Robot Motion Planning
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Lecture – 17
Controllability

Hello and welcome to this lecture number 17 of the course robot motion planning. In the last class we looked at the motion of a car like mobile robot and the kinematic constraints that such robots are subjected to, for example, they cannot move sideways. Now, today we will move on with the same discussion on kinematic constraints and then look at controllability that is if a robot is subjected to these kinematic constraints, then what happens to the controllability of the robot? So, in the last class, we were looking at kinematic constraints and I will very briefly go through what we were doing in the last class just to revise and then we will move on to controllability.

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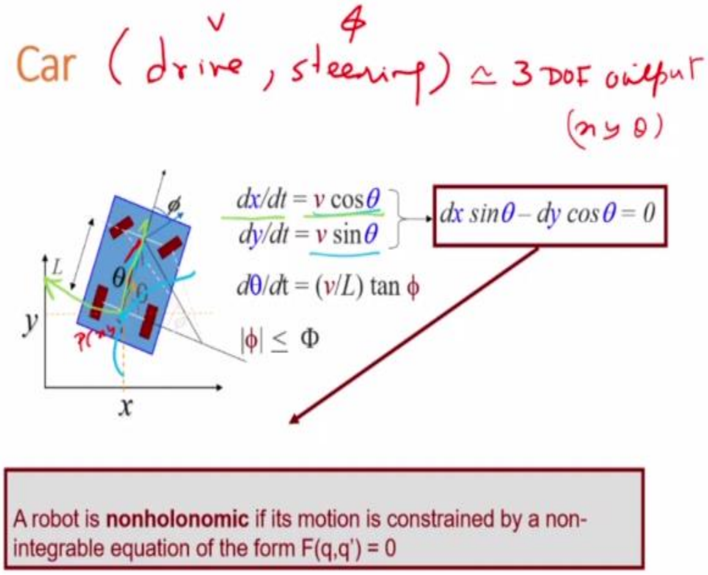

Now, we said that a vehicle like a car which has two control inputs, for example there is a drive at the back, so drive and steering. So, it has v and Φ as inputs whereas the output is 3 DOF output. That means we have x, y of the centre point P that is my point P having coordinates x and y plus there is an angle θ. Now, we said that when this car moves, it has a particular velocity v and the direction of the velocity vector is always in that direction.

And because of which if the car is taking a turn, for example if it is turning about this point like this, what is going to happen is that at that point, so, let us say the car is turning like this.

So, at this point what will happen the velocity vector will be tangential to the point on the curve. Now, because of which because this is my velocity vector v, if I take this projection in the x direction and y direction what we get is $\frac{dx}{dt} = v\cos\theta$ and $\frac{dy}{dt} = v\sin\theta$.

Now, the local axis of the car is in this direction, let me draw the local axis also. So, the local axis is in the side that is my local axis. Now, because of these two $\frac{dx}{dt} = v\cos\theta$ and $\frac{dy}{dt} = v\sin\theta$, when I divide both of these equations, this by this, what I do get is that $dx\sin\theta - dy\cos\theta = 0$. Now, this is what is the non-holonomic constraint with this robot is subject to and what does it mean?

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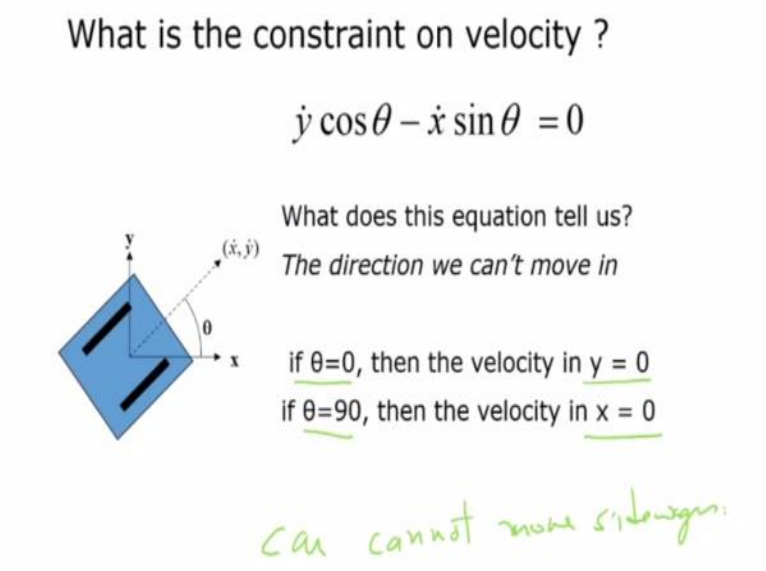

It basically means that for conditions where θ = 0, the velocity in the y direction will become equal to 0 and when θ = 90 then the velocity in the x direction will become zero, which essentially means that the car cannot move sideways. So, this is something that we have seen in the last class.

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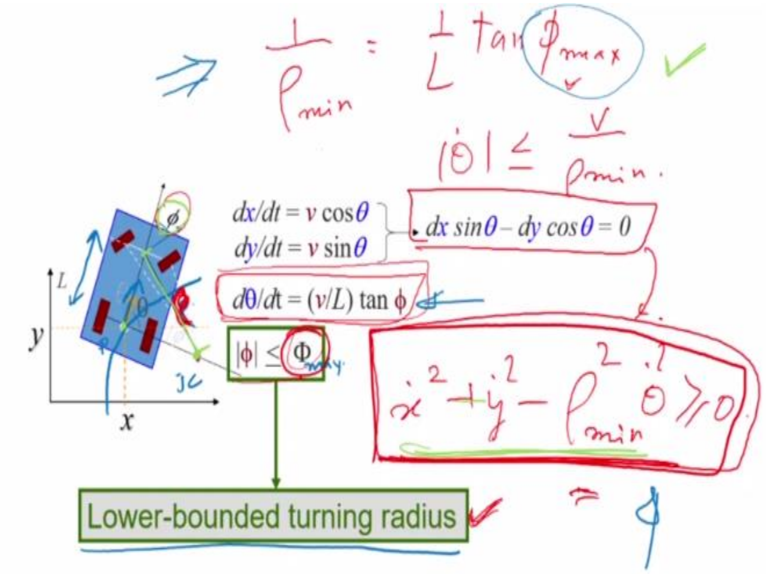

Now, something to remember very carefully is that at this point the velocity vector is tangential to the curve because of which the car is turning, right, is taking a curve. Now, if it is turning, then it has the instantaneous centre of rotation which is located at this point and ρ is the length from the instantaneous centre of rotation to the centre of the front wheel there. Now, if that is the case and ρ also has the turning radius Φ it would be having a maximum and a minimum.

For example, a car can turn only maybe 40^0 left or right, it can turn more than that. So, because of which we end up with a constraint that $\dot{x}^2 + \dot{y}^2 - \rho_{\min}^2 \dot{\theta}^2 \geq 0$. Now, this comes from here $\frac{1}{\rho_{\min}} = \frac{1}{L}\tan\phi_{\max}$ and we can get this from geometry. Now, just from geometry, we can get this equation in terms of the L the length between the centre of the front wheels and that of the rear wheel.

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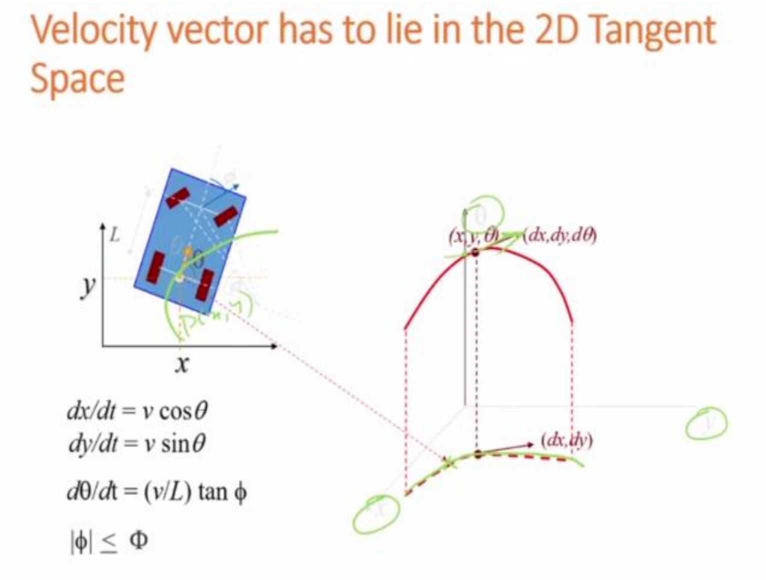

Now, because of this constraint what is happening is that the velocity vector that is there, now, let us look at this again. So, this is my centre point P and this is my P (x, y). So, when it is taking a curve like this on my X-Y plane, this is my X-Y plane, from the X-Y plane this is the curve right it is taking. Now, at that point what is going to happen is depending on the θ, this is my θ axis, at that particular location the velocity vector is going to be tangential to that point depending on whatever θ you have. Now, this is one.

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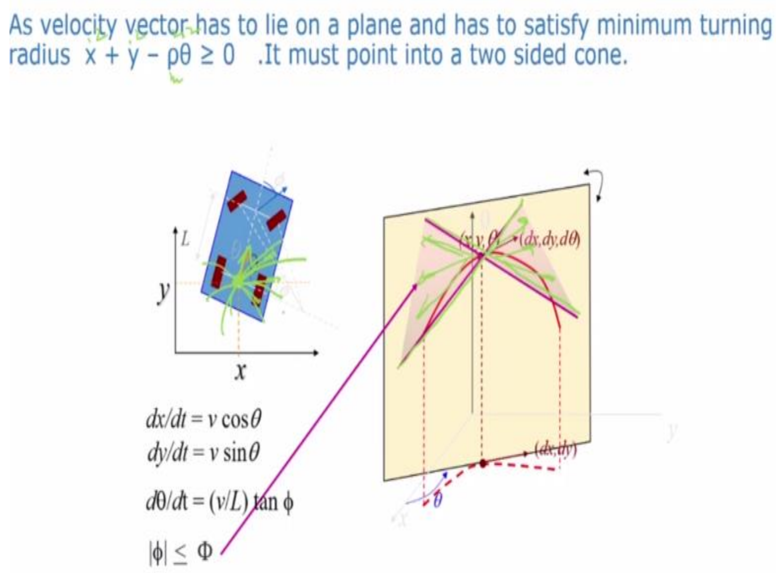

Now, we also saw that the velocity vector is bound by this equation here because of the turning radius by this equation. So, what will actually happen is that this velocity vector is also constrained to lie inside a cone. So, this velocity vector first of all has to lie on this plane for a particular θ and because of the maximum turning angle constraint that is put the velocity vector has to lie inside this cone which is constrained by the equation which is here $\dot{x}^2 + \dot{y}^2 - \rho_{\min}^2 \dot{\theta}^2$, which means that the vector has to lie inside here, it can apply anywhere outside because the turning radius is constrained.

