Robot Motion Planning
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Lecture – 16
Motion Planning with Kinematic Constraints

Hello and welcome to lecture number 16 of the course robot motion planning. In the last class we looked at basic search algorithms that is when you have multiple paths, which are possible from the initial point to the goal point avoiding obstacles, which path are you going to take? Now, when we are saying the best path, it could be best in terms of the shortest length, it could be best in terms of least energy consumed or it could have some other cost function in terms of optimality.

Now, some assumptions that we have made so far is that one is that the robot is a point robot. So, when we are doing path planning and C-space, the robot is a point robot and it can move in any direction. Now, you know that in the real world, the robot may not be able to move in any direction and that means there are some constraints or kinematic constraints to its motion. For example, a car that you see, automobile in the road.

It can go forward, it can turn at a particular angle, but it cannot go sideways. Not being able to go sideways is because of a kinematic constraint, a normal limit constraint there. So today, what we will look at is when we are going to the real world and there are robots which have kinematic constraints, then how do you do the path plan? So, today's topic is path planning or motion planning with kinematic constraints.

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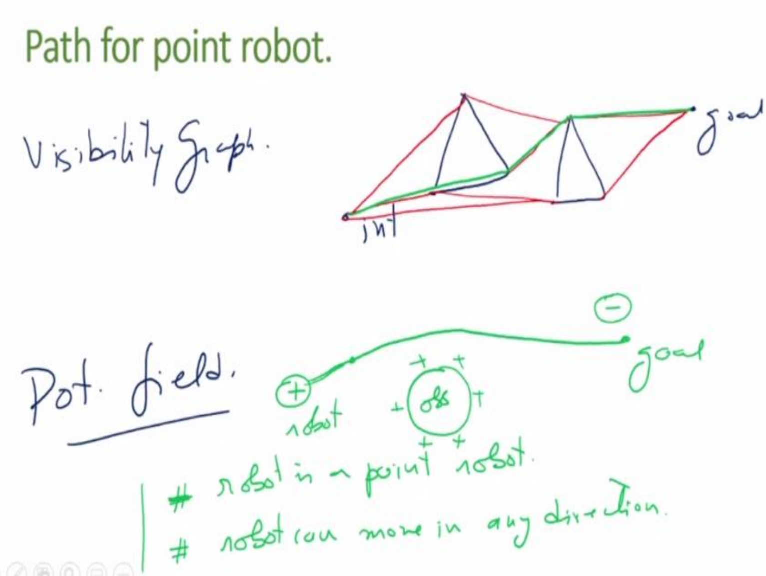

So today we will be talking about motion planning with kinematic constraints. Now kinematic constraints basically mean there are some constraints to motion or to moving in a particular direction. Now, just very quickly explaining what I am talking about here is that when we talked about other methods like visibility graph for example or the potential field method, in visibility graph we have obstacles and then we have an initial start point.

Let us say this is my initial point, this my goal point and what we do is we connect each of the vertices by straight lines. So whichever vertex is visible from one vertex, we just connect it by straight lines and then try and see how can I go from the initial point to the goal point. So, in this particular case, for example this could be a path like this, like this, that could be a path that will take me from the initial point to the goal point.

In potential field method, we saw that we have obstacles, for example this is an obstacle and these plus repulsive field is there on the obstacle. And this is my goal which has an attractive field that is a minus and the robot has plus, this my robot and what we have seen is that the robot is attracted towards the goal, repelled from the obstacle and as a result it goes to the goal. So, at every point in the vector field the robot is subjected to a repulsive field and an attractive field because of which the robot finally goes to the goal point.

Now, here also we are making the assumption number one is that the robot is a point. Now, the second assumption that we are making is that the robot can move in any direction. Now, I am sure you are aware that the robot and many of the robots like mobile robots, cars, they cannot move in different directions at the same time, especially they cannot move sideways. But there are robots which can move in any direction which are holonomic robots.

But there are also robots which cannot move in all directions which are nonholonomic robots. Say for example, a car can go straight and it can turn, but it cannot go sideways. So, today we will be looking at path planning and especially whether it is possible to take this kind of nonholonomic robots and position them wherever you want that is what we talk about in terms of controllability.

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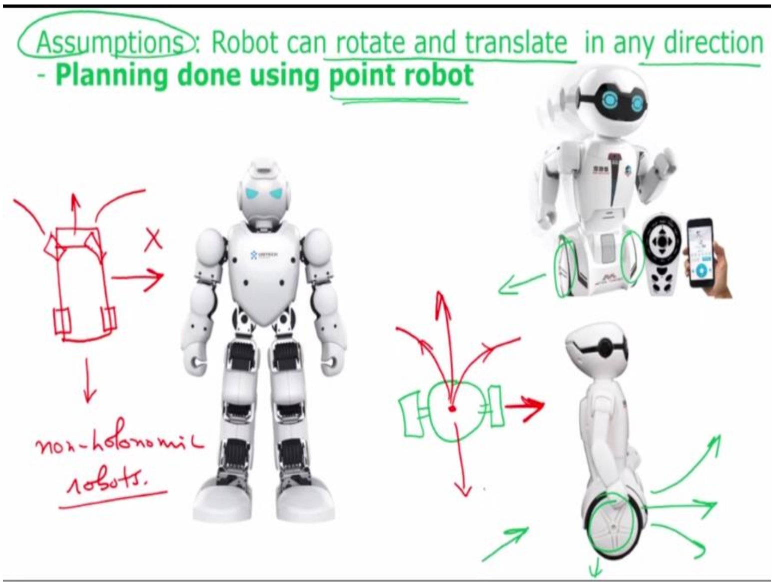

So, these are some examples. So, the assumption is that robot can rotate and translate in a direction. So, this is an assumption. And planning has been done using a point robot assuming that the robot can rotate and translate in any direction. But these are some examples of robots for example this humanoid robot. Now it can rotate, it can translate and rotate in any direction. This humanoid robot can rotate and translate whereas this robot which is here which is mounted on wheels.

For example, you look at this one on the right side, this is mounted on wheels there. Now this robot can go front, it can turn at a particular angle, but it cannot go sideways. It cannot go sideways like that. So, suppose I draw the top view of this robot, so this my top view there are two side wheels on the sides and these are the two wheels, so this robot can go straight. So, it can go straight, it can go back, it can turn at a particular angle.

But it may not be able to go sideways, it cannot go sideways, it can rotate at an angle. It can rotate at this point, but it cannot go sideways like that. Now in such cases where there are kinematic constraints, then we will see how we do the path planning for such robots and such systems. okay. So, this is an example, there are many examples. The other example is that of a car. So, we have four wheeled car or automobile that we see in the roads.

So, this has got a steering in front. So, there is a steering in front, so this car can go front, it can turn at a particular angle, let us say it can turn like this or turn like this, it can go back, but it cannot go sideways. So, such systems are basically called non-holonomic systems and they are non-holonomic because they are subjected to a nonintegrable kinematic constraint.

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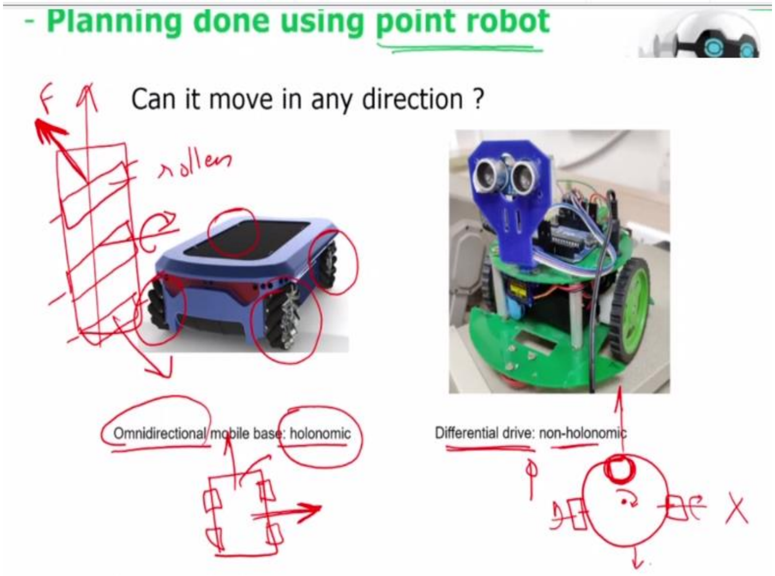

Now, there are robots which can go in all directions. For example, this is an omnidirectional mobile robot which is called the holonomic mobile robot. If you look at the wheels of this robot, these are what are called magnum wheels. And if I just draw the enlarged view, so this is a wheel, up on the wheel there are rollers, you can see these rollers. So, these are rollers which are mounted at an angle to the axis of the wheel. So, these are rollers like this.

There is a roller here, there is another roller here, maybe another roller here. Now, what happens here is that because the rollers are having an angle to the main axis, this is my forward direction, there is a resultant force which is in this direction F. So, when the wheel is rotating, the robot is subjected to a force which is not straight, front or back but has an angle to the axis to the forward axis because of which the robot can move in any direction.

So, now, if you have a robot like this as shown here and there are four wheels; 1, 2, 3, 4. So if I draw the top view, it would be something like this. And we said that this four depending on the angle at which the rollers are mounted, so these are rollers are mounted on the wheels, what can happen is that we have a resultant force in a particular direction. For example, if I rotate it in the clockwise direction, the force is going to be this side.

If I rotate in the anti-clockwise direction, the force will be this side now. So, by manipulating the direction of the force, I can make the robot move in a particular direction. For example, such robots can also go like this. They can go forward, they can turn and they can also go sideways simply by manipulating the direction of the forces on those wheels. So, such robots are called omnidirectional mobile robots.

