I am just presenting the final product. What is going to happen? Cylindrical polar coordinates also allows for the separation of variables.

So what it means is that we can probably

$$p(r, \theta, z) = R(r)\, \theta(\theta)\, Z(z)$$

Let us for simplicity, consider only the case of stationary medium cylindrical duct circular cylindrical duct with a stationary medium that,

$$M_0 = 0$$

We can obviously add more terms.

Now this particular form has to be substituted back in here in the above in this equation. And when once we do that I will sort of avoid or probably fast forward to the separable form. What is going to happen? The most important thing is your radial solution for in the radial coordinates.

Now you know this is not as simple as your x and y coordinates because you will probably get

$$\frac{d^2R}{dr^2} + \frac{1}{r}\frac{dR}{dr} + \left(k_0^2 - k_z^2 - \frac{M^2}{r^2}\right)_R = 0$$

This is called your Bessel's equation very famous equation that appears in almost all physical problems like I mentioned a while back.

When we solved things in cylindrical polar coordinates or circular you know polar coordinates. So here in this case we will get $k_0^2$of course and k_z^2which is nothing but your $Z''(z)$. We will you see that soon and m square is arising when we do the separation in the theta direction.

So this is like this is multiplied. Now this is one equation

$$Z''(z) + k_z^2\, Z(z) = 0$$

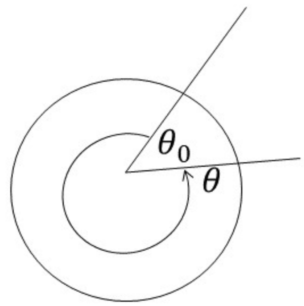
$$\theta''(\theta) + M^2\theta(\theta) = 0$$

So it is a solution of this obviously there are different ways to write.

Let us begin with the θ case because then that would help us get the solution of Bessel function. So we have sorted out with the transfer direction and then z direction is straight forward. So θ-direction obviously it is trivial to see that this is nothing but your trigonometric function.

$$\theta(\theta) = C1\cos M\theta + C_2\sin M\theta$$

But, well n can n be integer well for a circular case or n can; or n can be any real number or is it confined to be only integer. Well, we have a circular duct and if we fix our theta system here. So when we go back and come around so theta it increase from 0 to 2 pi.



$$M = integar$$

So if you consider any generic theta here, theta naught if we go around and come back theta naught plus theta 1 plus 2 pi will be the same. So basically what I am trying to say is that pressure. $\tilde{p}(\theta) = \tilde{p}(\theta + 2\pi)$

So what that would mean is that m must be an integer. Otherwise this condition will not be satisfied. No, its pretty straight forward if you let us say if you like if there is no variation M is 0; then that is about it.

Well, you know you are just getting unity, but when m is 1 you can easily verify cos θ and cos of θ+2π. They are the same thing with m = 2, 3, 4 and the same arguments will hold for the sin function. But if you try it for any other integer you know maybe try 1 by 4; 0.3 or whatever it is half. You will probably not see this is fairly straight forward to figure out, because of periodicity requirement m must be an integer.

There are actually cases when you have some semi circular duct semi circular muffler. Or you have problems in applied physics where you applied maths and all that where you have wave propagation in a pi kind of a sector or its a like a vibrating diagram of a π kind of a shape.

Then of course you need to worry about rigid velocity at fix this at θ =0 and this is θ =θ_0. Then of course m need not been integer, but for our circular case perfectly circular duct where you there is no; there is no partition or there is no restriction to the wave propagation along the circumferential direction angular direction.

Then an m has to be an integer it cannot be anything else. Otherwise there might be a non integer discrete values based on the boundary conditions along the angular direction. Right now there is no boundary condition along the radial direction except for periodicity.