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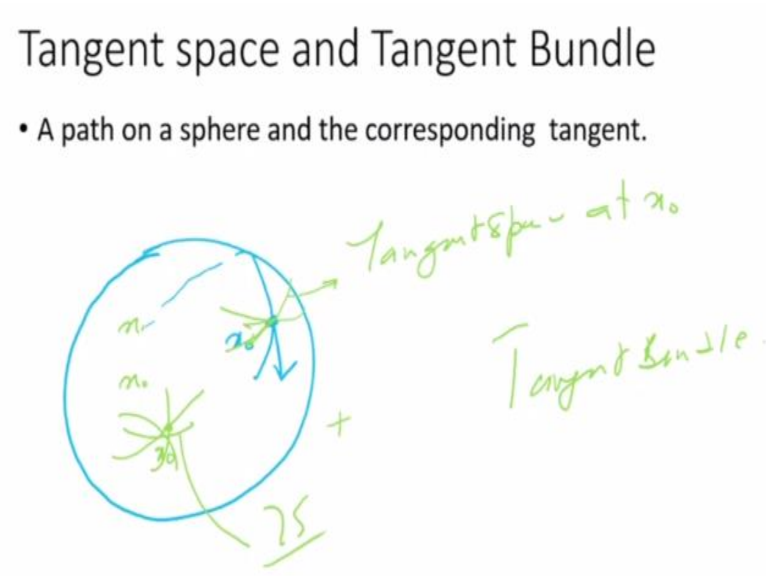

Now, that basically tells us that we can think about this as a tangent and a tangent space. So, you understand very nicely that this is a tangent, right. So, this is a tangent to the curve at this point. Now, at that point there can be many curves, right. This curve can take different turns, so there can be different theatres, so there can be turns in this direction or passing through that point, but these tangents would be in different directions.

Now, the collection of all those tangents makes up a vector field. And this is better to or easier to understand like this if we have a sphere like this you can imagine that there is a path which is coming from that side coming like this and coming this side. So, this is a curve on a sphere, it is on the other side. Now, at this point, let us say x_0 we have a tangent which is like this. Now, on this at this point x_0 there can be other curves.

So, you can very well imagine that they could be other curves here. Now, each of those curves are going to have tangents. Now collection of all those tangents are what makes up the tangent space, tangent space at x_0 at the point x_0. And now, there can be different x_0, x_0 can be different at different locations on the sphere which means at each of these x_0, there can be different curves and each of them will have a tangent space.

So, the collection of all these tangent spaces together makes up what we call a tangent bundle. So, now we have seen the concept of a tangent space and the concept of a tangent bundle.

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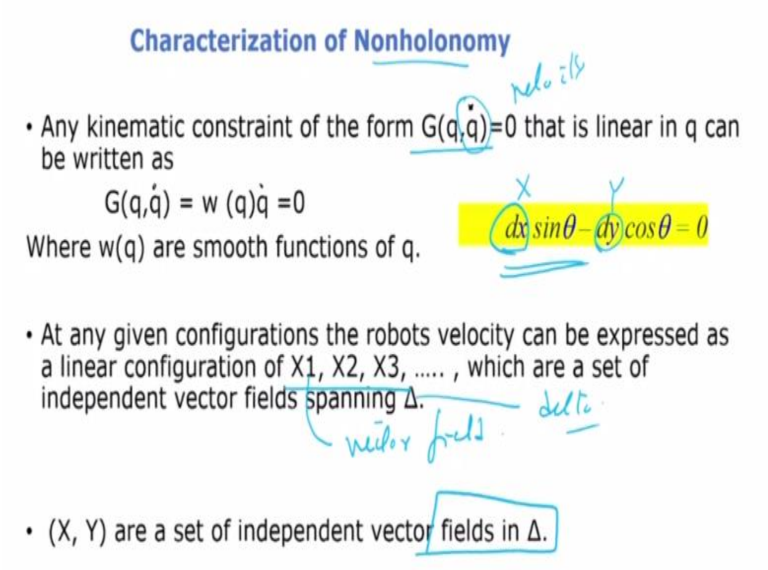

Now, we have also seen that any kinematic constraint can be written in the form $G\left(q,\dot{q}\right)$ in the case of non-holonomic kinematic constraints basically it is written in this form and what we have seen that these are functions of velocity like in this case. There we have a dx and dy, these are incremental velocities. Now, at any given configuration the robot velocity can be expressed as a linear combination of different vector fields.

Each of this is a vector field which are a set of independent vector fields spanning at Δ. So, this basically tells us that at any given configuration, the robot's velocity can be expressed as a linear combination that is what we have seen that they are vector fields and which are a set of independent vector fields spanning all of Δ. Now, X and Y are a set of independent vector fields in Δ. Now, what is this X and Y?

This is my X in the X direction that is my Y in the Y direction. So, there are different vector fields at any given configuration of the robots and this velocity can be expressed as a linear combination of X_1, X_2, X_3, X_4 which are a set of independent vector fields spanning Δ.

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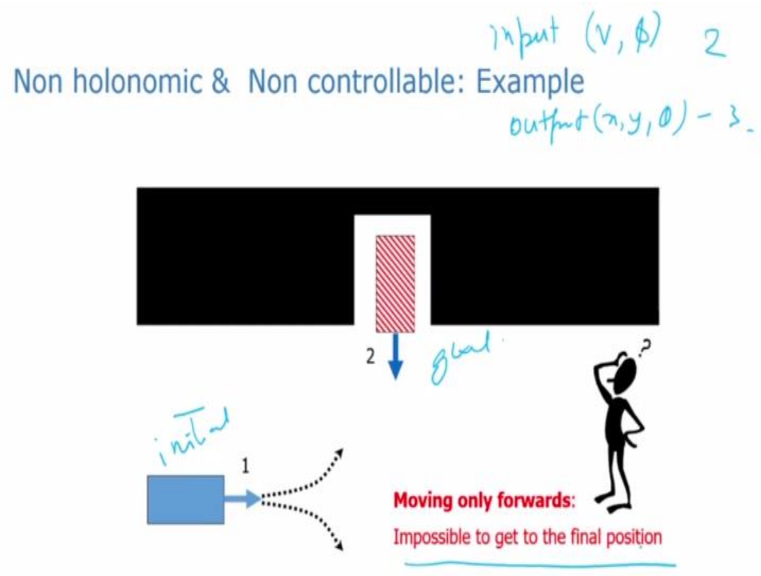

Now, we move on to what is controllable and non controllable. Basically, why this question comes up is essentially because we have seen that there are two control inputs. The control inputs are wheel velocity and steering angle Φ, but the outputs are x, y and θ they are 3. So, this is 2 and that is 3. So, for example if you look at this figure here is it possible for this car to go and park there? So, the car is here. This is my initial point and this is my final point or this my goal configuration, let us say this is my goal configuration, is it possible to go and park? How do you know? Now, if the car is only moving forward then it is impossible to get to the final position.

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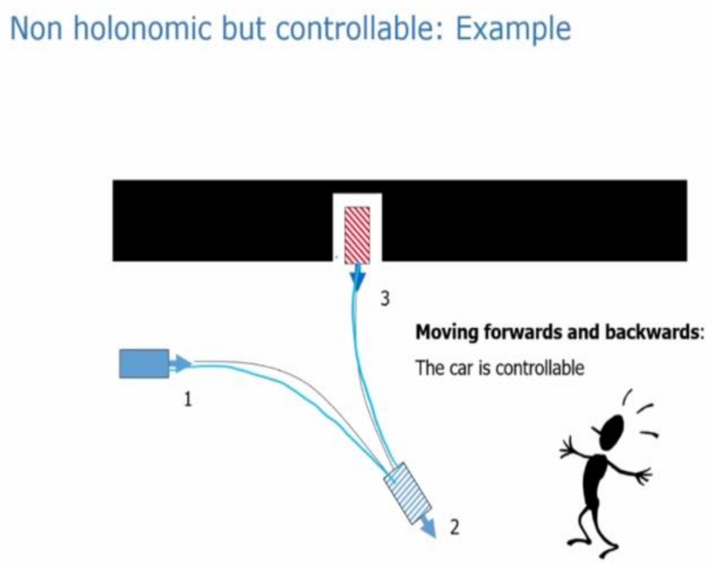

Whereas if the car can move forward and can move backward then the car can go and park like this. For example it goes front like this and then it goes back like that and it is parked in this configuration. Now in this case, the car is controllable, in the previous case the car is not controllable. Now, I hope you understand the meaning of not controllable means you cannot get to the desired position.

Now, this was an easy example and we have eyes so we can see and try and figure out that okay maybe we can move like this and move like this and go there. But you would remember that in path planning program which is doing this automatically, so how would the program figure out is there any mathematics behind that?

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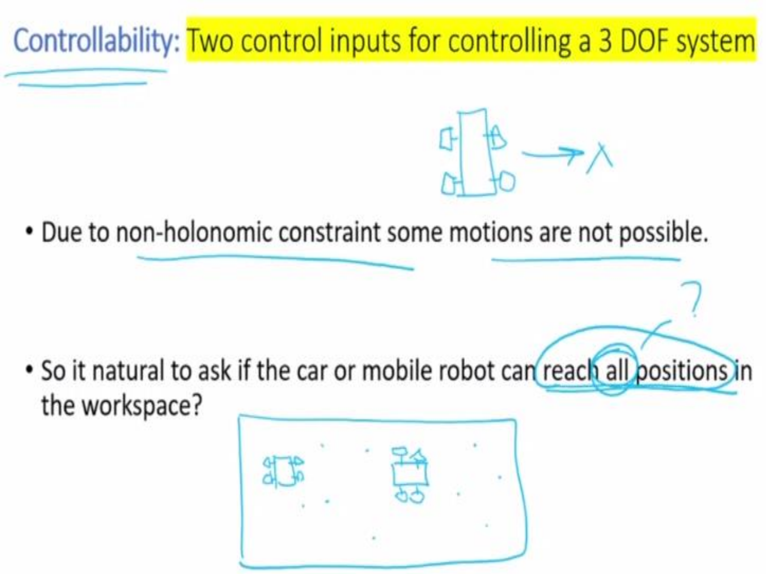

Now, so controllability is two control inputs for controlling 3 degree of freedom system. Now due to a non-holonomic constraint, some motions are not possible this we have seen. For example, a car cannot go sideways. So, this car cannot go sideways direction that is not possible because of this non-holonomic constraint. So, it is natural to ask if the car or mobile robot can reach all positions in the workspace, can it reach all positions? So, I have a workspace like this and I have a car.