Whereas this one here which is a differential drive and yeah, so this is a non-holonomic mobile robot in which we have two side wheels and a caster wheel in front. Now, in this case what would happen here is that depending on the direction of rotation of the wheels this robot can go forward, it can go backward and it can turn, it can turn about this axis also, but it cannot go sideways, sideways motion is not possible. Now, such a robot is a non-holonomic robot.

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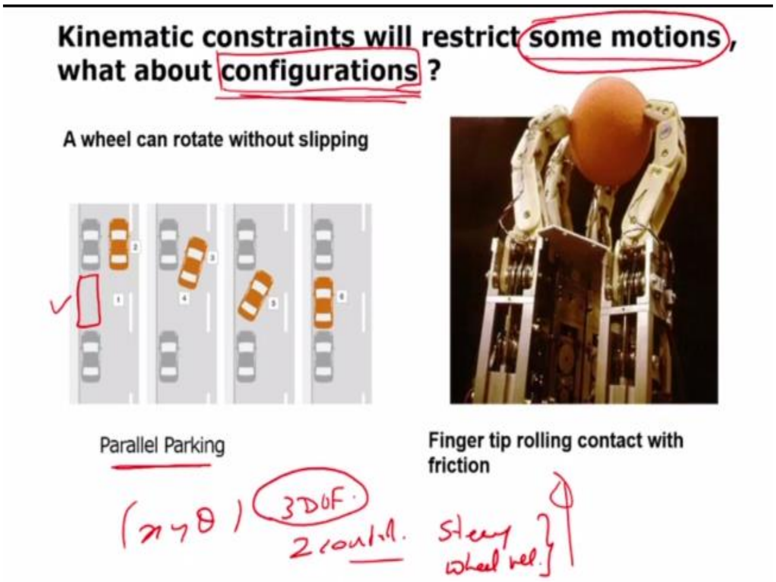

Now, there are many examples of such systems in the real world. For example, the kinematic constraints. Now what do these kinematic constraints do? These kinematic constraints will restrict some of the motions. Now what about configurations? For example, let us take the case of this parallel parking of this car. So, I have a car. This car wants to go and park here like this. Now is it possible for the car to go and park like that?

Now in one shot it cannot go like that obvious but can it have a number of manoeuvres by which the car can actually maybe going front, turning going front, turning or going front and back and turning it can go and park as you can see here, it is able to park like that. So, basically it turned, went like this, then again turned went like this and then parked like this. Now, this is basically what is called parallel parking.

Now, there are two questions here being asked. Number one is because this car has a kinematic constraint which does not allow the car to go sideways, now will it restrict some of the motions? Yes, for example, the car cannot go sideways. So, it is restricting some of the motions of the car. But what about configurations? Now, the question is interesting because although the car cannot go sideways, can it go and part like this?

Is this configuration possible? And that is what comes under the study of what we call controllability. We will see that although there are kinematic constraints, but the system is fully controllable. That means it is possible to go and obtain any configuration, may not be in one motion, but in a series of motions that is interesting. Now, this is another example, fingertip rolling contact with friction. So, you have the multi-finger hand which is manipulating an object.

Is it possible that the motion of this object is constrained in some directions? Are they kinematic constraints? And the second question whether it is possible to obtain all configurations for this object? That means may not be in one shot, but is it possible to orient this object in any way you want? So, the question is there are kinematic constraints, yes. So, some motions are restricted, yes, but the system is fully controllable. That means we can obtain any configuration we want.

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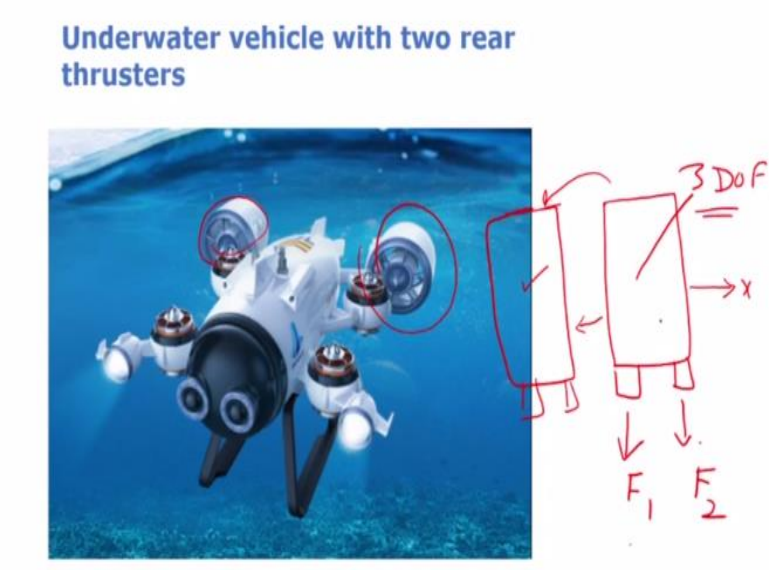

Some more examples. For example, here underwater vehicle with two rear thrusters. So, this is an underwater vehicle. So, these are interesting examples that there are two thrusters here. There is one thruster here, another thruster here. So let us say this applies a force F 1, this applies a force F 2. Now, because there are two thrusters on the rear side, there is one here and one here, this cannot go sideways because some kinematic constraints are there.

But is it fully controllable? That is an interesting question. That means although it cannot go sideways, can I go and park it here that is almost like saying it is going sideways? So the answer is yes. It cannot go sideways, but I can have manoeuvres by which, not with one manoeuvre, I can have a number of paths by which the robot can actually go and park on sideways. So, it appears that it can go and park here.

So, this configuration is possible, although not in one shot. So, in our path planning we will talk about these kind of problems that what are the different paths it has to follow such that it can go park sideways.

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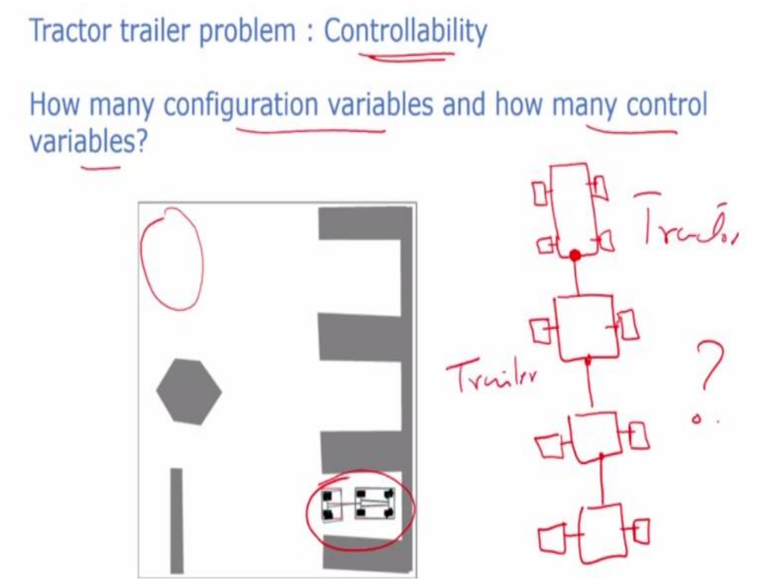

Now, this is another example. The tractor and trailer problem and there is a problem. This is also a question of controllability. So, in the previous case, let me just go back here, we can ask the question of how many control variables are there and what is the degrees of freedom of the system. For example, let us go one step back here. This car has an X, Y and a θ. That means there are three degrees of freedom.

However, the control variables are only two, so only two control variables are there. Which are the control variables? The steering on the front and the wheel velocity, wheel velocity on the rear. So, there are only two control variables, but it has three degrees of freedom. So, is it controllable? Can you control these three degrees of freedom having only two inputs? That is the question of controllability now.

Similarly, here the input is only F_1 and F_2 because you have only two thrusters. But if you consider only planar motion, then this also has three degrees of freedom. It also can go up and down, yes, let us ignore that. But on this plane, it only has three degrees of freedom. So, can you control three degrees of freedom having only two inputs? Now tractor and trailer problem, I am sure you have seen these tractors which carry trailers on the highways.

So, there is a tractor in front and there is a trailer at the back and it is pulling the trailer. Now, in this case the tractor was parked here, now can you go and park here? So simply by looking at this problem it is difficult to say. But from controllability point of view, we need to look at how many configuration variables are there and how many control variables are there? And is this system fully controllable?

If it is fully controllable, that means you can go and park in any configuration, not in one step but in multiple steps or by following multiple paths. So, the question of controllability is very important here because the number of actuators or control inputs is less than the number of degrees of freedom of the system. This is an interesting problem. Now if you want to make it more interesting, so I have a tractor which is pulling a trailer and there is a trailer.

You know the trailer is connected with the pivot and it is just connected there and it has two wheels. So, this is a tractor trailer problem as shown in the figure here. Now, suppose I connect one more here, is this still controllable? Is it still possible for this to go and park somewhere? Suppose I connect one more, can I keep doing this and it will still be controllable or there is a limit which after which you say that you can park it anywhere? So, these are interesting questions. So, this is the tractor trailer problem.