So I will write it in board periodicity in theta direction ok. So m is integer another way well this you can write this in you know also in different way, which is basically,

PERIODICITY in θ direction

$$C_3 e^{-jM\theta} + C_4 e^{-jM\theta}$$

So we are we should be happy with m and now that is why let me write down your Bessel's equation again. So you know this is a vast vast topic, m I am writing as the suffix here because you are trying to solve for m. And this is a vast topic which arose at least a century back from the time of Bessel more than a century back I would say.

$$\frac{d^2R_M}{dr^2} + \frac{1}{r}\frac{dR_M}{dr} + \left(k_0^2 - k_z^2 - \frac{M^2}{r^2}\right)R_M = 0$$

This is your Bessel's equation. What do you do for solving it? You notice the difference when you had let say m is 0. And then, of course this term is not there and, but you still have a different term.

I mean an additional term that is this term it decays. Clearly this is like a damping or gradual decrease if you go along the radial direction. And other terms appear to be familiar to your regular you know sort of wave equation or Helmholtz equation in Cartesian coordinates. So, this is the term that makes it probably quite difficult to solve it.

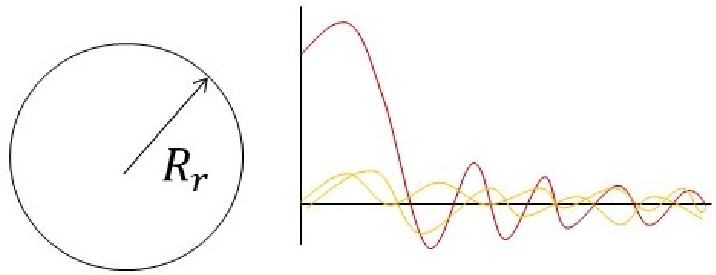
And of course when you have this term $\frac{M^2}{r^2}$ then of course additional difficulties arise. So there are several books written on Bessel Function. One of the very classical books on important books on Bessel Function is standardize on Bessel Function and then lots of engineering mathematics books like Kreyszig, or Arfken Weber or there are number of such important text which talk about give a detailed account on development of Bessel function.

So there are four classes or probably we can sort of begin with the is second order equation really. So you have your two solutions or and these are called ordinary Bessel functions of integer order m. So they are given by

$$1^{st}\ kind\ -J_M(k_r r)$$

$$2^{nd}\ kind\ -N_M(k_r r)$$

So you have your duct just like you have wave variation along the x direction or y you have your kr and r is the radial coordinate. Now this is your ordinary Bessel function of integer order m. Well, because m is an integer and this is a second order differential equation we must also have another solution just like cos sin.



So such a function is called Neumann function. What is the speciality, about these functions? Well like I said, obviously you can you are not, definitely not going to digress into talking about in detail about the properties and or proving the properties of the functions of proving any identities for that matter. Our job here is you just tell talk about perhaps the most salient or most important property. One of that is the ordinary Bessel function.

This is called or the 1st kind ordinary Bessel function of integer order of the 1st kind and this is finite at the origin, where m is equal to 0. That is the zero-order ordinary Bessel function is the 1st kind. That is unity at the origin and it basically MATLAB has inbuilt algorithms to compute this.

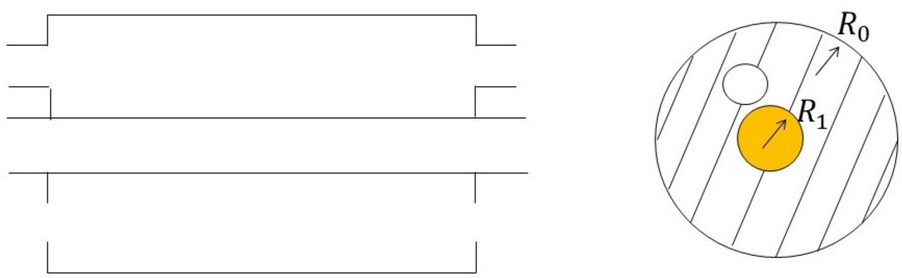
Priority to that, you know they people had to really look up tables before cheap computing power available and then there have been lot of asymptotic developments in formulas for such things. And also, you know this, actually there is a series solution that you can do is a Frobenius series solution that one can obtain for a general order N and we it can be shown when m is an integer.