So, the question being asked is, this car is subjected to a non-holonomic constraint, which means it can go sideways, but can it go and park anywhere in this workspace? That means are all configurations possible? How would you figure that out? Is there some mathematics behind that that is what we are looking at here now. So, I have a mobile robot which is subjected to non-holonomic constraint and I want to find out if this non-holonomic mobile robot can go and park at any configuration in this workspace.

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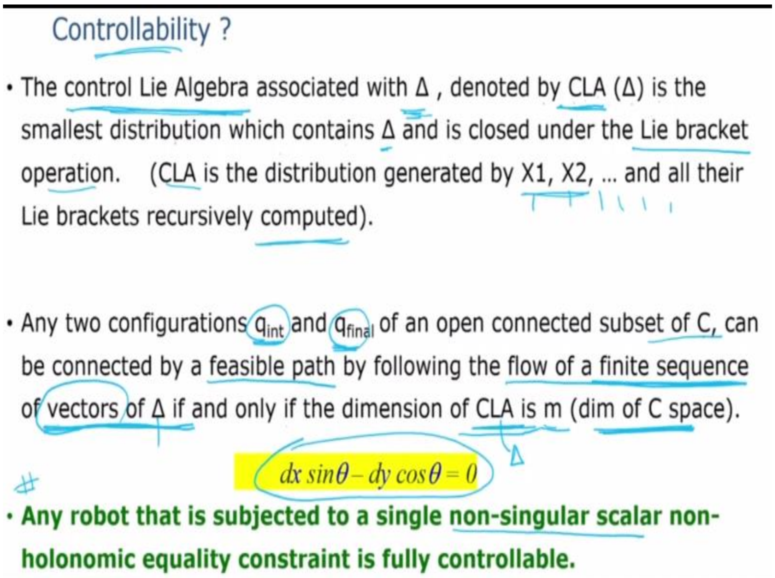

So, we're trying to answer this question of controllability. So, the control Lie algebra associated with Δ denoted by CLA, so CLA stands for control Lie algebra, is the smallest distribution which contains Δ and is closed under the Lie bracket operations. So, we will just come to what is Lie bracket. CLA is a distribution generated by X_1, X_2 and all their Lie brackets successively computed. What this means I will just come to an explain.

So, we are seeing this new term control Lie algebra associated with Δ denoted by CLA Δ is the smallest distribution which contains Δ and is closed under the Lie bracket operation. CLA is the distribution generated by X_1, X_2 and all their Lie brackets recursively computed. Now, any two configurations $q_{initial}$ to q_{final} of an open connected subset of C, C is the configuration space, can be connected by a physical path by following the flow of a finite sequence of vectors of Δ if and only if the dimension of CLA is m.

That means the control Lie algebra of Δ is m, which is the dimension of the configuration space. Let us try to understand this by looking, I will just come back to this page again.

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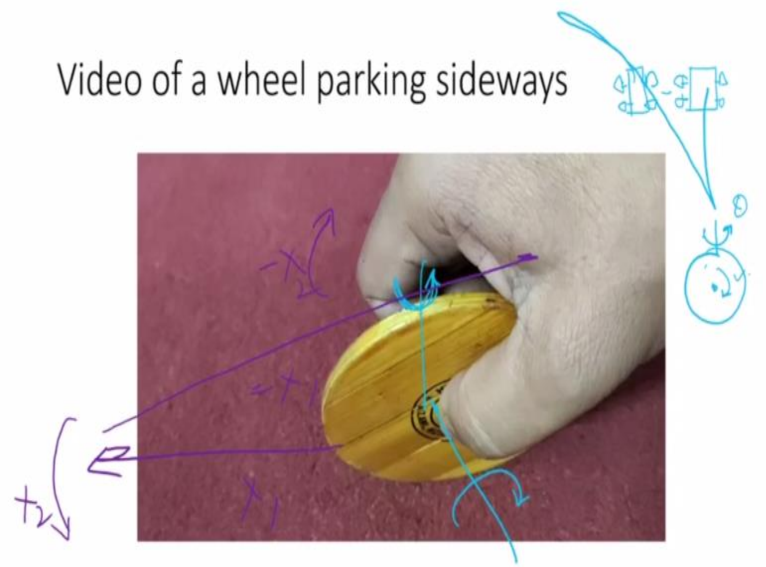

But let us look at this example, what I am trying to explain here or what this mathematics is all about. So, first of all it says that suppose there is a mobile robot which cannot go sideways, in that case by a combination of motions can it actually goes and park sideways. For example, this is a mobile robot, this is a car kind of a mobile robot. Now we know it cannot go sideways. But then by making or following different paths, can it actually successively have different parts and then finally go and park sideways.

So, in that case, it is not going sideways exactly and parking there, but it is probably making a series of manoeuvres and then going there, say for example it goes like this, then comes like this, then goes like that. It has actually gone sideways, but by following different paths not exactly there. Now let us take the case of a wheel. Now this is a wheel as you can see in this figure, this is the wheel. Now in this wheel, this wheel can rotate about this axis like that.

So, this might suppose I say there is an axis there, so it can rotate like this and it can twist about this axis. So, there is one more accuracy on which it can turn. So, this wheel can rotate and it can steer it that is my steering axis, then it can also change its direction there. So, it can rotate and can become in some other direction. So, there is a θ there and there is a velocity here. Now let us look at this video and it becomes clearer there.

(Video Starts: 13:36) So this vehicle or wheel can go forward like that. It can roll forward, it can rotate forward and then go back like that. So, rotate forward and go back like that and it can twist about its main. There is a steering, it can steer about the main axis. So, this is my steering, it can twist like that. This is like a unicycle model. So, it can twist about the axis. So, it has two motions. One motion it can go front and back you can see that.

The number two is it can twist about his axis, this is what it can do. Now, the wheel cannot go sideways, see the wheel cannot go sideways, this sideways motion is not possible because of the single non-holonomic constraint. So, because of this non-holonomic velocity constraint, this robot cannot go sideways. So, my question here is we know the robot cannot go sideways, but it has these two motions, it can rotate and it can, so in that case how would you make it go and park sideways? So just see what is being done.

It is going forward, so this is my first motion. Notice this carefully, it goes forward. It rotates forward then it twists, the steering angle changes. Then it goes straight back, then the steering angle changes again. I will reply it. So here what it is doing is it is going forward that is one velocity vector field. It can have different velocities, so that is a vector field that is one, let us call it X_1. Next it can steer, let us call it X_2.

Once it goes there, it is turning, this one you can see it is turning there so that is my X_2 that is another vector field. Then it is going back in a straight line. Now because it is at an angle, you can see it has gone at an angle that is my vector field 3. And it has again steered that is my vector field 4 forward. **(Video Ends: 15:28)** I can say X_1 is the forward direction, X_2 is the rotation, X_3 is back, and X_4 is minus X_2 again.

That is what I mean by saying that this what we were talking about here that any two configurations $q_{initial}$ and q_{final} of an open connected subset of C, C is the configuration space, can be connected by a feasible path by following the flow of finite sequence of vectors. So, we have a finite sequence of vectors which is it was going forward, it was changing the steering angle then going back, changing the steering angle again.

So those are finite, it is a flow of finite sequence of vectors of Δ if and only if the dimension of CLA is m and where m is the maximum dimension of the C-space. So this basically tells us that this car which is subjected to a single non-holonomic velocity constraint can actually go and park if we can get a sequence of vectors, X_1, one X_2, X_3 vector fields like this and using control Lie algebra if we can generate vector fields such that this can go and park there.

And that can only happen if the dimension of Δ is equal to the dimension of the C-space. Now, there is a corollary here. The corollary is any robot that is subjected to a single non-singular scalar non-holonomic equality constraint is fully controllable. That means if you have a single constraint like this which is given here that a car is subjected to such systems are fully controllable means you can take and place them anywhere inside the workspace. Now, let us look at an example which will make it a little bit clearer.

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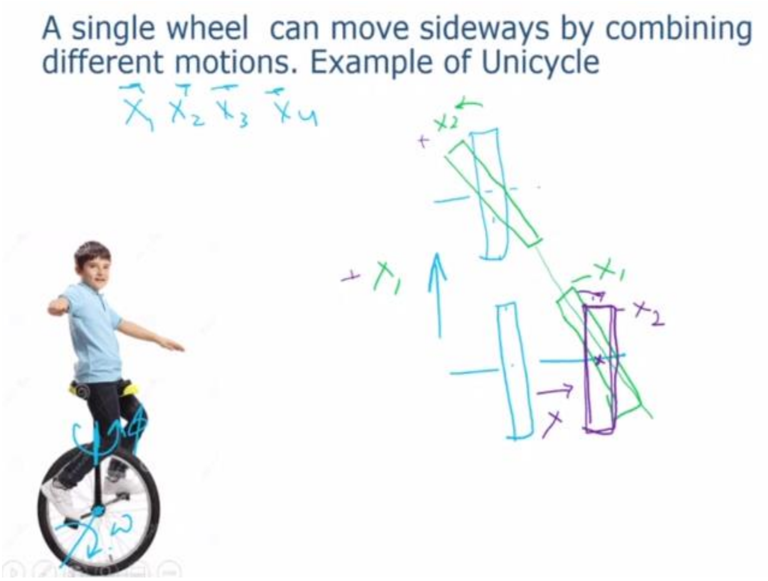

Let us look at an example which will tend to make this clearer. So, what this is basically telling is that a single wheel can move sideways by combining different motions. This is X_1, X_2, X_3, X_4 like this, each of this is a vector field. So this is a single wheel. Now, it can rotate forward similar to the wheel that I just showed, this can rotate about the axis and it can steer about that axis. So, it has two inputs now.