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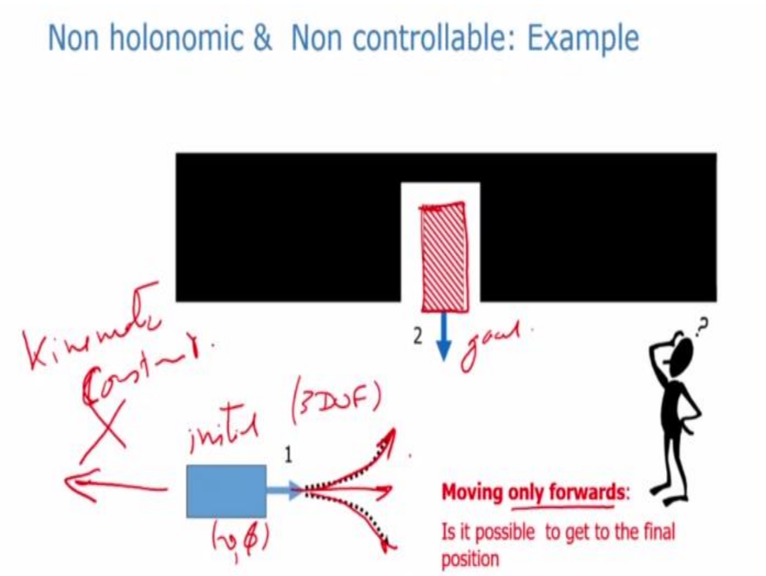

Now, systems can be nonholonomic and they can be noncontrollable. For example, in this particular case, this is the car and I want to park the car here. So, this blue car is my initial point and this is my goal point, I want to park it there. Is it possible to park it there in one shot? In which case I am moving only forward that means the car can go forward and it can turn at a particular angle, it cannot go backwards, let us say I have a kinematic constraint or yeah we have a kinematic constraint that the robot cannot go backwards.

So, in that case is it possible to go? So, again the car would have two variables which is the wheel velocity and the steering angle Φ and it has three degrees of freedom. So, is this still controllable or is this not possible? Now, if it is possible, then what are the ways by which you can do that? In this particular case, it is not possible. Simply by going forward the car will not be able to park there. Whereas if the car goes forward and backwards, then it is possible to get into that position.

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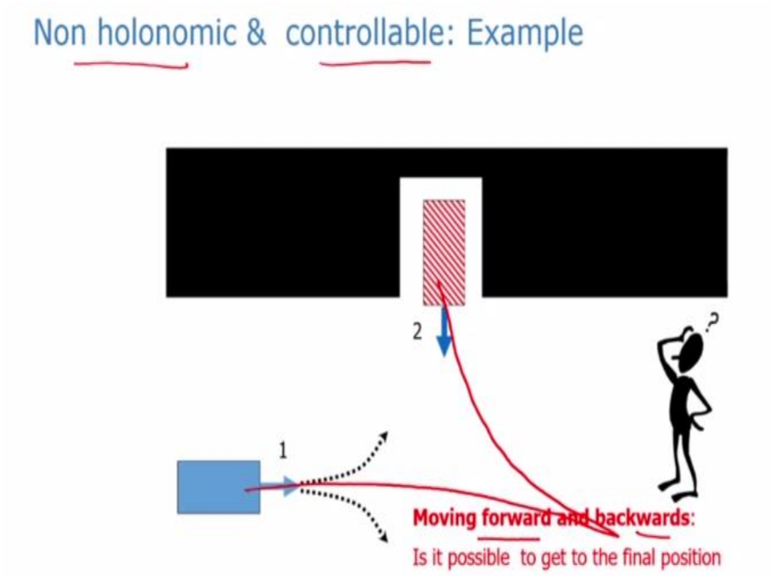

So, this interesting problem where simply by moving forward, it is not possible, but by moving forward and backward it is possible to go and park in that position. So, in this case, it is nonholonomic but it is controllable. So how can I do that? For example, it can go like this and then can come backward like that. So that is one way by which it can go and park that way. So, this is a case which is nonholonomic but is controllable because it can move forward and it can also go backwards, so it is possible to get to the final position.

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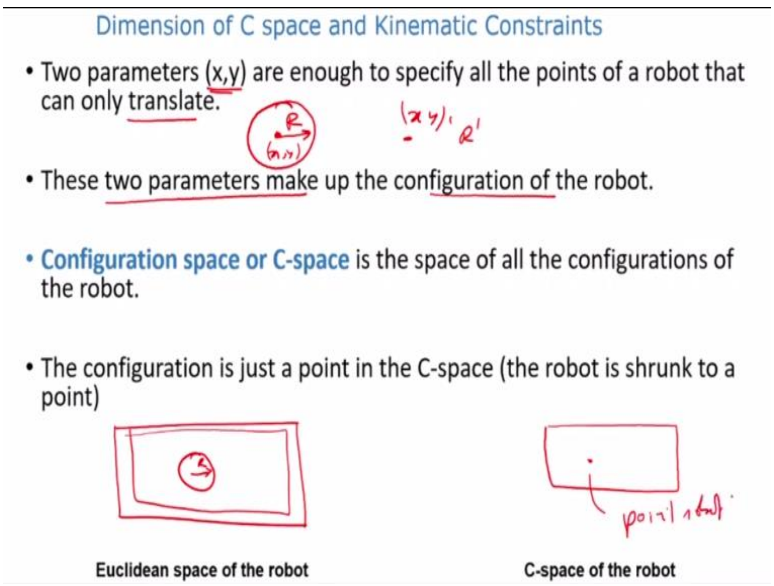

Now, let us look a little bit into the dimensions of C-space and kinematic constraints. Now, two parameters x and y are enough to specify all the points of a robot that can only translate this something we have seen. So if a robot can only translate, then we need only x and y. And these two parameters make up the configuration space. So, if I have a robot like this and I have R, then x, y; if I tell you the position x, y and I know what is R the radius of the robot, the robot can only translate then two is only enough.

So, these two parameters make up the configuration of the robot. So, configuration space or C-space is the space of all configurations of the robot and the configuration is just a point in the C-space, the robot is shrunk to a point, that is something we have done a few classes back that the robot is here the robot has a radius R. So, what we basically do we shrink the robot, so this gets shrunk, the work area gets shrunk by R.

So, it becomes smaller and the robot becomes the point that is what we have done a couple of classes earlier, the robot becomes a point. Then I do the planning for the point robot, so this is a point robot.

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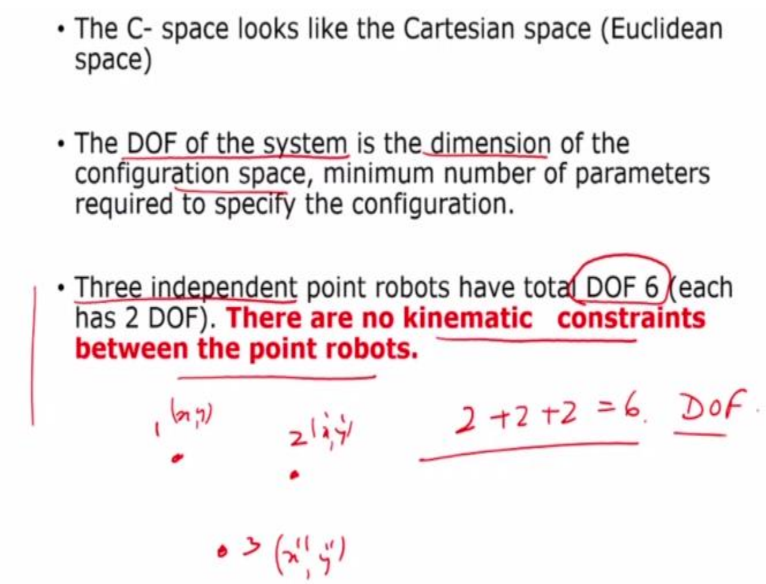

Now, the C-space looks like Cartesian space in this particular case, but they are different spaces. Now, the degree of freedom of the system is the dimension of the configuration space that is something we have seen earlier. Now three independent point robots have total of 6 degrees of freedom. So, what we are trying to derive here or the direction which we are going is that we are trying to see that what is this kinematic constraint and what are the different kinds of kinematic constraints that are there.

Based on which we can classify a system as holonomic or nonholonomic. So, three independent point robots, so we have three point robots; there are three of them say 1, 2, and 3. Each one of them have two degrees of freedom, we have seen that in this particular case only x, y is required for a point which is translating. So, here each of them has two. So, this is x, y, this is (x', y') and this is (x'', y'').

And they are independent, means they are not connected in any way, there is no connection between them. So, the total degrees of freedom is 2 + 2 + 2 which is equal to 6. So, there is total of 6 degrees of freedom on this system and there are no kinematic constraints between the point robots that they are not connected in any way. So, this is a particular case where the robots are independent, each has 2 degrees of freedom and the total there are 6 degrees of freedom of the system.

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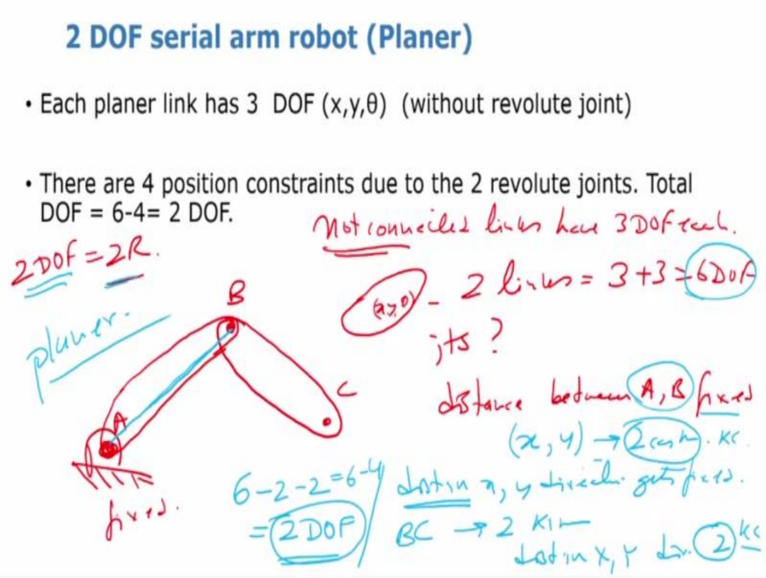

Now, let us look at the case of a 2 degrees of freedom serial arm robot. Now this is a serial arm robot, let us say this is one arm and this the other arm, two serial arm robot like this. So, this is a revolute joint; one revolute joint here, one revolute joint here. So, these are 2 degrees of freedom to our robot. This is fixed space, so, this is fixed now. Now each of these links has 3 degrees of freedom. So, each of this has three DOF that we know.