These two things are Neumann function, and these things are linearly independent and otherwise, if m is a non-integer then it will be like Jm or J - m or something like that. But we will probably worry about these things only and the plot of that will form 0 order will look like this decaying thing. But if you have an m is not equal to 0, 1, 2, 3, 4; it will always start at the origin.

So it will be decaying further and then going like this and something like that. So this is always is finite at the origin Bessel function of zero order. Now what about the N thing? This is called Neumann function, or the Bessel function of 2nd kind named after the person Neumann who first proposed the series solution for this.

So this is also called the Bessel function, ordinary Bessel function of the 2nd kind and what is the peculiarity or special feature of this? This is not finite at the origin; this is infinite at the origin. So now if you have a problem, often you have you know mufflers where you know you have an annular propagation area.

Let me just draw it for you. Suppose you have a rigid pass tube it is called rigid pass tube and you have a muffler like this. And say you have a inlet like this and outlet like this. So if you draw few ways to visualize these things like this and the idea is that only this particular area where I am shading only that is available, this area is not available of wave propagation.



So under such a case where the origin is excluded from the wave propagation domain, there you have to include your Neumann function because then there are two boundary conditions. One is your rigid boundary condition. Well, at r is equal to the outer radius R_0 and here that also at R_1. But in cases where the origin is included, then you are left with no choice but to drop Neumann function altogether. You know the full solution.

$$R(r) = AJ_M(k_r r) + BN_M(k_r r)$$

Now because the origin is included this is infinite this is going to 0. So A times this thing. So what does the finite solution then look like? And of course, we need to apply the rigid walled boundary condition. So how does that happen? Now we can again apply use your Euler equation.

$$\rho_0 \frac{\partial U_r}{\partial t} = -\frac{\partial p_r}{\partial r}$$

So when you put this solution we will get

$$\frac{dJ_M(k_r r)}{dx} = 0$$

We will get this rigid wall condition. So they are roots of the derivative of the roots k_r of this equation or the solution of the equation is basically the roots of the derivative of the ordinary Bessel function of order m. And you know, eventually just like you have m is equal to if we saw for a rectangular duct that,