There is a velocity let us call it ω and there is a steering angle which is Φ. Now, with these two inputs, can it go to every position in every configuration in the workspace? Now, the configuration has three dimensions, right. So, the vehicle has, a normal car has x, y and θ, right. So, now what we can see is let me take the top view of this wheel. This my top view is three axes. So, if this is the top view, it can rotate forward which has gone forward like this what I just showed in the video.

So it has gone like this, it has gone forward then what we do is we rotate or we change the steering angle. When I change the steering angle it comes like this. So this was X_1, this is X_2 it has turned in that direction. Then X_3 is it comes backwards like that. I can say this is $-X_1$.

So it has come backwards and the wheel has positioned here now. And then what I do is I do $-X_2$ again. I do $-X_2$, so it comes here. So, it goes this side to become $-X_2$.

So you can see that this was $+X_1$, that is $-X_1$, this is $+X_2$, this is $-X_2$. So what it did is basically it used different manoeuvres to actually go sideways and it has gone in park sideways like this. That means this point here, Lie configuration space is accessible. The robot cannot go sideways or this wheel cannot go sideways, but then by using different manoeuvres it can actually go and position there. That means the system is fully controllable because it can go and position at every point in the workspace.

Now, let us go back to the definition again and try to make sense out of it. So, it says that the control Lie algebra associated with Δ, denoted by a CLA Δ, is the smallest distribution which contains Δ and is closed under the Lie bracket operation. I will just come to what is Lie bracket operations. Now CLA is the distribution generated by X_1, X_2, and all their Lie brackets recursively computed. We have seen what is X_1, X_2.

Now any two configurations $q_{initial}$ to q_{final} of an open connected subset of C can be connected by a physical path that means it can go from $q_{initial}$ to q_{final} by following the flow of finite sequence of vectors of Δ if and only if the dimension of CLA is m. That means the dimension of the control Lie algebra is equal to the dimension of the C-space. In that case the system is fully controllable this what it is saying.

So, here we saw the example that this wheel can actually go sideways just by making different manoeuvres. First it goes like this, then it turns that side, then goes back and then it turns in that direction. So, this is X_1, this is X_2, this is $-X_1$ and that is $-X_2$. Now, so this is the case of a single wheel that can move sideways by combining different motions.

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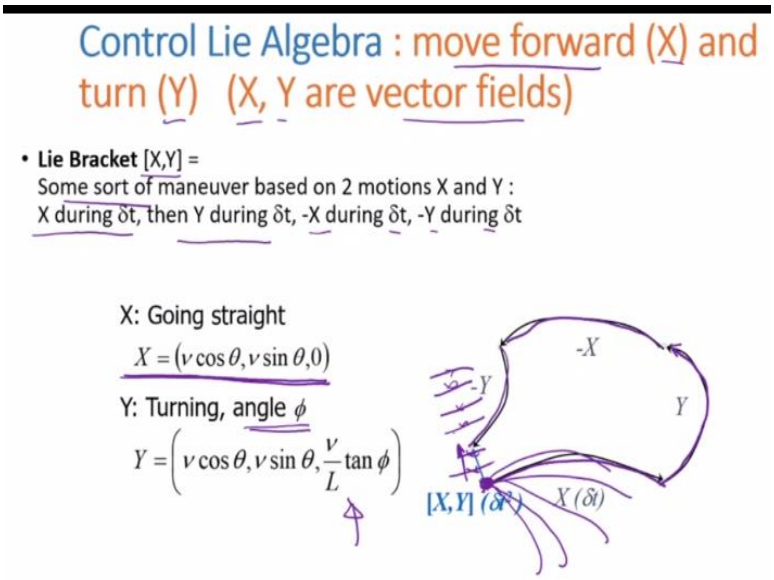

Now, let us look at the mathematics of control Lie algebra. Now what is control Lie algebra? In this particular case, it was moving forward by X and turning Y. So, X and Y are vector fields. Now the Lie bracket operation is written this way with bracket X and Y. This is some sort of manoeuvre based on two motions X and Y, X during δt, very small amount δt during that time and Y during Δ t, –X during δ t and –Y during δ t.

Now, what this basically means is that this wheel can go forward. This is my velocity vector v cos θ, v sin θ and the turning angle is Φ, so, it can turn by this much. Now, let us look at here. So, this is my starting point. So, it went like this X, then when turned Y, then turned –X, then turned –Y that is exactly what I did here. So, there was a plus X, X_1 is plus and X_2 is plus, then it went $-X_1$, then it went by this amount $-X_2$.

So, this is exactly what it is doing, and in the process it is going from here to here, which is actually sideways motion. So, by changing this combination, now there can be different axes right, it can go like this, like this, like this, like this, like this; all axes are possible, whatever axes are possible. So, by using different axes, it can go sideways and park in different places that mean all points here are actually controllable. So, the wheel can go sideways and park anywhere.

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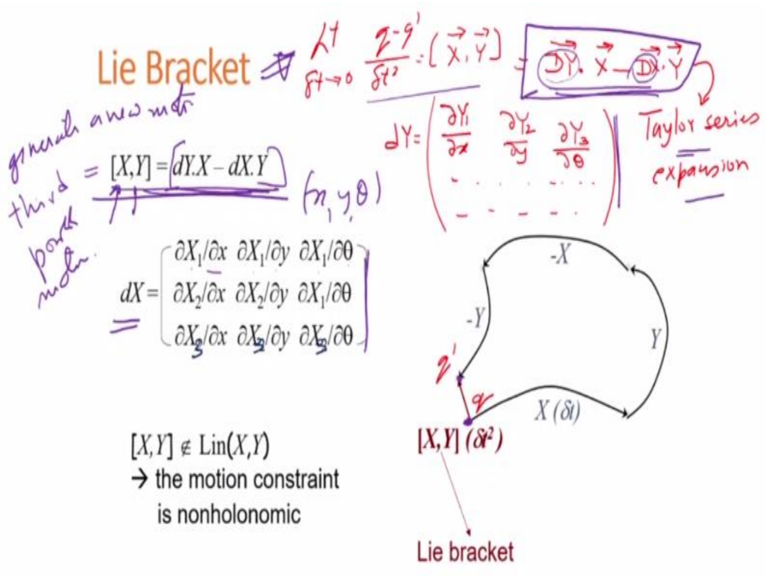

Now, let us look at this again. What control Lie algebra is actually doing is by using Lie brackets we are generating another motion which is given this way. So, this is basically generating a new motion. So, it is taking X vector field, Y vector field and then generating another motion by using Lie brackets. Now, what this basically means suppose you are at q and you want to go to $q^{'}$, there are basically four motions that are required.

And it means from here if you look at the $\underset{\delta t \to 0}{Lt} \dfrac{q-q'}{\delta t^2}$. If I take the Taylor series expansion of that what I get is this term. So, what I do is I use Taylor series expansion on this and I get $\left[X,Y\right] = dY \cdot X - dX \cdot Y$ using Taylor series expansion where $dX = \begin{bmatrix} \partial X_1 / \partial x & \partial X_1 / \partial y & \partial X_1 / \partial \theta \\ \partial X_2 / \partial x & \partial X_2 / \partial y & \partial X_2 / \partial \theta \\ \partial X_3 / \partial x & \partial X_3 / \partial y & \partial X_3 / \partial \theta \end{bmatrix}$.

In our case, remember for the car it is x, y and θ these are the three variables that are there.

Similarly, $dY = \begin{bmatrix} \partial Y_1 / \partial x & \partial Y_1 / \partial y & \partial Y_1 / \partial \theta \\ \partial Y_2 / \partial x & \partial Y_2 / \partial y & \partial Y_2 / \partial \theta \\ \partial Y_3 / \partial x & \partial Y_3 / \partial y & \partial Y_3 / \partial \theta \end{bmatrix}$. So, what this is telling us is if you have two vector fields X and Y, by doing this operation which is called Lie bracket, you can generate a third motion, this is a third possible motion and successively you can generate different motions simply by taking the Lie bracket of these vector fields that is what it is telling us.

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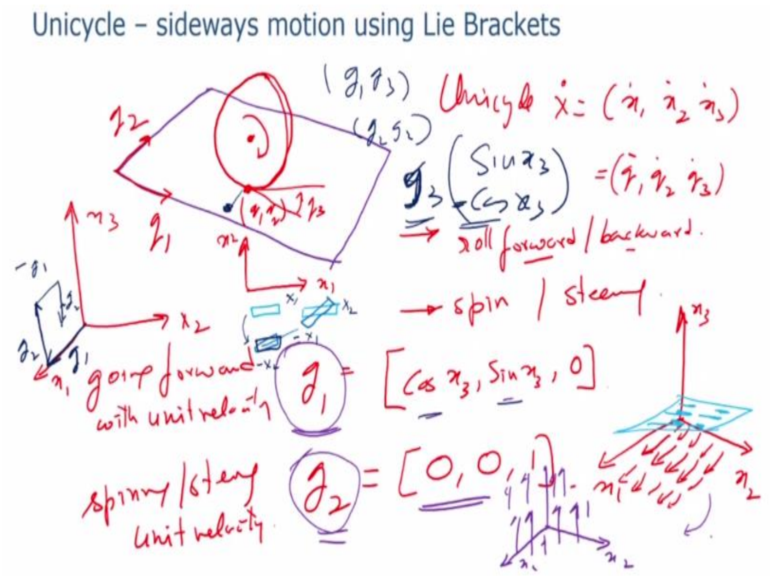

So, let us look at the analysis of this unicycle. Now for this unicycle let us look at it this way. So, this is my unicycle that one wheel and the kid was sitting on top of it that is this one. So, this is a unicycle. So, this is one wheel, which can rotate about the axes and it has a steering about the vertical axis. So, if you look at that one wheel, then basically this is my wheel, it is making contact there. So, I can say the unicycle $\dot{X} = (x_1 \quad x_2 \quad x_3)$ these are vectors x $_1$, x $_2$, and x $_3$.

So x, y and θ which I can write in terms of q as q $_1$, q 2 and q $_3$. Now in terms of velocities, I can simply put a dot there indicating with these I can also find the velocity. Now, in this particular case let us write down the frame of reference, this is q $_1$ that is q $_2$. So the contact point it is q $_1$, q $_2$ at that contact point and it is pointing in this direction so that is my q $_3$. So, it is like x, y of this point and θ is the angle, the steering angle.

Now, this wheel can roll forward or backward, and it can spin about a point or we can also call the steering. Please note that it has to go forward and backward, in the case of the car we saw if the car can go only forward and not backward, then it is not controllable. So, it has to be able to move forward and backward and it must have a steering. Now, suppose I define two vector fields when the robot is going forward and turning.