So, if I simply leave this object in space, suppose just one link, then each of these links if they are not connected, so not connected links have 3 DOF of each, not connected, no joints, right.

So, this object persists a link, it has x, y and just θ. So, there are two links. So, there are 3 + 3 = 6 DOF now if they are not connected. But the moment I make joints, what do those joints do? These joints put some kinematic constraints, what kind of kinematic constraints?

For example, if this is my point a joint A, this is my point B joint B, and this is C. Then the moment I put this joint what would happen? The distance between A and B gets fixed, why because I put a joint at A. So, this B can only rotate, the distance between A and B is now fixed. So, this distance gets fixed now, that means how many constraints have been put. So, the distance between A and B means fixed the x and the y are fix that means there are two constraints.

So, the distance $\sqrt{(x_a - x_b)^2}$ this is 1, the distance in x direction and y direction get fixed. So, there are so many constraints? So, two kinematic constraints have been put. Now, what about B and C, this is A and B. Now, B and C are also constrained by two kinematic constraints. That means the distance in x and y directions, so there are two kinematic constraints there. So, kinematic constraints are 2, kinematic constraints are 2.

That means what is happening here is we have 6 degrees of freedom if there is no joint, the moment I put these joints, we get 6 – 2 – 2 which is equal to 2 degrees of freedom and that is how we have 2 degrees of freedom system. So, when I said 2 DOF, 2 arm these 2 degrees of freedom comes from here, each of this. If it is not connected, it has 3 degrees of freedom each of the links, consider this to be planar, this is a planar example.

And the moment I put the joints, the distance between the joints gets fixed now. So, that means there are 2 constraints, 2 constraints, so 6 – 4 which is equal to 6 – 4 = 2 degrees of freedom. Now, these constraints that I put is a distance constraint. That means the distance between A and B is fixed now.

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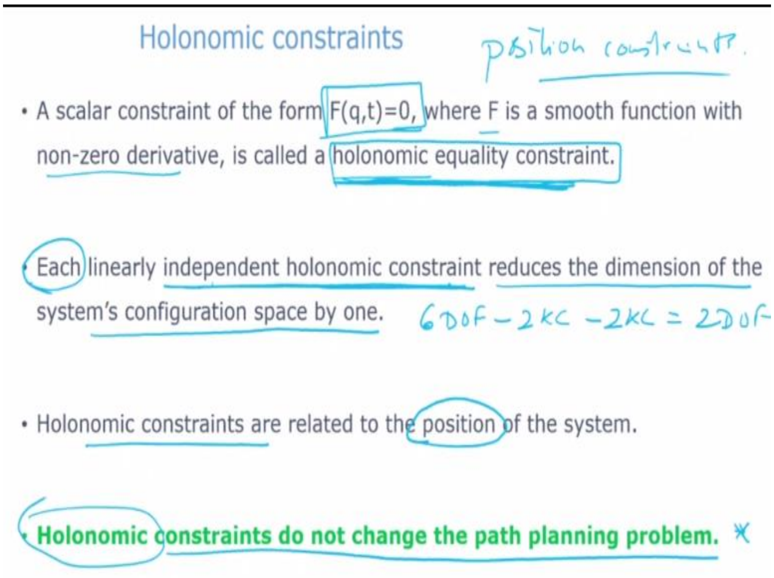

So, a scalar constraint of the form F (q, t) where F is a smooth function with nonzero derivative is called the holonomic equality constraint. So, this position constraints that have been put because of this joint is a position, it is basically putting a constraint on the position of A and B, right. So, a scalar constraint of the form F (q, t) = 0 where F is a smooth function with nonzero derivative is called a holonomic equality constraint.

And each linearly independent holonomic constraints reduces the dimension of the system's configuration space by one. In the previous case, it was 6 degrees of freedom and each of the links had two-two kinematic constraints. So, it was –2 and it was –2 kinematic constraints and then become 2 degrees of freedom. So, each linearly independent holonomic constraint reduces the dimension of the system's configuration space by one.

So, if there are 4 constraints, it becomes 6 – 4, so we have only 2 constraints left. So, holonomic constraints are related to the position of the system. In the previous case if you remember we talked about the position of A and B, the distance between A and B are basically position constraints. Now holonomic constraints do not change the path planning problem, please note this very important.

The moment you have a holonomic constraint, it does not change the path planning problem. So, whatever we have talked about the path planning of this serial arm manipulator, for example obstacle avoidance, going to C-space and then finding the path, so that problem is still remains the same, it does not change the path planning problem. So holonomic constraints do not affect the path planning problem at all.

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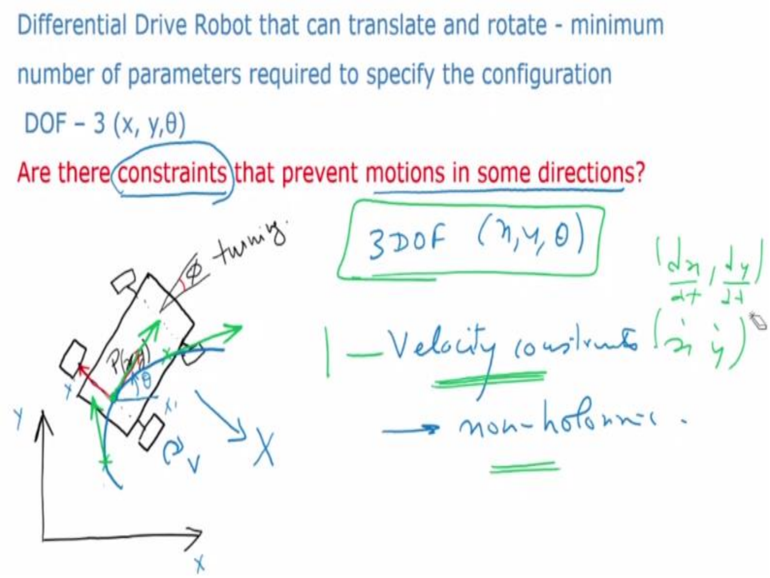

Now, what about differential drive robots that can translate and rotate? For example, this robot that we see here this is a differential drive robot that can translate and rotate. Now, how is this robot going to move? It has a velocity there, let us call it a velocity v, velocity of the wheel and this will mean that it has a velocity of that centre point P and there is a turning angle Φ. So, basically what we have here is that this has 3 degrees of freedom.

This is something we have seen x, y and θ, θ is the angle that the x of the robot is making with the global x'. Now in this particular case, we know that the robot cannot go sideways. So, other there are constraints that prevent motions in some directions and those constraints are called holonomic constraints. Now, if you look at this, basically we can see that when the car is rotating, when it is taking a turn like this, the velocity here has to remain tangential at that point.

And because of which what would happen is a kinematic constraint would be imposed on the car and because of this kinematic constraint, which is a velocity constraint, this cannot go sideways. So, here we have constraints that prevent motion in some directions and these are velocity constraints. These velocity constraints are called non-holonomic constraints. Now, what is the velocity constraint that has been imposed?

The velocity constraint that has been imposed is that when we are taking a turn, the velocity of the car of the point P has to be tangential to that curve at that point. So, if I go to the next point here, it will be tangential there. So, if I come to a point here, it will tangential there. So, when the car is moving at every instant its velocity is tangential to the curve at that point and that is causing a velocity constraint now because the velocity is being constrained in the direction.

I will come and explain a little bit more as we go along. Now, this is a velocity constraint and this is a non-holonomic constraints and non-holonomic constraints ensure that this robot cannot move sideways because its velocity is constrained.

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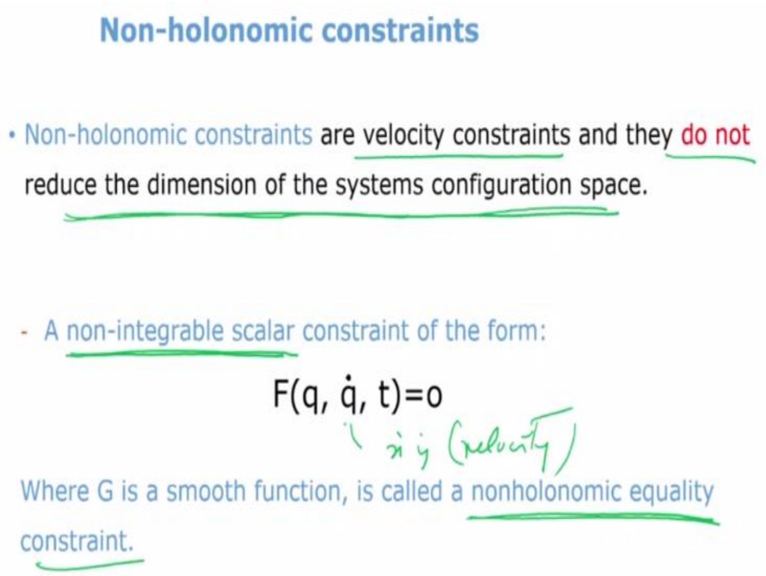

So, non-holonomic constraints are velocity constraints, we just saw that the velocity is forced to move in a direction which is tangential to the curve at every point. They do not reduce the dimension of the systems configuration space. For example, in the previous case, it was a 3 degree of freedom system and we have one velocity constraint, but it does not reduce the dimension of the systems configuration space.