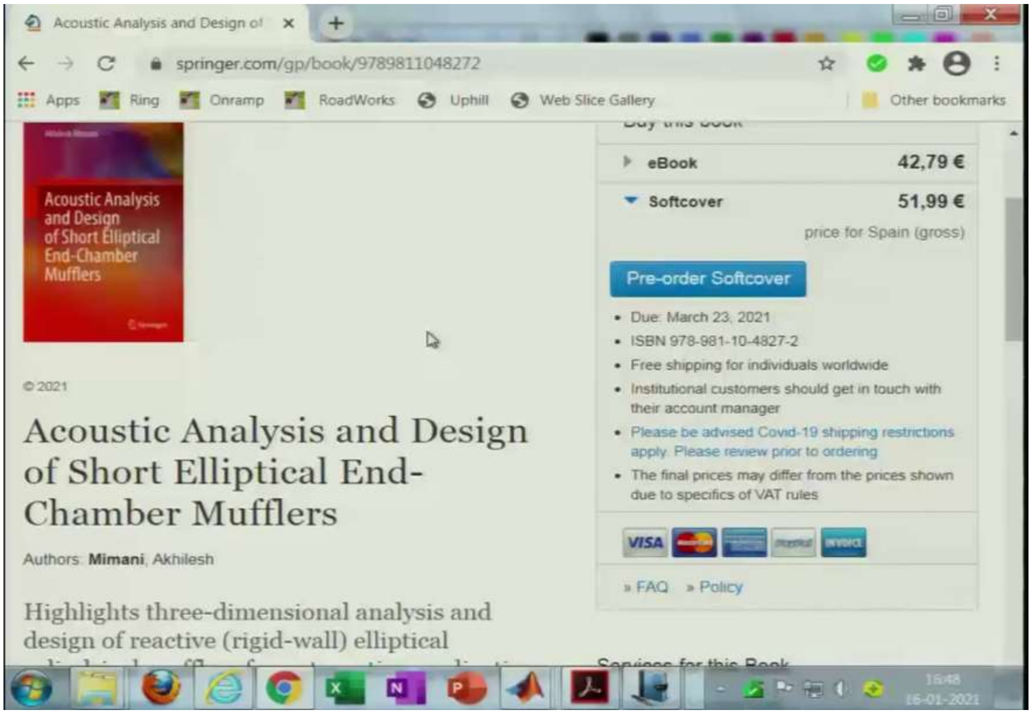
$$\frac{N\pi}{h}$$

$$\frac{M\pi}{h}$$

for a rectangular or a square waveguide.

It's much harder to solve that is usually solved numerically you know this particular equation. And you often see these numerical values at the back of the you know textbooks on you know mathematical physics or applied math's or even in acoustics.

One such place where you can find these values is of course, the end of the book by Professor Munjal Acoustics of Ducts and Mufflers and other engineering applied textbooks. Another place where you can find the detailed development of these documentation of these roots is my upcoming book where I can show you the website of such a thing.



Its on elliptical mufflers acoustic analysis of a specialized muffler configuration. But in the initial chapters a second chapter of this book upcoming book I have talked about I have documented not only, of course the circular waveguide also electrical resonance frequencies of elliptical rigid wall elliptical ducts. And which are very useful, you know to evaluate the translation loss in terms of the analytical formulation.

But, at the very last of 2nd chapter you know I have given a table of this kind where you know m is your order and n is the root number; because I will let me try to explain this thing a little more in details.

M / N	0	1	2	3	--------
0	0	1.84			
1	3.83				
2	7.05				
3					
. .					

.

Generally, you start with 1, but in this case we will begin with to maintain the consistency with rectangular waveguide we will call the 0th root, 1st root, 2nd root, 3rd root you know and so on. Now for this particular equation this guy for if we fix m is equal to 0. So, if we have derivative of the 0th order Bessel function.

$$\frac{d}{dr}J_0(k_r r) = 0$$

You know what are we going to get? So we just I just briefly discuss the variation of the Bessel function of 0th order right, derivative of that would actually start from 0. So if we like something like this you will always find one the 1st root of the derivative of 0th order Bessel function starting from origin or r is equal to 0. So that is why you know your again is consistent with our analysis that you know for if you set k r is 0.

And m of course is 0, then k z will be equal to plus minus k naught or plane wave mode. So regardless of whatever shape you deform the rectangular waveguide into it will always the plane and if you do not line it with any matrices or this is rigid walled thing. Then of course, your plane wave will always be there if there is you know dissipation you know involved.

Now of course, if you go up you have your 1st root here, here, here. These are all your radial modes so you know this is of course and let me write it 0 we will get some this value. I think its probably about 3.83 and so on. Like this you keep this the 2nd root that is your you know where the purple curve intersects this y axis here is 7.05 or something like this do not remember.

Similarly, instead of 0 if you have 1 this guy will start from origin, but you will not consider that root that will be a trivial solution. So you will find out the first derivative is somewhere here, so you will get the first value is 1.84 or something like this. But then there will be sequence of such values because, just like you have m π/b where m can be anything 0, 1, 2, 3 any integer it can be in 10000.

So similarly, derivative of the Bessel function if you fix $(RR_0) = 3.83$, they are infinite number of roots. So basically,

$$\frac{d}{dr}J_1(k_0 r) = 0 \quad \left.\frac{dJ_0(3.83)}{dr}\right|_{R_0} (RR_0) = 3.83$$

So this is a non-dimensional value $k_0 r$ is equal to 3.83 where the derivative of the Bessel function goes to 0.