So going forward is equal to let us call this, this is one vector field $g_1 = [\cos x_3, \sin x_3, 0]$. So this is going forward with unit velocity. So, this wheel is rotating about this axis with unit velocity. So in the x direction y direction it is going to be cos x_3 and sine x_3 and there is going to be 0 because this contact point is always there. Now, what about the spinning or steering? Steering is another vector field $g_2 = [0, 0, 1]$; let us say unit velocity.

So we have two vector fields, one vector field for going forward and another vector field for turning for steering the steering angle. Now, I can draw this vector field in terms like this, this is my x_1, x_2. This is x_1, x_2 and this is my x_3. This corresponds to x, y and Φ if you want it that way. So, if it is going only in the x direction these are the vectors or vectors field which are pointing in this direction, it is going only in the x direction.

If it is going only in the y direction, then I can have another one here which is going only in the y direction. It is going on in the y direction. And if it is going only in the z direction, then I can also have a set of vector fields which are going only in the x_3 direction. This is my x_1, x_2, in that direction. So basically, what we are saying is that the final motion of this wheel is going to be made up of all these vector fields.

And to perform the Lie bracket I am going to take one vector field of going forward, one vector field of steering, and I am going to perform the Lie operations and see if I can generate another vector field. So, what we do here is basically I am going to use this formula. This formula I am going to use here as given here.

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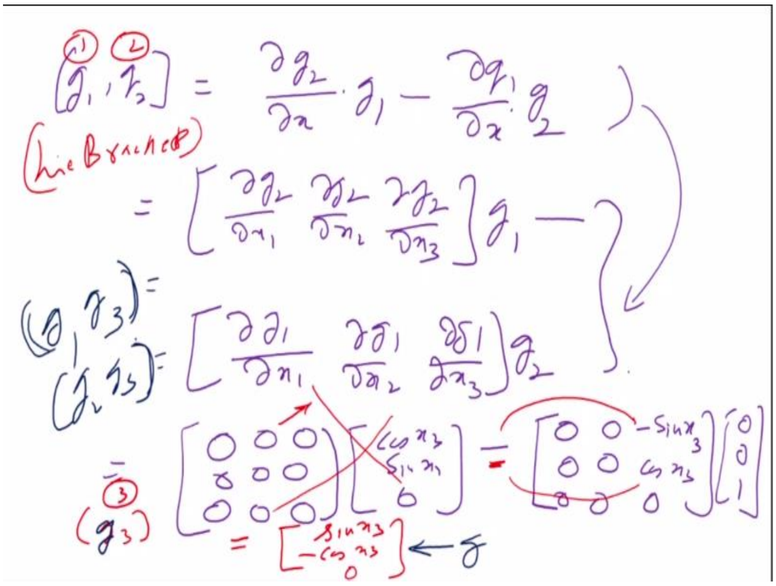

So, the way we write it is $[g_1, g_2] = \frac{\partial g_2}{\partial x} g_1 - \frac{\partial g_1}{\partial x} g_2$. This is exactly what we wrote there. So, this is if you just go back and see what is written here is this derivative with respect to the second field into X and derivative of the first field into Y. So, this my first field and this my second field, the way we have written it here is like this.

Now remember what is g $_1$ and g $_2$? The g $_1$ and g $_2$ are given here, g $_1$ and g $_2$. So, if I take the derivative now, this is $[g_1, g_2] = \begin{bmatrix} \frac{\partial g_2}{\partial x_1} & \frac{\partial g_2}{\partial x_2} & \frac{\partial g_2}{\partial x_2} \end{bmatrix} g_1 - \begin{bmatrix} \frac{\partial g_1}{\partial x_1} & \frac{\partial g_1}{\partial x_2} & \frac{\partial g_1}{\partial x_2} \end{bmatrix} g_2$. So this is equal to this. Now all I have to do is put in the values. So, let us look at I am taking the derivative of g $_2$ with respect to x $_1$, x $_2$, x $_3$. $[g_1, g_2] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \cos x_3 \\ \sin x_3 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 & -\sin x_3 \\ 0 & 0 & \cos x_3 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

So this term will fully become 0 because this matrix is fully 0.

So this term will fully become 0 and what we will be left with is $\begin{bmatrix} \sin x_3 \\ -\cos x_3 \\ 0 \end{bmatrix}$. So, I have generated a third, this is my g $_3$ now. So this was one vector field, this was another vector field, I have generated a third vector field now by using the Lie bracket. So this is what I call by Lie brackets.

So, I have taken two vector fields, performed the Lie bracket operation and I have generated a third vector field now which is sin x_3 and cos x_3. Now, let us go back and look at what is the meaning of this. So, this is the axis. So let me draw the figure again or can we do it here itself. So basically, what we are doing here is if I look at the top view, the top view is in this direction. No, I just draw it in the other figure, there is no space. I will just write here.

So if I draw my axis here x_1, x_2 and x_3. Now, how did this wheel go sideways. Basically, if this is my x axis, x_1 and that is my x_2, then the wheel was here. The wheel first moves forward, so this is my wheel, this is my top view. It moved forward, it went there, then it rotated, it rotated like this, then it came backward like this and then it rotated again. Rotated means there was a steering angle change, it became like this that is how it went sideways from here to here.

So, this was one vector field x_1, there is another vector field x_2, this is $-x_1$ and this is $-x_2$. If I draw it in terms of axes, it went in this direction, this is my g_1 vector field, it went forward. Then what happened is it rotated. So, if it rotates means it is using the x_3 axis so it is rotated by some angle, the steering has changed. So, this my g_2. Now, it has gone back again, it has gone back this is my g_1.

This is $-g_1$ and it has come here now this is my $-g_2$ because it is having a steering and rotating it will not come back to x_1, x_3 plane, this is going to be a slightly different point. So, this is my $-g_2$ now. Now if we look in terms of what was the third vector field that we obtained, what was the vector field that we got, we got sine x_3 $-$cos x_3, this might be the third fellow, this is my g 3. So, sine x_3 cos x_3.

So, it is what we got the third fellow g_3 as sine x_3 and cos x_3 minus. So, if you look in the x direction, this is my x, x is in this direction right. So, it has gone minus in the y direction which is indicating that it has gone in the minus direction here. So that is my x_1, x_2 axis so it has come here now, it has moved sideways. So, this is my $-$cos x_3. So, now this is indicating that wherever we started off from.

We started from cos x_3 and sine x 3 and we generated another field which was a sine of x_3 and minus cos of x_3. Now because this is a minus it is showing that has gone sideways actually. So, this basically is telling us that the vehicle has actually gone sideways and it has parked by using a third vector field which is g_3 now. So, successively I can have another vector field g_1 and g_3 now or g_2 and g 3 to generate another motion.

So, by successively generating these motions, the car can go and park anywhere this is what it is saying. So, basically, let us go back to that definition again that we talked about. So, the theorem is very important there. So, it says that any two configurations in $q_{initial}$ to q_{final} of an open connected subset of C can be connected by a feasible path by following the flow of a finite sequence of vectors using control Lie algebra.

And the system is controllable if and only if the dimension of control Lie algebra is m which is the dimension of C-space. So, this is what it is saying in terms of controllability. And this also we can use to prove that the single wheel is actually going sideways because by using Lie brackets we are generating a third vector field which is actually making it go sideways, means it has a velocity in that direction which can actually go sideways and go and park anywhere it wants. Now, this is basically to do with control Lie algebra.

And by using Lie brackets we can prove that there is a forward direction of motion which is given here, there is a steering motion here, but Lie bracket is giving you a third motion which is also possible which is actually sideways. Now, the sideways fellow is the combination of other two vector fields. Now why successively using differently brackets, so g_1 and g_3, g_2 and g_3 I am going to generate more vector fields. So, each of this will become a new vector field, it may be new, may not be new, but I can generate more vector fields, right.

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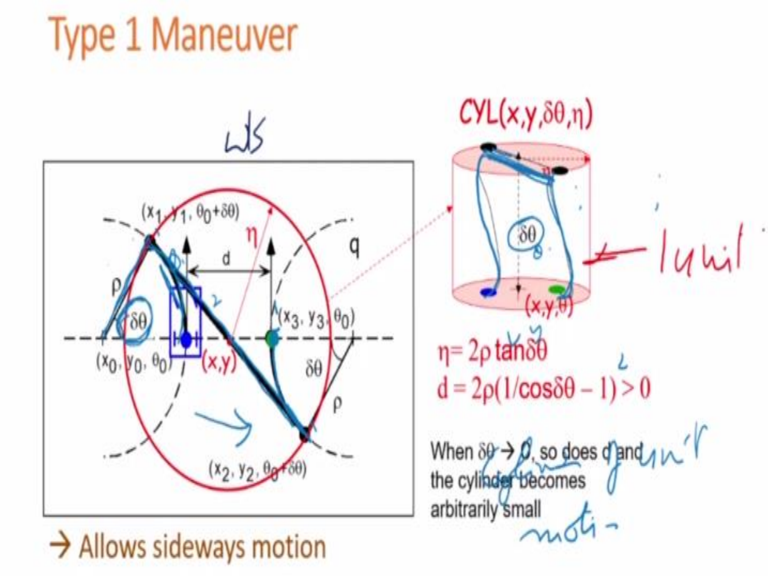

So, this is how we use Lie brackets to try and generate other motions which apparently seem to be constrained by the non-holonomic constraints. So, if the systems like a car or a wheel is fully controllable because again they are subjected to a single non-holonomic constraint and hence some systems are fully controllable that is what we are told. That means, if this is my workspace and there is a car there, you should be able to take this car and park it anywhere you like, not with one motion, but with many motions following many paths.

Let us look at one example here. The car is here, I want to park it sideways. I want to park it exactly here. So, I want to take this car and I want to park this car sideways. We looked at the example of the single wheel robot, now this is a case of that. How is it possible? It cannot go straight like that, but Lie algebra tells us that there is another vector field which can actually take it sideways. Now, how do we do that? It takes a curve like this, the car is here now.

So, it has taken the steering wheel, it has taken a particular angle Φ, the car is here now. Then the car goes backwards, it is here now. So, it has gone straight back like this. Now, here it turns the steering wheel that side and then goes like this and it is parked here now. So, the car is actually gone and parked there. So, there was one motion here, second motion here and third motion there. Now, because the car can have different steering angles, of course there is a limit to that.