Now, a non-integrable scalar constraint of the form $F\left(q,\dot{q},\mathrm{t}\right)= 0$ is called a non-holonomic equality constraint. So, it is written this way. So, in the previous case if you note this was F of (q, t) the q is position, this is position, it is (x, y) basically whereas in this case in the second case what we are seeing is that there is a $\dot{q}$, so it is $\dot{x},\dot{y}$, so there is a velocity constraint now because the velocity of the robot is constrained to move in a direction which is tangential at every point on the curve.

So, it is a velocity constraint, hence it is $\dot{x}, \dot{y}$ or we can say dx dy. So, this type of constraint is called non-holonomic equality constraint, now that is one. The second this is a non-integrable scalar constraint. If it was integrable you could integrate it and it would become a position constraint, so, it will become holonomic. So, one of the conditions for this to be a non-holonomic constraint is that it should be non-integrable scalar constraints that means you are not able to integrate this.

And the non-holonomic constraints do not reduce the dimension of the systems configuration space. These are some things that we have to remember as we go along. So, we have seen two kinds of constraints holonomic which depends on position, non-holonomic which depends on velocity.

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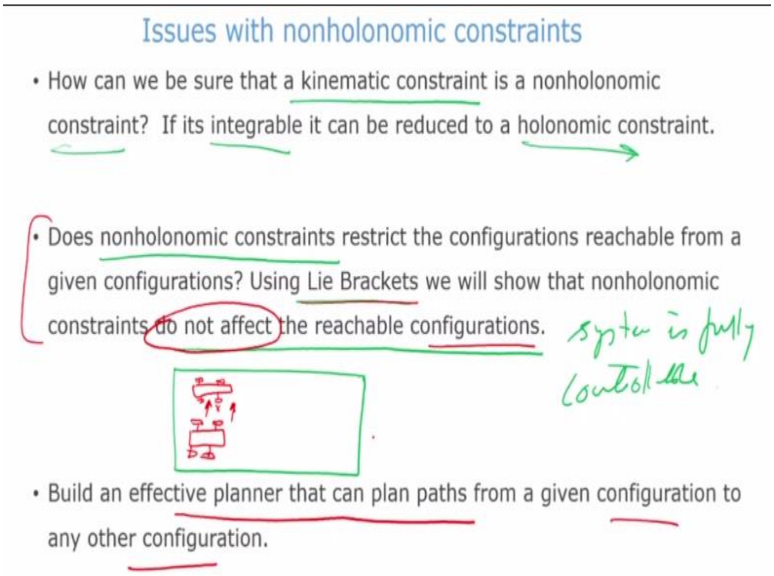

Now, issues with non-holonomic constraints, first of all is that how can we be sure that kinematic constraint is a non-holonomic constraint? If it is integrable it can be reduced to holonomic constraint. So, the first thing is that we have to be sure that it is a non-holonomic constraint. Now, does not hold me constraints restrict the configurations reachable from a given configuration?

Now, using Lie brackets we will show that non-holonomic constraints do not affect the reachable configuration space, which basically means that the system is fully controllable. That means suppose, this is just an example, I have a car which has 4 wheels. Now, the car cannot go sideways, right. It cannot go sideways because of this velocity constraint. But suppose I want to go and park it here, is it still possible? That means is every configuration achievable?

So, the theorem basically tells us that although we have a non-holonomic constraint, but the non-holonomic constraints do not affect the reachable configurations. Using Lie brackets we will be able to show that not non-holonomic constraints do not affect the reachable that means it can actually go sideways also. It cannot go sideways, it can go and park sideways by using a number of manoeuvres.

So for example, it can use different manoeuvres to go and park sideways. So if that is the case, then we need to build an effective planner that can plan the paths from a given configuration to other configurations. It cannot go sideways, but then I plan a number of paths which will take me from one configuration to another configuration in such a way that all configurations are reachable.

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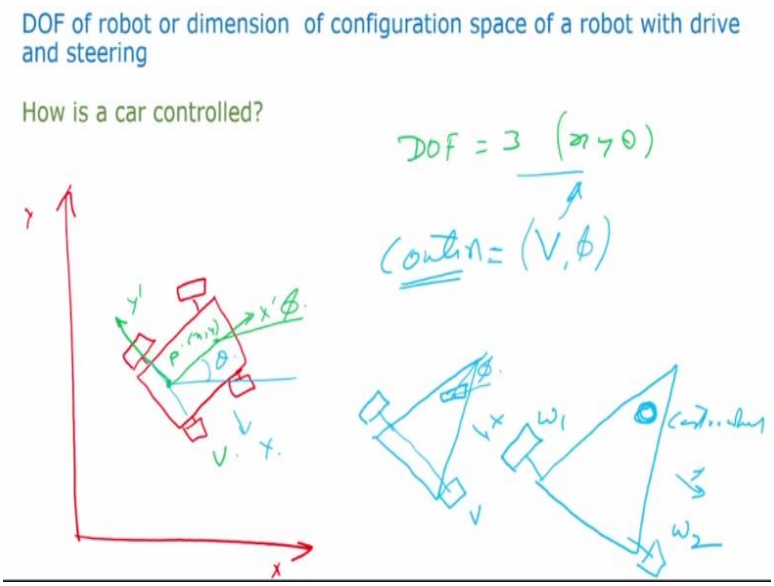

Now, degrees of freedom of robot or dimension configuration space of a robot with drive and steering, so how is a car controlled? So let us look at this again here. This is a car. So, these are four wheels. This wheel let us say has a turning and we have the global x this the global y and we have a local x and a local y which is given here. So this is my local x, this is x' and that is y'. So, it has to be the centre point.

There is y', this is my point P which has coordinates x and y. Now, how is this car controlled? There is a wheel velocity which is ω because of which there is a linear velocity which is v. At this point P has a velocity v and there is a turning angle which is there. So there is a turning, so let us say this is my angle Φ. So, degrees of freedom of a car are equal to 3 which is x, y and θ, θ is the angle the x' is making, so this is my angle θ.

So it has 3 degrees of freedom. How are we controlling it? So, trying to control it is by only with two inputs, one is v the wheel velocity and the other is my angle Φ. So, these are the two control inputs. So, I am controlling the car, the 3 degrees of freedom of the car by using only two inputs now. So, the question here is can the car go and park anywhere? So, are all in the configuration space can the car go or all the points in the space achievable for the car? That is what we are looking at here.

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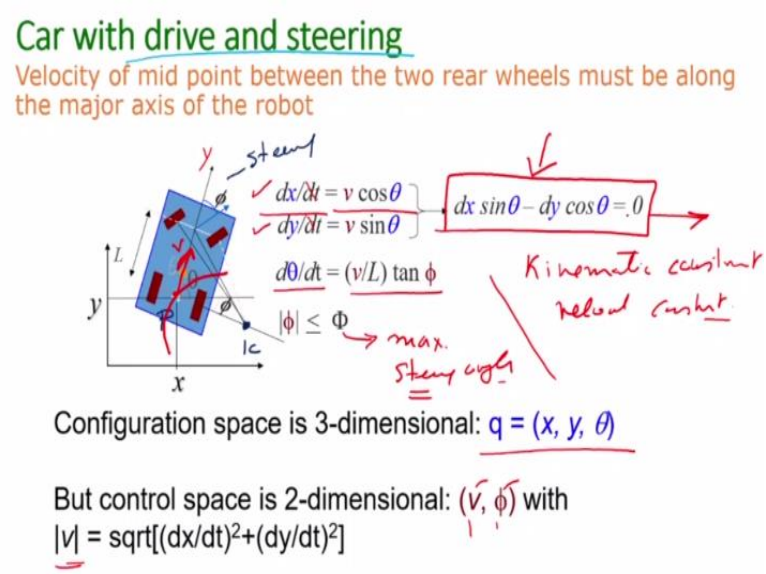

Now car would drive in steering. Now there can be different kinds of, let me just as we are talking about this. The other kind of robot that is there is this type of robot. You can have a robot which is like this, this is joint right. So, we have a robot which is, again it has velocity v and there it has a single let us call this Φ. Here also it is v and Φ. So, this is a mobile robot which has driver at the back and it has a steering angle in front.

You can also have a robot which has this kind of robot where you have differential drives and there is a caster wheel in front. So, there is ω_1, ω_2. Now all of these subjected to kinematic constraints. That means this robot cannot go sideways, this cannot go sideways, this car cannot go sideways that means it has a velocity constraint. Now, let us look at what this velocity constraint looks like. So, car would drive in steering.

So, let us look at this carefully. So, this is the car and this is the point, let us call this point P, I had talked about this point P. So, let me just write P, the point is the centre point between the two wheels which is here. Now, Φ is the steering angle. This is steering angle and θ is the angle it makes with the x axis, the local x axis of the car is making with the global x that is my angle θ. Now Φ, when the car is rotating, there is instantaneous centre of rotation which is here.