So such formally or such tables are very nicely explained or documented in lot of books and I have also included as a special degenerate case of an ellipse. Elliptical solution is very its quite formidable and it requires much more detailed consideration it is the topic in itself. So we will not be covering that in the course. It is just a brief mention that I made and those tables of the resonance frequencies of rigid walled elliptical waveguide and also clamped elliptical memories is given in my upcoming book so the interested reader can a value you can read that.

And so the you know the final solution then we will look like the following

$$\tilde{p}(r_1\theta_1 z)$$

$$= \sum_{M=0}^{\infty}\sum_{N=0}^{\infty} J_M(k_{r,m,n})e^{jM\theta}$$

$$\times\,(C_{1,M,N,}e^{-jk_{z,M,N}z} + \; C_{2,M,N,}e^{jk_{z,M,N}z})e^{j\omega t}$$

So now this is what we are going to get. Now kr of course, is fixed by the rigid wall condition. So this is your solution in rigid wall thing. Incidentally there is also solution for soft wall waveguide. But in that case the Bessel function will be evaluated to 0 at the radius R naught instead of the derivative.

$$\frac{dJ_M}{dr}(k_r r)|_{r=R} = 0$$

But for most practical purposes you know all wave guides are rigid in nature. So we have your only this condition. Like let me write it down again for you evaluated at r is equal to 0. So this will give us infinite number of roots for a given order and we need to consider only the first few based on the convergence.

How much is a solution converging? Of course, at higher frequencies, more number of roots have to be considered. Because the ways become even the higher order mode starts getting propagating parts instead of being evanescent. They also contribute in the modal summation, so this is the modal summation solution and k r is you know given by the table is we already know from this condition.

$$k_z^2 = k_0^2 - k_r^2$$

$$k_{z,M,N} = \sqrt{k_0^2 - k_{r,M,N}^2}$$

$$k_0 > k_{r,M,N}$$

So,for mmnth mode it is given by clearly. You know the rectangular waveguide was a simpler thing to consider because of the associated thing that, because of cos and sin functions. Here, we have Bessel functions that is the only difference, but the life is quite formidable in case of this.

So clearly, the k m nth mode will propagate only when k naught is greater than your k r m n. Otherwise, it will not propagate. So what is the one question that arises in our mind is that, you know for a circular waveguide we can probably like to know some you know numerical value or some handy expression instead of just being quite abstract.

So the you know as it turns out like I have mentioned in this preceding table that this is the lowest value. So basically what it means is that the first mode is your the circumferential mode like your with variation like this.

So this is your corresponding to

$$J_1(k_{r,1,0}r)k_{r,1,0}R_0 = 1.8412$$

So at that value we can figure out that is the cut on frequency.

$$k_0R_0 = 1.8412$$

$$\frac{2\pi}{\lambda}R_0 = 1.8412$$

So because we have non dimension values, you multiply both sides by r and kth. We know the value of $k_{r,m,n}$ into r which is 1.84410 mode so k_0R_0 is your this particular thing.

Now we probably can do the algebra two or what is diameter and your what it will give us

$$\lambda > \frac{\pi}{1.84}\ D\ or\ f < \frac{< 1.84}{\pi 0}C_0$$

Or basically your cut on frequency

$$fC = 0.5857\frac{C_1}{D}$$

If you do you will get that is 0.5857 times C_0 / D. We will sort of get that you will get that. So now the thing is that it we might be interested in knowing that how do the waves things they how do they; how do they look like? So for that, I will directly get on with the book by Professor Munjal.