It has different steering angles, so for different steering angles it can have this kind of a circle or it can have slightly what is a wider circle or it can have maybe a slightly smaller circle there depending on Φ now the turning radius. From here it can probably come backward like that and then go again. So by doing this set of motions, the robot can actually go sideways and park wherever it wants because the system is fully controllable.

And we are able to prove it by showing that if it has a set of vector fields like this, we can use Lie brackets and then generate other vector fields x_1 and x_2, will give you a third one x_3 which can be in some other direction, it can be sideways. Now this is mathematically showing us that the car can actually go and park sideways using Lie brackets. Now, in terms of an algorithm how do you make the car go in park sideways that is what is more interesting to us.

Now let us look at this example a little bit more detail. What did this car actually do? The car actually changed this angle δ θ. So, this is the angle that it is making at the centre depending on whatever is the turning radius, this my turning radius that is Φ, right. So, depending on this turning radius there is going to be a δ Φ. So, if I look at my graph if this is x, y and Φ, if this is x, this my x axis, y axis and that is my θ axis this side.

Then the first thing it did is it changed the θ angle. So, basically it changed the θ angle. So, from here it went up like that, so it is changing this one now. So, this is what it is doing it is going here. Then it went back straight. So, this is my straight motion. So, it went straight and then it changed the θ angle again to come back here. So, what it is doing it is basically having a cylinder kind of motion which basically means that it is taking a curve like this which is on the X-Y plane with a particular δ θ.

Then it is going straight it is on that plane itself it is going straight there for that particular δ θ and then this changing the δ θ coming back here, changing the δ θ and then going. So, basically this is a cylinder, we can call this a cylinder of unit motion.

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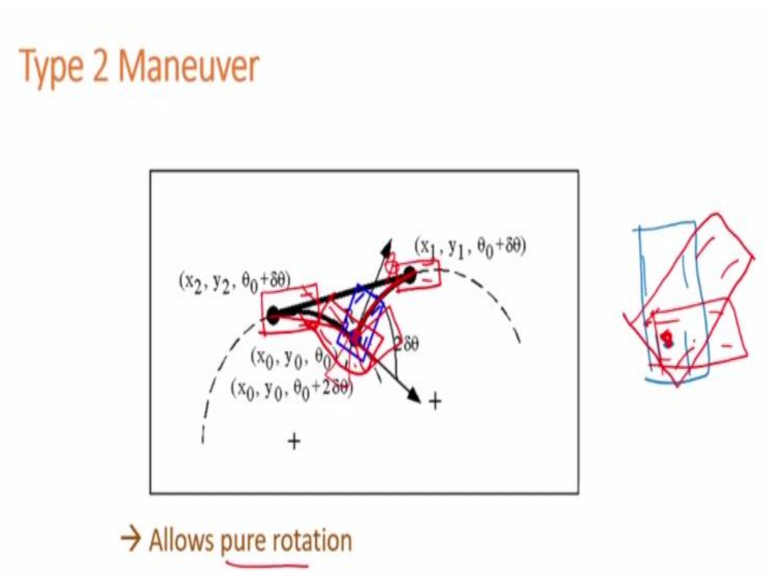

Now, by having a connection of the cylinders, we can make it go in different directions. Now, let us look at this example again. The car is here, what about pure rotation. I am wanting to rotate the car and face it this side. In the previous case, in this case it was going sideways, we are talking about the system going sideways. Here, we want to rotate the car. So, the car is like this, let me draw and explain it again. This is my car. So, this is the point P.

I want to rotate the car, let us say I want to rotate it and make it like this at that point. It cannot rotate like this about this point, in one motion it cannot because of that non-holonomic constraint, but what we can do is see the car is here initially the blue one. It goes forward like this, it has a particular steering angle Φ. Once it has reached this point it is like this now. Then comes backward and becomes like this. Then you turn the steering angle like this, take the turn and comes in. So, what happens?

The car is now here. So, this is actually showing you that there is a pure rotation about the point P and you can again prove this using Lie algebra by using Lie brackets, the car can actually rotate. You can take any angle, the car can actually go in position to any angle. The previous one showed the car is actually going sideways. This one is showing you that the car is rotating about that point now that means it can take any rotation. That means it is fully controllable, so it can take all rotations.

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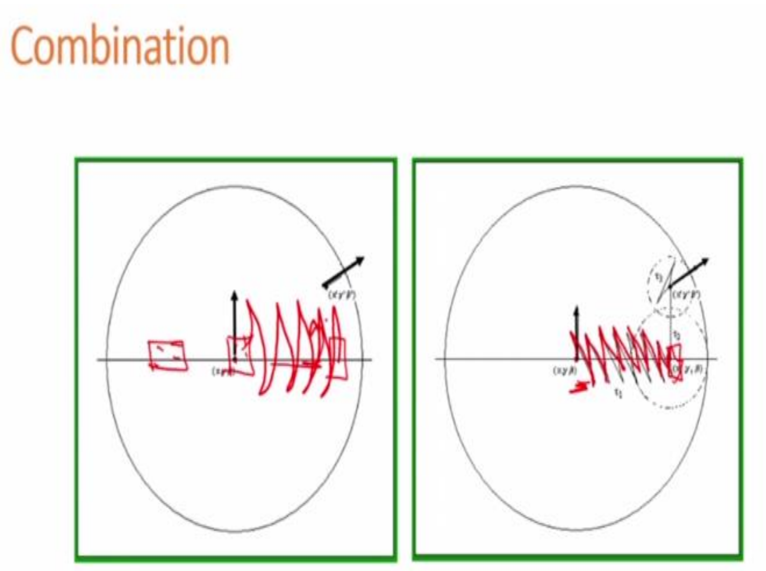

Now, what about a combination of this, for example suppose the car has to go and park here. It can take a combination of these motions like this, like this, like this, like this, like this, like this and then go there. Each of this one unit was one motion, one motion that I showed here. So this is one unit. So, this is my one unit and by having a number of this we can actually go and park anywhere.

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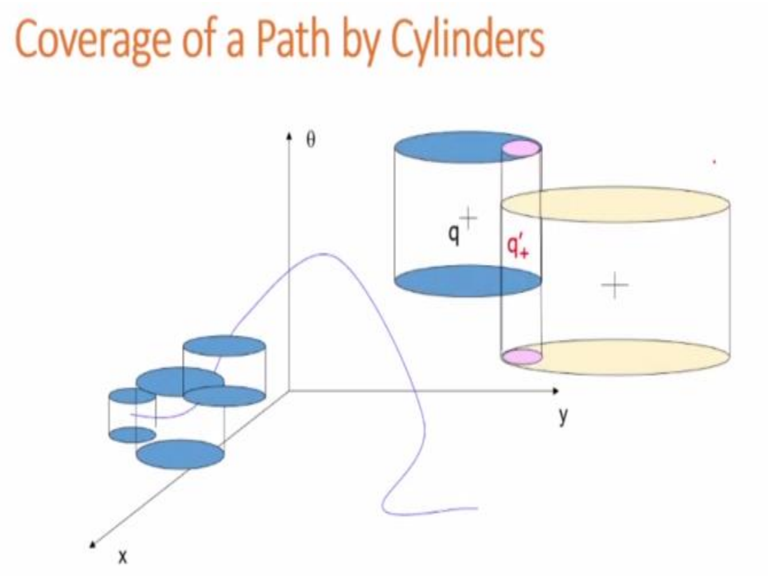

Now coverage of a path by cylinders, each of this path can be covered by small cylinders, each one corresponding to one manoeuvre. So, here we have small cylinders which are covering and it is going from one point to another point and achieving.

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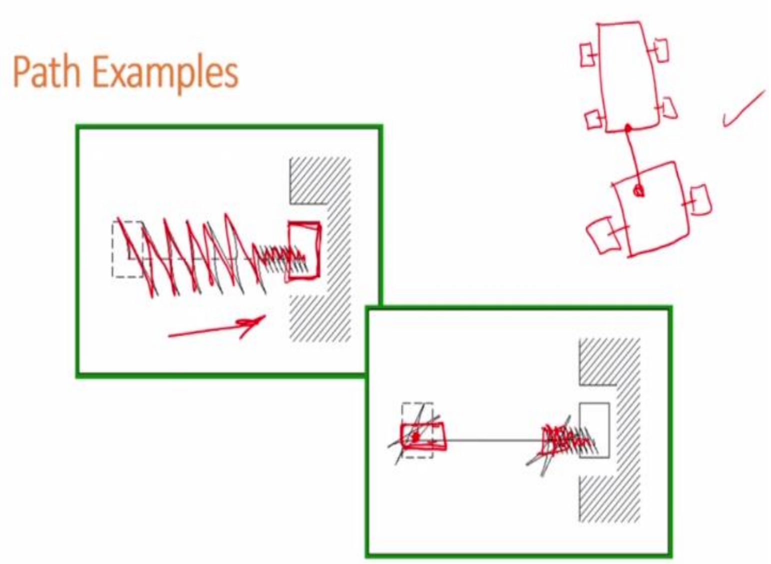

So, here for example, example of a path. Suppose the car was to go and park there, what it is doing is it is taking one set of cylinders like this then it is taking a small set of cylinders like this and then going and parking there. So, although it could not go like this directly, it went there. Now the next important question here is, are there other ways? Now for example, if it does this kind of rotation can manoeuvre, you can rotate about this point go here.

And after going here, you again manoeuvre and rotate yourself. So, what I am saying is that in the previous case what we were doing, so in this case the car wanted to go sideways, it was like this, it had to go there. So, it is using these manoeuvres to go there like this. It is pointing in the forward direction. Now suppose I make it turn, make rotate it like this, let it go till there and then upright it again what about that.

So, if you look at this what is being suggested in this figure is that it first there was a pure rotation about that point, it went forward and then it corrected itself again and then took small motions. So, it may be possible to get into a particular configuration by following different paths, but the important path here is that the system is fully controllable, so you can go to any point and park. So, simply by looking at this it would appear that it is very difficult and may not be possible, but strangely this is a fully controllable system.

Now more interesting as we come along is what about this case? I talked about the tractor and the trailer. This is one car, what if I add a trailer? So, I add a trailer there now, that is my trailer. Now what? Now is still controllable? That means you can take the tractor and park it wherever you like? Now, simply by looking at this is difficult to say, but if you follow this control Lie algebra, we will see that the system is also fully controllable. So actually, you can go and park wherever you like.