So, this is my IC, instantaneous centre of rotation, so the car is rotating about that point. Now, let us say that the car is taking a turn like this. The car is taking a turn like that, now at this point when the car is taking a turn the velocity vector is going to be in this direction. The direction is always along the y axis of the curve, right, in the front direction. Now, if that is my v, then, $\frac{dx}{dt} = v\cos\theta$. So, I can break it up into two parts, I can break v into two parts.

One will be $\frac{dx}{dt} = v\cos\theta$ and $\frac{dy}{dt} = v\sin\theta$. And when I divide this by this, I divide this what would happen? The dt, dt will cancel and I have dx sin θ – d y cos θ = 0. We have the direction of v, I hope you understand this. Now, when I take the v cos θ, I get the velocity in the x direction and v sin θ is the incremental velocity in the y direction. Now, what we get is this is what we call my kinematic constraint or velocity constraint you know.

Now, this velocity constraint is the one that ensures that the robot cannot go sideways, why because at every point on that curve the velocity has to be tangential to that curve. And what is the constraint? It is dx sin θ – dy cos θ and $\frac{d\theta}{dt} = \left(\frac{v}{L}\right)\tan\phi$, now tan Φ is less than this is my maximum steering angle. So, every car can steer only by a particular angle right, the front steering wheel cannot become 90 ^{0}, right.

So, there is a limit on that by how much you can steer and that is my maximum steering angle. Now, the configuration space is three dimensions which is x, y and Φ, but the control space is two dimensions you have v and Φ which is my drive and my steering and the velocity is constrained this way $|v| = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$. So, I hope you understand where this constraint came from.

So, this is the kinematic constraint that the car is subjected to because the velocity has to be in the direction. And it is very easy to derive that if you take the components in the x and y direction you get this equation and this equation. You divide both of them you get the kinematic constraint.

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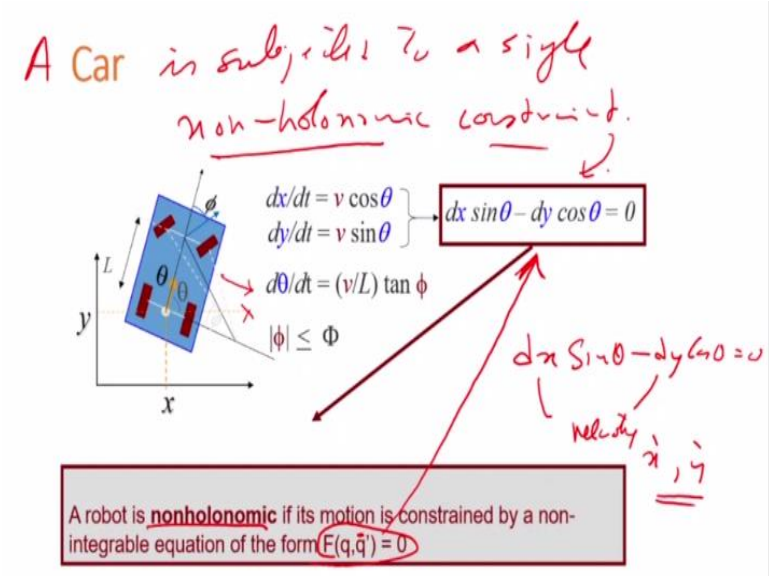

Now, this is the same car. What we are saying that a robot is non-holonomic if its motion is constrained by non-integrable constraint of the form $F\left(q,\dot{q},\right)$ and in this particular case this is the non-holonomic constraint. And you can see it says dx sin θ – dy cos θ = 0 and dx and dy are velocity. So, it is x dot and y dot. So, basically what we are saying as the robot is non-holonomic if its motion is constrained by a non-integral equation of the form this and this equation is this one.

So, we are seeing that a car is subjected to a single non-holonomic constraint. So, a car is subjected to a single non-holonomic constraint of this form, it is non-holonomic constraint. So, this non-holonomic constraint will cause the car not to be able to move in some directions. So the car cannot go sideways, but it is fully controllable. That means it can go and park, it can achieve every configuration.

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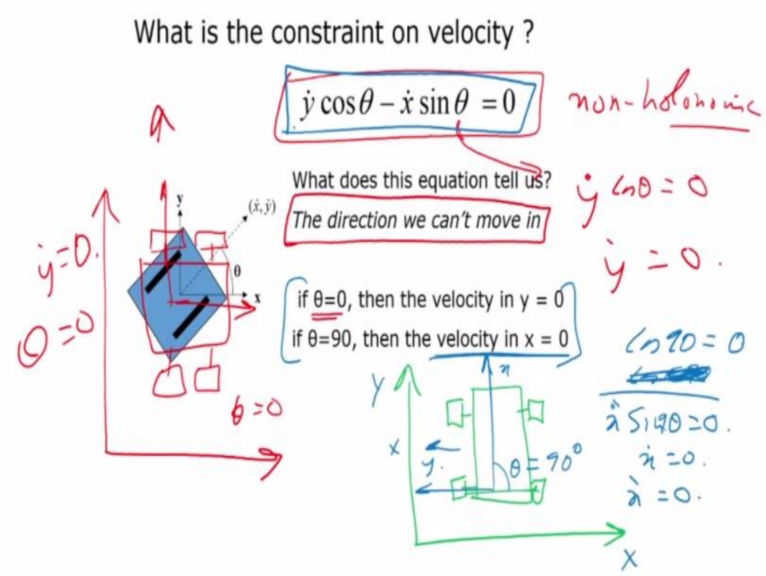

Now, what does this equation actually tell us? First of all is that it is a non-holonomic constraint. So, it is a velocity constraint, so it is non-holonomic constraint. Now, what does this equation actually tell us that there are some directions in which you cannot move. I said the car cannot move sideways, but how can you say that in terms of the equation? Now, in this equation if θ = 0 so θ being equal to 0 means if this is my global axis, so if θ = 0 it basically means the car is like this.

θ is equal 0 means the car is this side now that is the car now, that is the car if θ equal to 0. So, the x axis is also in this direction, of the car's x axis that is x = 0. Now, if θ = 0, then velocity in y direction is equal to 0. So, in this equation if I put θ = 0 now what will happen? This will become 0. Then what we are going to get is $\dot{y}$ dot cos θ = 0, that means $\dot{y}$ = 0, that means the velocity in y direction = 0.

What is the velocity in y direction? It is the same, it is 0 which means when θ = 0, $\dot{y}$ = 0 that is what it is saying that means it can go sideways. Whereas on the other hand if I draw it again here, let me draw it again here if the car is facing this side now, which means that the local axis of the car is this side, this and this, x axis, this is my y axis. So, in this particular case, what is θ? θ = 90^0, this is my x and this is my y.

So, θ = 90 in the second case, now if θ = 90 then what happens is $\dot{x}$ sin θ = 0 because cos θ is 0. So, cos θ = 0. This means that $\dot{x}$ = 0 now which basically ensures that the the velocity in the x direction is 0. So, it means it can go sideways, it can go this way now. So, what we are seeing is that when θ = 0, it cannot go in the y direction. Please note the correction θ = 90, then this is equal to 0.

So, when θ = 90 ^{0} then $\dot{x}$ sin θ = 0 and $\dot{x}$ = 0. This is 90, so $\dot{x}$ = 0, please note that. So, these two basically tells us that we have this constraint and what this constraint does. It ensures that the car cannot go sideways. Now let us see this in a little bit more detail. I am going a little slow because you need to understand this very clearly.

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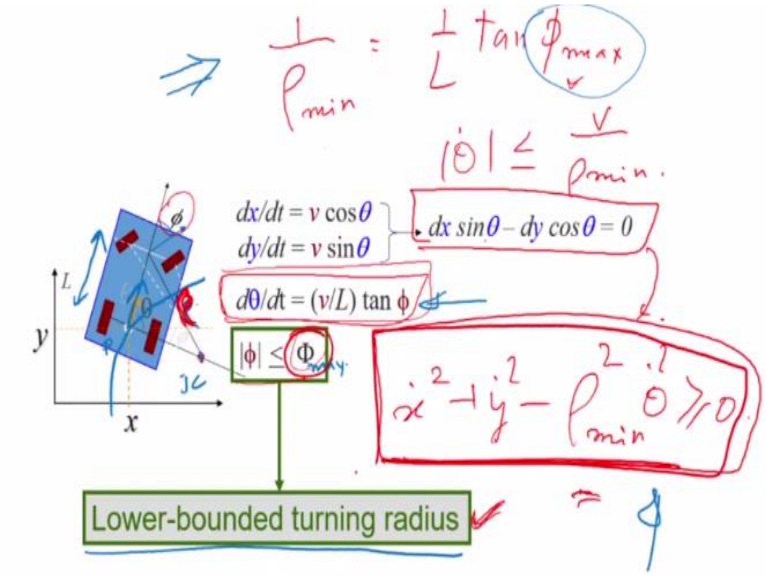

So, we have my turn which is like this, it is taking a curve. At that point, this my point P, the vector is in this direction. Now, we get one equation from that dx sin θ – dy cos θ, we get another one which is here. Now, in terms of the car which is rotating about this instantaneous centre of rotation IC is there which is rotating about the instantaneous centre at that instant. In that case, if this is my ρ, instantaneous centre of rotation, this is the radius, then $\frac{1}{\rho_{\min}} = \frac{1}{L}\tan\phi_{\max}$.

What is L? L is the distance between these two wheels and Φ_{max} is that maximum turning radius. So, this is Φ_{max}. So, it is constrained by this equation $\dot{x}^2 + \dot{y}^2 - \rho_{\min}^2 \geq 0$. So how do we get that from? This is my ρ. That is my ρ instantaneous centre of rotation. So, the radius of rotation is $\frac{1}{\rho_{\min}} = \frac{1}{L}\tan\phi_{\max}$.