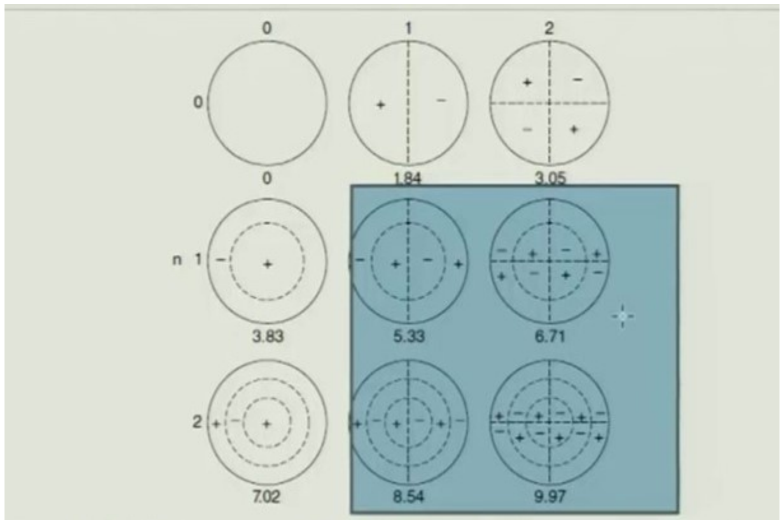


Figure 1.4 Nodal lines for transverse pressure distribution in a circular duct up to m = 2, n = 2 (Reproduced with permission from [5])

It is the its given in here. So we see that it is the planar wave that because that is the 00 mode; that is k r 00 and this is your spatial variation of the acoustic pressure field across the circular cross section of the duct; so each half is an opposite phase to the other half plus minus. So like this we have alternating minus plus minus plus here you have plus minus plus minus plus.

So these modes are they have they are called axi-symmetric modes, also known as radial modes where I am pointing you know in this direction; in this direction. So they have no theta dependence they only vary or they show variation only along the radial direction.

So whatever angle you go, you know you have the same pressure value. But if you very, if you but if you whereas you go along the particular circle, you have 0 variation and this dotted lines are the nodes. So along these are called pressure nodal circles, so at this value sorry at this radius pressure is identically 0 and they are alternating phases.

So similarly, here we have two nodal circles. On the other hand these modes along the row are called circumferential modes. So, its not like they do not have variation along the r direction, but they have but their regions or lines of demarcation which divide the circular region in the regions of opposite phases.

So here you have basically,

So you definitely have variation along the r direction, but then your angular direction has also variation. So the but you know there is no nodal circle there is only they divide they are called nodal radii or nodal diameters which divide the region circular region into regions of alternating phase.

Similarly, here you have cos 2 theta that is j 2; second order Bessel function multiplied by cos 2 theta. So it divides into regions of opposite or quadrants of opposite alternating phases. Like this we can have you know circular circumferential modes along the r direction and these are you where I am highlighting.

These are your cross modes your radial and this variation are both here. So like this we can sort of analyze the wave propagation in a circular duct stationary medium and the modal solution is something that I have already sort of talked about. And the cut on frequencies have also discussed but obviously requiring all these things.

Computing all these things using modal solution I mean incorporating higher order modes will require access to computer. Basically some computing system MATLAB or FORTRAN or some programming language so that you can get the modal solutions and there are lot of interesting muffler configurations. We will begin to talk about that from the end of the or the middle of week 4 where you talk about certain area discontinuities or extended inlet and outlet. There you have such an axi-symmetric configuration. So all these things specially the Neumann function things will be very useful.

But we will not do that in the week 4. We will probably try to do one of those cases in the later parts of the ends of the end part of the course. But for now, I guess its time that I stop this consolidated set of lectures for 3D analysis.

It was just an introduction to the more important topics that we will be covering. So, as usual as I mentioned we will be considering beginning only from the lumped system analysis going to the plane wave propagation, talking about transfer matrix using planar waves. And number of important muffler elements muffler.

How do you quantify the muffler performance and so on? And then talk about perforates and its only towards the end that will sort of begin to use this is a glimpse to the more important things. So with this, I will sort of stop this lecture; lecture 3 to 5 of lecture 2. And I will see you in week 3 where we will talk about all this electro acoustic circuits, lumped system analysis and all that I talked about. Thanks a lot.