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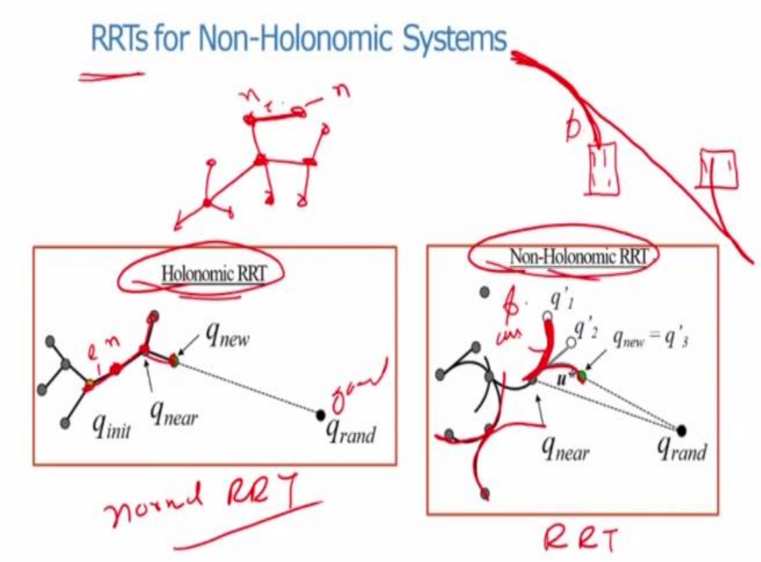

Now, let us see further how do we plan the path for such kind of motions? Now, the car has to follow a sequence of motions to get to wherever you want it to go. So, one way of doing it is by using RRT, so rapidly exploring or rapidly expanding random trees, this is something we studied long time ago. So, this is the normal RRT where we start off from here, we take a node out. So these are nodes and these are edges, something we studied quite a few lectures ago.

And we generate another node which is feasible, another node connects with paths and try and see and connect them. So, if I have a goal point and initial points, I can connect this tree kind of structure. So, I start from one point, get an x node, get another node, get another node, get another node here okay. And then finally cover the whole workspace and then see if I can connect the initial point to the goal point. Now, these are straight lines.

These are nodes, these are edges, but in the case of a car what we see is that we were talking about motions which were like this. So, this was my car and the car actually took a turn like this, then it went straight, then it took a turn and came here. So we are looking at this curvature that the car can take so we are interested in the turning radius five okay. So, the steering wheel Φ is very important for us.

So, in the case of RRT with nonholonomic RRT, so these are holonomic RRT, in case of non-holonomic RRT each of these edges are curves, the maximum curve that the car can take. So, I can go this side, I can go this side, this can go this side, this can go this side. That means this is the maximum steering angle the car has by which the car can go forward or backward because if I go backward, it will be like this. So please note that these edges are now curves corresponding to Φ of the steering angle of the car. So, this is a holonomic RRT, this is a non-holonomic RRT.

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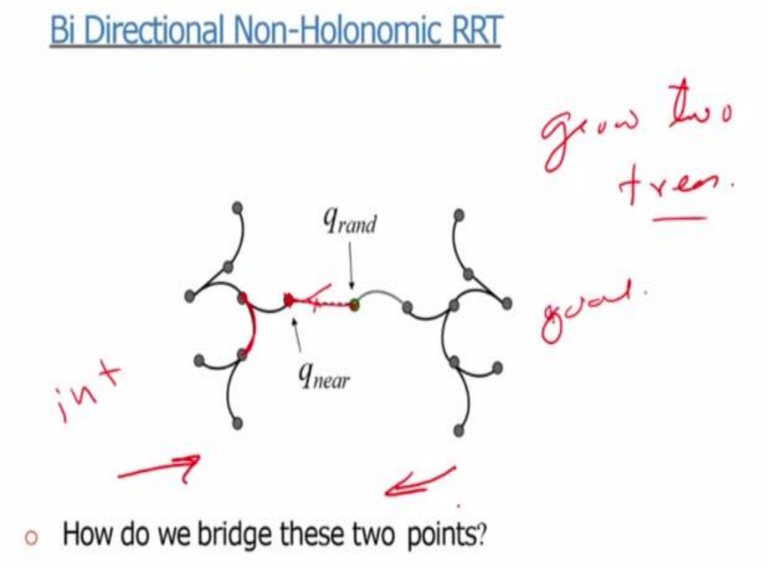

Now, let us see how do, we proceed further. So, we can have bi directional non-holonomic RRTs in which case exactly the same way we are starting from the initial point and the goal point in two directions. So, we grow two trees and see where they connect. And so like this is coming from this side this coming from the side what we can do is from here we can see which is the nearest node connect with a straight line and then figure out how to make small nodes and try and connect that is connecting or bridging two points from two different trees.

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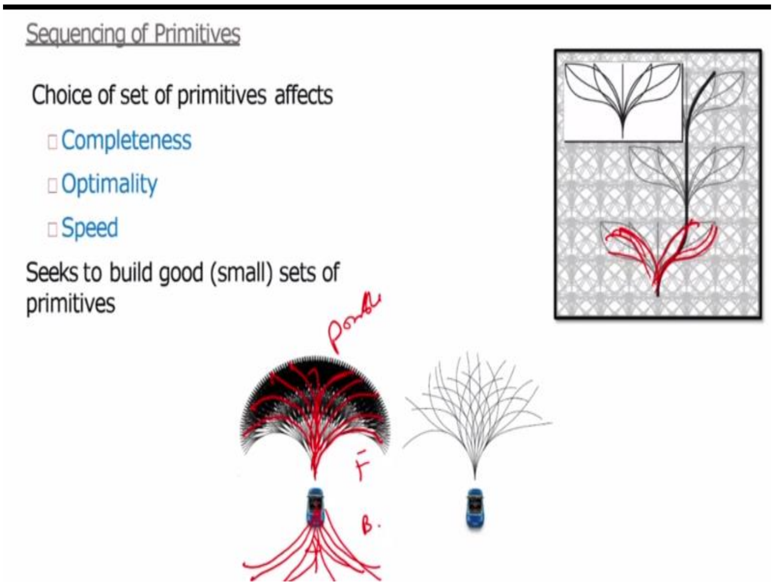

Now, we can also generate a sequence of primitives. For example, this car as shown here has a turning radius like this which means that every radius there is possible, so all these are possible. Similarly, in the backside there is going to be one more which will be like this, so all these are possible. That means this car can take any path either in the forward direction or in the backward direction. So, if I cover the workspace with this sequence of paths and then connect the initial point and the goal point to this set of points, then I have a path because all of this is possible for the car to go and that is basically done by collecting a sequence of paths.

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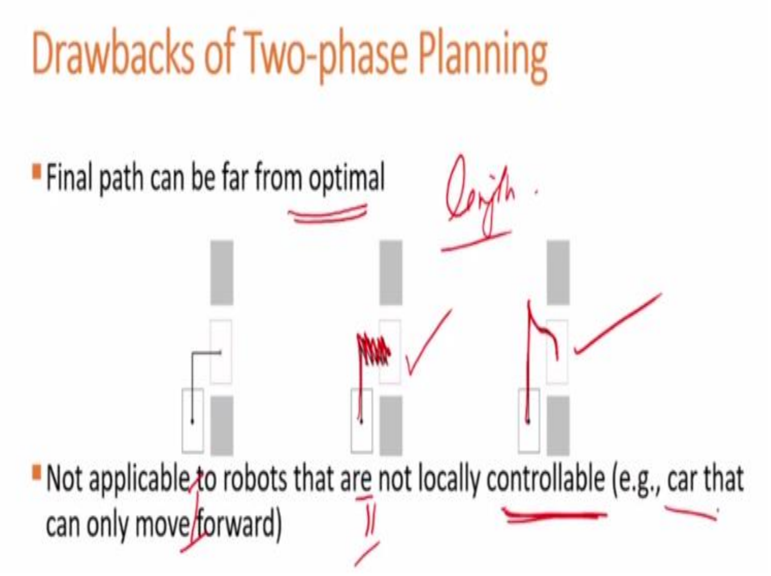

Now for example here parking sideways, it goes forward, then this is my motion 1, this is my motion 2. So, it goes here like this, like this, like this, like this, like this and then parks. Now, this path may not be optimal. I talked about optimality that when we talk about optimal here basically what we mean is that it is optimal in terms of path length, it should be a shortest length. So, in this case it need not be the shortest length.

Now, this is another manoeuvre by which it has gone and parked sideways, this is the one normal human drivers use, straight and like this. But this is also possible, this is also possible. Now, this is not applicable to robots that are not fully controllable. That means a car that can only move forwards it cannot do this.

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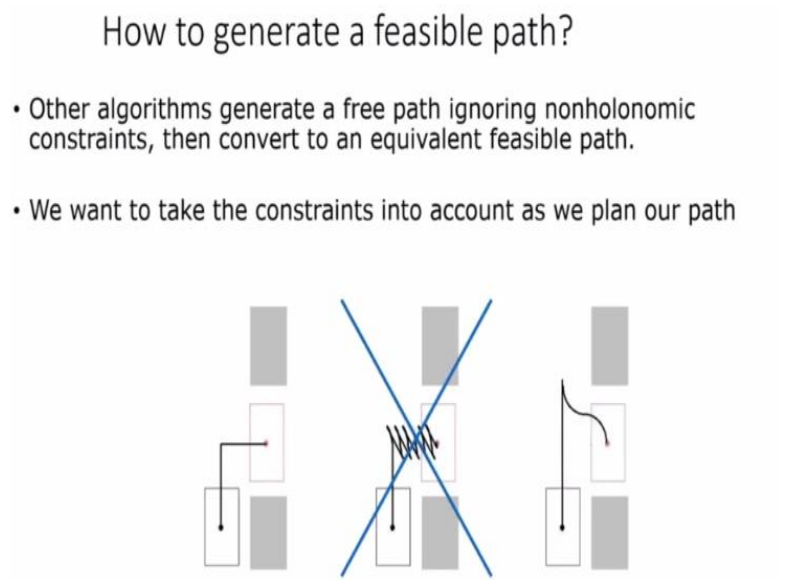

Now, the next question that we come up with is how do you design an algorithm that is going to give you this? So, we are now talking about designing an algorithm that will automatically generate the sequence of paths and will take a robot from any point to any other point, let us see how that can be done.