And from that we get this equation that $\dot{x}^2 + \dot{y}^2 - \rho_{\min}^2 \geq 0$. Now, this ensures that there is a lower-bound turning radius of your car and this condition has to be satisfied. Now, what does this mean?

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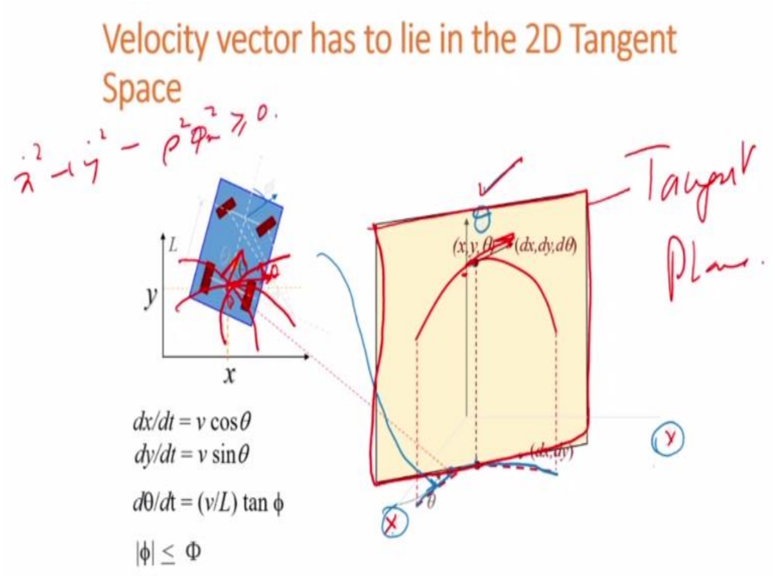

Now this basically means something very interesting, please look at this very carefully. So, we have a radius here. This car is rotating. This is the velocity vector which is in that direction. Now this is my x axis, that is my y axis. So, I am bringing my x axis and y axis here. So, on the ground this is my curve that the car is taking. This is the curve that is taking, now at that point, this the point, this is my θ x. Now, at that θ the vector is pointing in which direction?

The vector is pointing in this direction, is tangential to the curve there at that particular θ, for a particular θ. This is my x axis, y axis and this is my θ axis. So, at that point on θ, the vector is in this direction, that is my dx dy. That means that every point here there is going to be vector. Now, for different θs, there can be different θs of the car, the car could be pointing in different directions. For example, this is another curve, in this particular case the car is pointing this side now.

Let me draw with some of the colour, this is blue, let me use red colour. So, at this instant when the car is like this, the car is like this. What would happen? The curve is like this now, so the vector is this side now. Now, through this point P there can be different curves, right.

But for each of those curves, the velocity vector will be tangential at that point. So, what it basically means is that for different θs when the car is turning this θ is changing right, this θ and the θ is different.

So, for different values of the θ here there will be different vectors which are going to be there, but these vectors will have to lie on this plane, this plane which is passing at this point. So, all the vectors will have to lie on that plane and this is also called the tangent plane. Now, if that is the case and there is a turning radius, we have seen that it is constrained to move in a particular direction. It is constrained by this equation why because there is a turning radius and there is a turning radius here.

And there is a turning radius of the steering wheel Φ max and ρ is there which means that this condition has to be satisfied, which basically means that just coming back and explaining this again. At this particular instance, we have a curve like this and let us say I have a curve like this and its tangential there. So, this curve is on my X-Y plane. This my X-Y plane. So, for this point, which is the point P, so for the point P we have a curve for a particular θ whatever was the θ here.

So, in this case let us say my θ was this much. So, for that particular θ at that point where this is my θ axis, my tangent was in this direction. But for different θs there can be different curves, there can be different curves at that point. But for each of those curves at that point the tangent will be in different directions, but they must all lie on this plane and this plane is basically called the tangent plane.

So, this plane is basically what we call the tangent plane, okay. So, please think of this carefully that why must the tangents lie on that plane. Now, if the tangents are lying on this plane and the tangents are bounded by the equation, $\dot{x}^2 + \dot{y}^2 - \rho_{\min}^2 \geq 0 0$.

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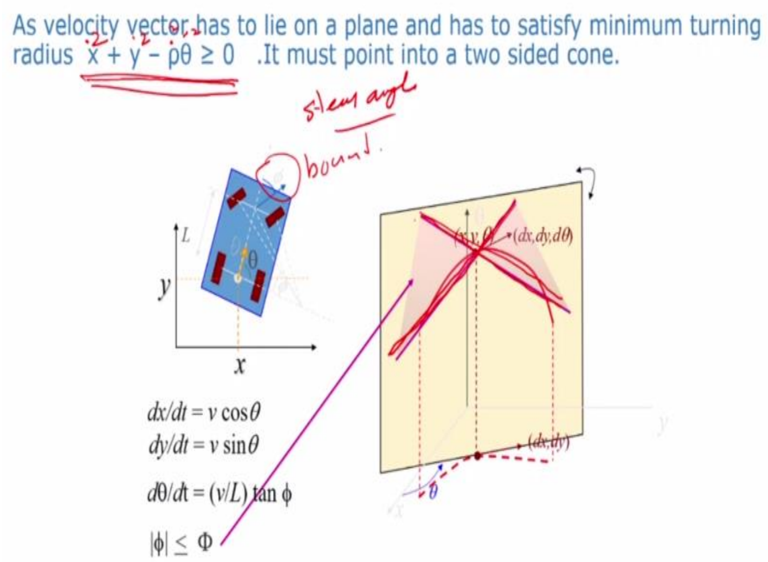

If they are bounded by this equation, then it basically means that all the tangents have to lie inside this cone. So, all the tangents have to lie inside this cone this is what it means that the tangent will lie on this plane called the tangent plane and because they are bounded that means the car cannot turn, this has a bound. The turning radius has a bound and this steering angle has a bound. Now, that bound will ensure that the tangents will lie inside this cone because it has to satisfy this equation. So, this basically gives us the idea of a tangent plane and also the cone inside which the tangents must like always.

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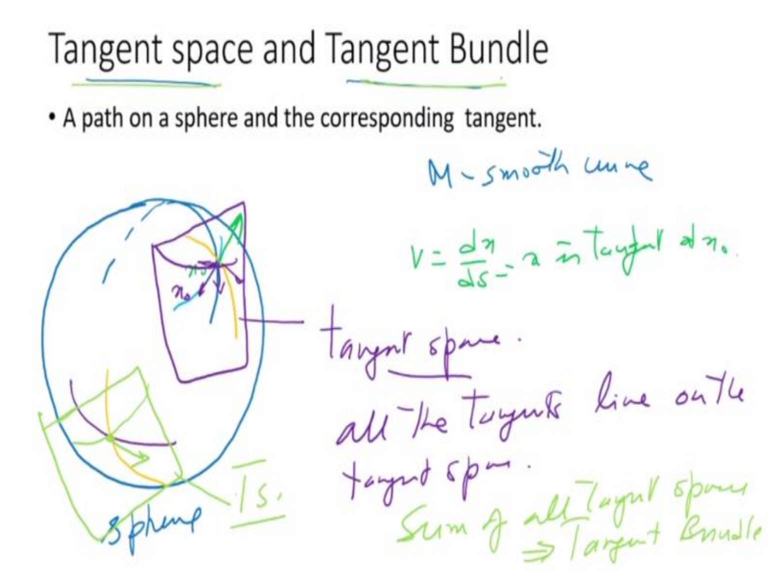

Now we need to think about this a little bit more before we proceed. So, let us talk about this tangent space and tangent bundle, let us think of it a little bit differently. So, suppose I have a sphere. Now on this sphere, I have a smooth curve, let us say M is a smooth curve. So, I have a smooth curve which is like this, that is a smooth curve. Now, at this point on the curve, I have a tangent. So, V is a tangent $\frac{dx}{ds}$, so x is tangent at x 0, at this point x 0 is a tangent.

Now, we can have different curves which are going to that point. I can have different curves, right. So, from here let me draw another curve here which is going like this, I can draw another curve which is going, these are all curves going through that point only. So, these are all curves which are going through this point the same point x_0. Now, each of those curves there is going to be a tangent and the tangent should be different directions because the curves are in different directions.

Now, all of these tangents will lie in a space. So, these are all tangents if you can imagine and this is basically called my tangent space. This is called my tangent space. So, we can say that all the tangents live or have to be on the tangent space which is this one, so this is my tangent space. Now, each of these tangent spaces is basically a vector field because it has tangents in all directions, all the tangents which are coming at that point are contained in this itself.

Now, at different points, there will be different tangent spaces on the sphere. For example, if I take some of the point here, we talked about x_0 only. Suppose I take some other point here now, I take this point which is somewhere else. It is on the sphere. Now, these are all curves which are going through this point now. Now, here also they will be tangents. Now, there is going to be a vector field here also, all those tangents will lie in there.

So, this is another tangent space. So, sum of all these tangent spaces basically make up the tangent bundle. So, this is the idea of a tangent space where the tangents lie on that point or all the tangents at the point lie. And sum of all the tangent spaces make up what we call the tangent bundle. The sum of all tangent spaces makes what we call the tangent bundle. So, this is exactly same as what we had just seen here.

But this is maybe a little bit easier to understand what is the transit space and what is a tangent bundle. Now, this kinematic constraint is the one that is causing all this because it is enforcing that the velocity of the car has to be in a direction which is tangential to the curve at that point and hence it has to lie in the tangent space and sum of all these tangent spaces make up the tangent bundle.