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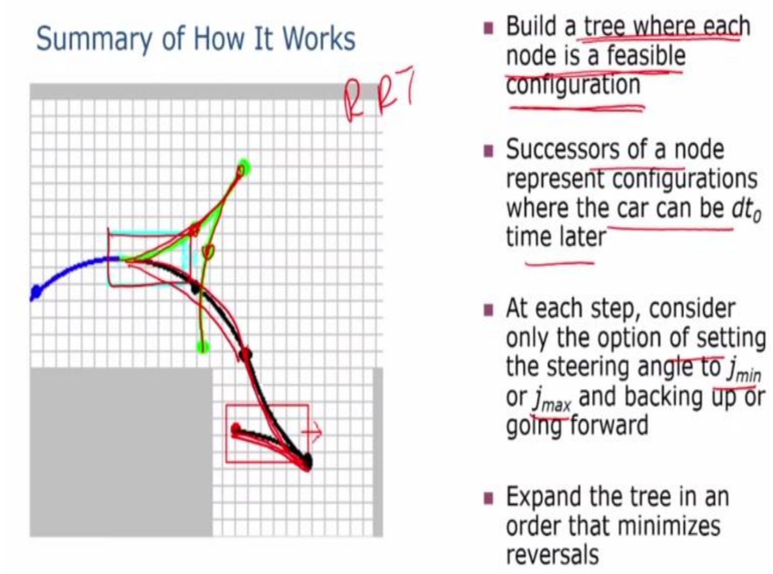

So, first of all how it works is it is something like RRT only. So, we build a tree where each node is a feasible configuration. So, we are going to build a tree from the initial point towards the goal point where each node is a feasible configuration. And successors of nodes represent configurations where the car can be at δt time later. At each step consider only one option of setting the steering angle to j_{min} or j_{max} and backing up or going forward.

Now expand the tree as it in an order that minimizes reversals. Look at this. So here, it was here. We took one curve like this and the curve like that. Then from here, I can take one more curve like this and then one curve back like this. So from here, this is one, there can be one more like this, this is one and that is another one. So, basically build a tree where each node is a feasible configuration. So, these nodes are feasible configurations.

Exactly similar to what we were doing in this case where we are building. So, this is exactly what we were doing. We were building nodes which are feasible and then trying to see whether I can connect the initial point to the goal point, it will become more clear in the next example.

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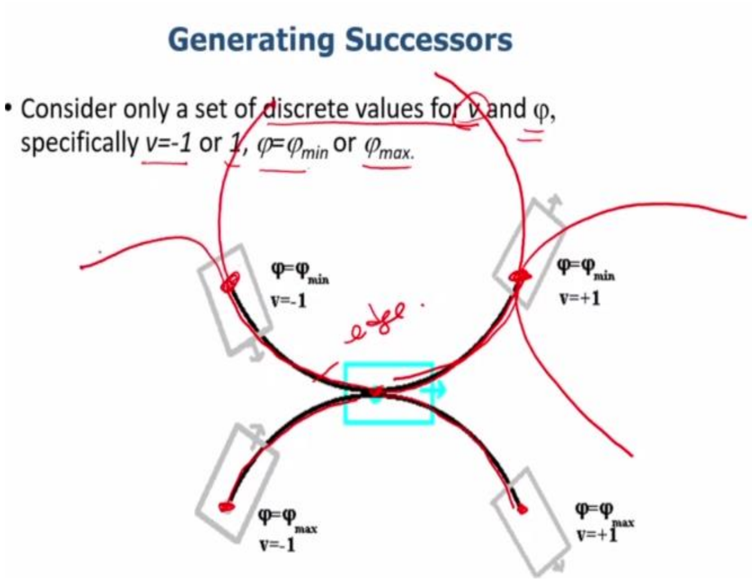

So let us look at this one. So how are we generating successors? Consider only a set of discrete values for v which is the velocity and Φ which is the steering angle. So specifically, v = –1 or 1, we say unit velocity and Φ equal to minimum or maximum, either way is okay. So here if you see from here, I am generating one node here, another node there. So, when it is going backwards, it is going like this, this is one node, this is another node, this is an edge now.

Now, suppose I generate like this and from here I generate another one like that, another one like this and go back side and generate one like this, like that. Similarly, from here I generate one that side one this side, exactly the same way.

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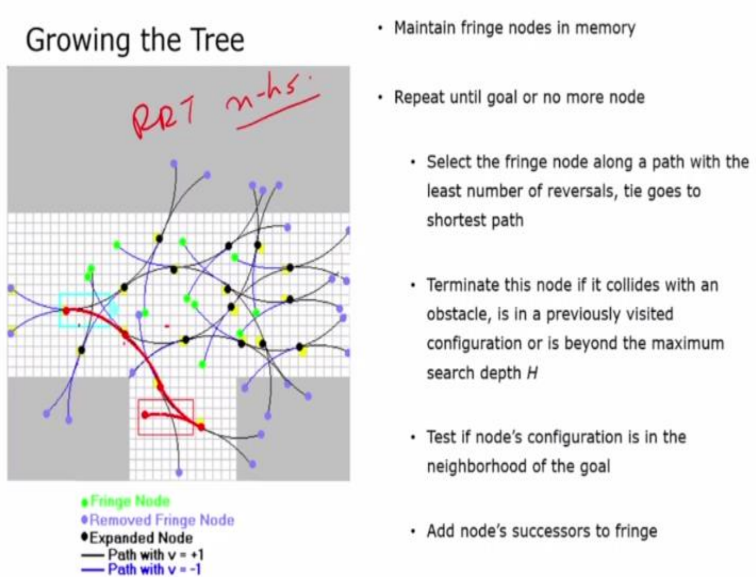

So by doing this, I can cover the whole space, I can grow the tree, right. So this is the first one. From here, I have generated one node like this. So let us have a look at this again. So, from here I am going to generate one node. Then generate one node here, one node here, one node here, then catches this node and generates another few nodes. So, it has generated another one here, so it got this one, generate another one, exactly the same way.

And you are catching one-one node at a time and generating nodes and you see that is covering the whole workspace. So, it has covered all the workspace, on all possible paths that a car can take now. Now you select one path which the car can take like this, like this and like this. Then it is a question of selecting a path. There could be more than one path. So, you have to select a path now which is going to take it from the initial point to the goal point.

And for selecting the path we can use A* algorithms. Now, this gave us an example of how we can do path planning for a non-holonomic robot to take it from one point to another point automatically. So, the algorithm can actually automatically generate its nodes and this is an automatic procedure which you can follow.

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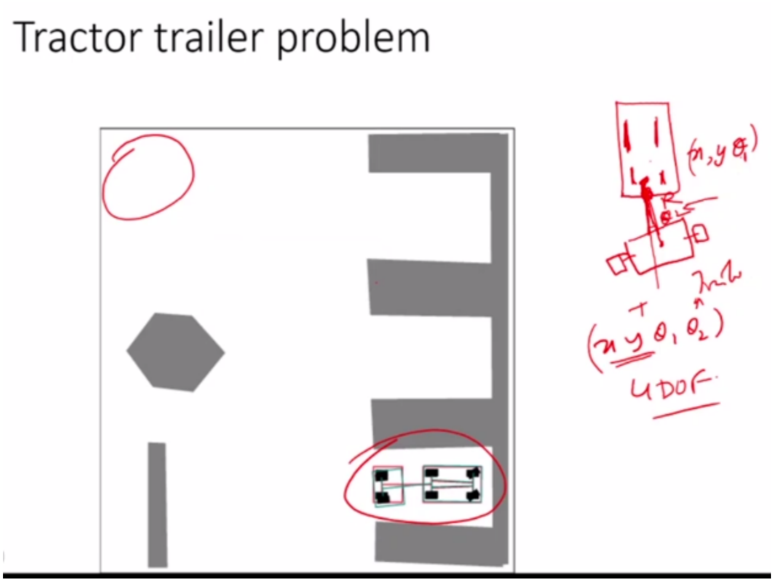

This is the case of the tractor and the trailer problem. So in the previous case, in this case we were interested only in the case of a car. And the car was having a single nonholonomic constraint that it cannot go sideways, it is a velocity constraint. And we showed that we can have an automatic path planning method similar to RRT. This is RRT for non-holonomic systems which can generate motions of the car.

Which; is going to take the car from one point to any other point simply by generating these possible nodes and then following it up with edges. So, you can see that the whole workspace is being covered. Now, in the case of a car it is simple because the theorem clearly states that robots subjected to a single non-holonomic equality constraint is fully controllable, so we do not worry. Now what about here? Now let us look at this case. So, we have a tractor and a trailer.

So, this is my tractor. This is my tractor here and this fellow is connected by revolute joint and there is a trailer at the back, that is my trailer. That is revolute joint there. So the first question that comes up is how many degrees of freedom does it have? Now what about this case? You simply look at it and you have to figure out whether it is starting from here and it is going in parking like that, is it possible? It is difficult to say actually.

But it is interesting that we will prove in this case also that this system is fully controllable system, so it can go and park anywhere actually like provided in fits of course. So in this particular case, you see we have more degrees of freedom. This car itself has so this car is having x, y and θ. So let us call it θ_1 because it is more than one now. Now, this trailer has a revolute joint there, so there is going to be another θ_2 to there.

So the car x, y and θ_1 is the degrees of freedom of the car in front or the tractor. The trailer has one θ_2 which is dependent on this angle here that is my θ_2. So, in case of this full system, we have x, y, θ_1 and θ_2. So, it is a four degree of freedom system. So the first three are for the tractor and this one is for the trailer because the angle of the trailer would depend on the θ_1 of the tractor now that is because there is a revolute joint here.

Now, in such systems, we will prove that such a system is also fully controllable. And then the next question would come how do you plan its park to take it from any point to any other point? So, what we looked at today is essentially the question of controllability that is when we are having a system subjected to non-holonomic constraints like this, how do we ensure that first of all is the system fully controllable?

If it is fully controllable, then how do we do the algorithms that will take it from some point to some other point and enable it to park? And we can do that only by using control Lie algebra. So, the other question we will look at in the next class that is the tractor and trailer problem, first we have to prove that it is fully controllable. Then the question of having an algorithm how would you move it to some position.

In the case of the car that was not required because it is already proved that car is subjected to a single non-holonomic constraint and is fully controllable, so you do not need to prove it. But in the tractor trailer case there are four degree of freedom system now, so that first we have to prove, then you can have an algorithm which is going to take it from one position to another position. So, we will stop here today. Thank you.