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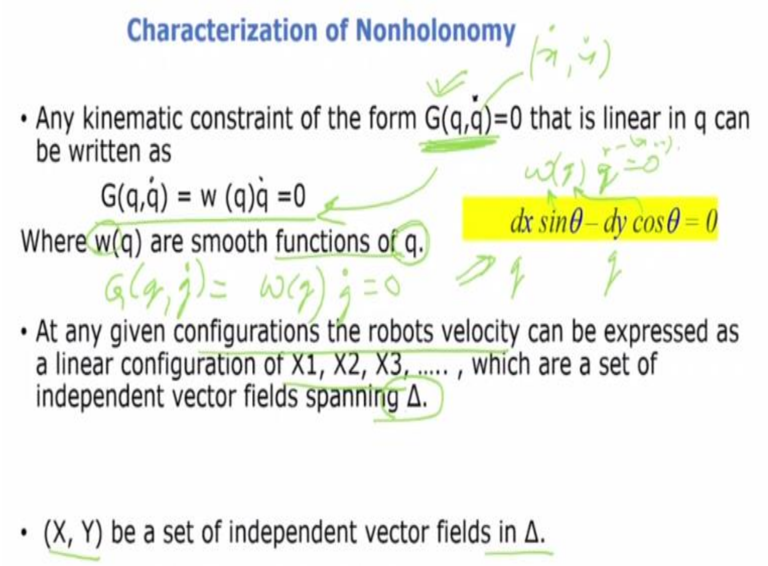

Now, any kinematic constraint of the form that is linear in q can be written as this. So, $G\left(q,\dot{q}\right)=w(q)\dot{q}=0$. That means a kinematic constraint of this form can be written in this form where w is a smooth function of q. Let us take the example here. So, we are looking at this where the kinematic constraint which is this one is written in the form $w(q)\dot{q}=0$, the q are x and y now, these are the x and y's. So, what is w here?

So, $\dot{x}$ and $\dot{y}$ is my q, so what is w w sin and cos? So, these are smooth functions of q. So, what we are seeing is that I can write the kinematic constraints of this form in the form $w(q)\dot{q}=0$ and any given configuration of robot velocity can be expressed as a linear combination of X_1, X_2, X_3 which are set of independent vector fields in spanning Δ. Let X, Y be a set of independent vector fields in Δ. What we are saying is that you can have $\dot{x}$ $\dot{y}$ which are basically directions or velocities in X direction Y direction.

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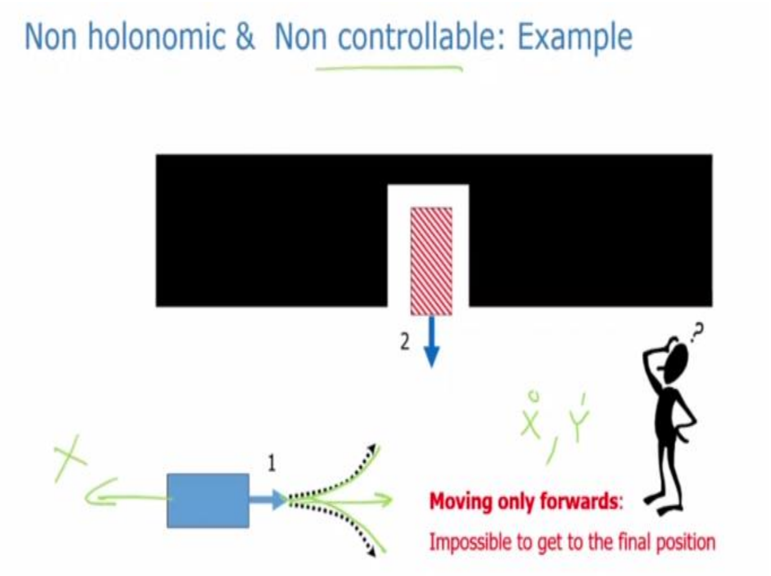

Now, by combining all these velocities in different directions, you can generate, this is what it is telling us that any given configuration of the robot velocities can be expressed as a linear combination of x_1, x_2, x_3 which are a set of independent vector fields spanning Δ. And if X, Y be a set of independent vector fields in Δ velocities in the X direction and Y direction, then what happens is you can move, the system becomes fully controllable and you can park or rather what this basically means is that the system is fully controllable.

Now, non-holonomic constraints of this form can be written in this form as $w(q)\dot{q} = 0$ and because of this the system is fully controllable. Why because at any given configuration, the robot velocity can be expressed as a linear configuration of x_1, x_2, x_3 which are a set of independent vector fields spanning Δ. What this basically means is that in this particular case, the robot can only go this side and the side and front, but not backwards. In this case, the system is not controllable.

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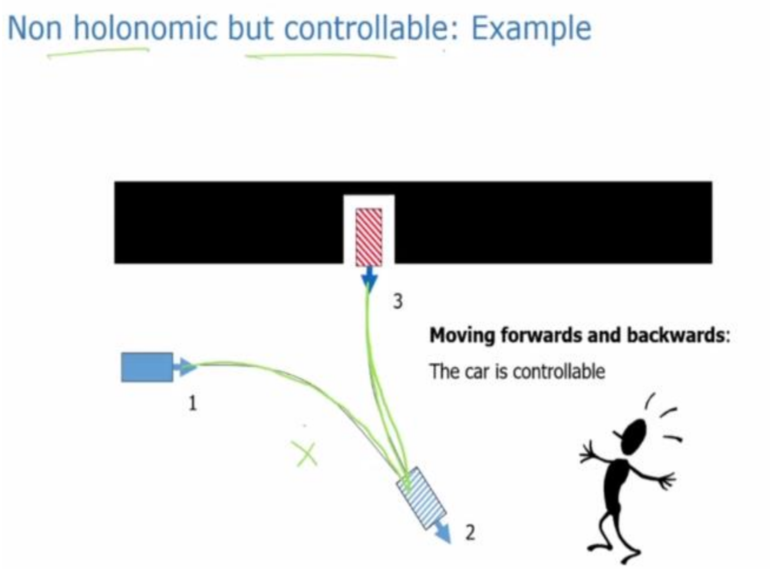

But in this particular case, the robot can go in x direction and it can go back also. Like for example it goes like this and then goes like this and parks. Now, the system is fully controllable. This is non-holonomic, but controllable.

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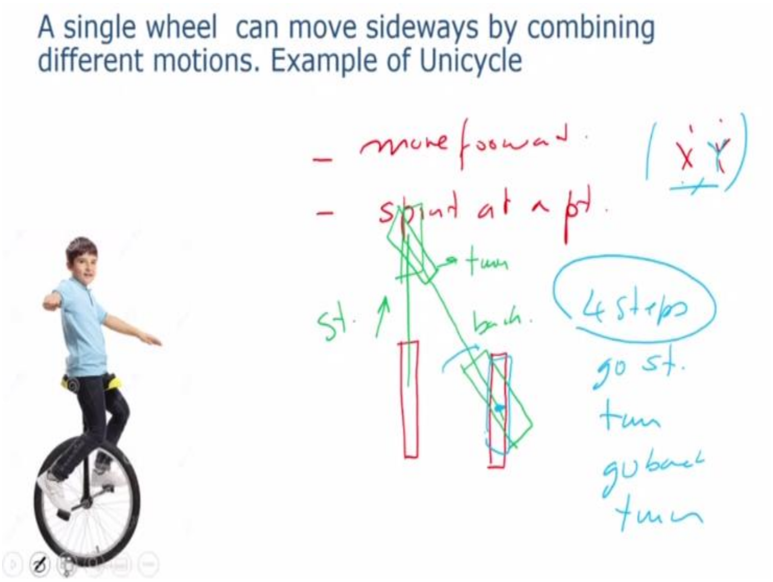

Now, what we are trying to see is how do you park sideways or how do you move sideways? That is something which we will be seeing in the next class by using Lie algebra that how can the system be moved sideways? Stop today with this, maybe just to keep you thinking is if you have just as a small homework. Can you think of this unicycle, you might have seen this unicycle where there is one wheel, the person can pedal and turn at a point.

So, this can move forward, this can move forward and it can spin at a point. So with this, this is my top view, I want to make it go here and park like this sideways. Now this cannot go sideways, right? But using a number of $\dot{x}$ and $\dot{y}$ motions in the two directions, can I make it go and park sideways? For example, can I go straight, just going straight, then I turn, I rotate at a point, can spin at a point, right?

Then I come back, then come back here, so it is like this. So go straight, turn, then back, and then turn again, then turn again, turn this side again, turn this side again, and it has parked like this. So, there are four steps. Go straight, turn, go back, turn, and in these four steps it is going in parking sideways. So, this configuration is controllable. Because although this single wheel cannot go sideways, we are seeing that this wheel has actually gone sideways and parked there; not in one step but in four steps.

This is the meaning of having different combinations of $\dot{x}$ and $\dot{y}$ to define the vector field. So please think about this. And in the next class, we will be seeing this in more detail especially in terms of Lie algebra that how is the system controllable, although it does appear that because of the kinematic constraint the system cannot achieve all configurations. In the next class, we will be talking about Lie algebra where we have to actually prove.

And after proving that the system is controllable, we have to derive or design path planning algorithms, which can take a robot and make it park anywhere including the tractor trailer problem in which case the algorithm should automatically be able to park the car wherever you wanted to park. And if the system is fully controllable, then you should be able to park anywhere. So, we will stop here today. Thank you.