Robot Motion Planning
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Lecture – 4
Transformations

Hello and welcome to this lecture number 4 on transformations. In the last class, we looked at the shape of the different work volumes for different robots like Cartesian robots or cylindrical robots or articulated robots and then we started off with the rotation transformation. So today we look at the rotation transformation in little bit more detail and we also solve a few problems as we go along. And after that, we look at the homogeneous transformation matrix that is when you have a rotation and a translation together, then how do you relate one frame to another frame.

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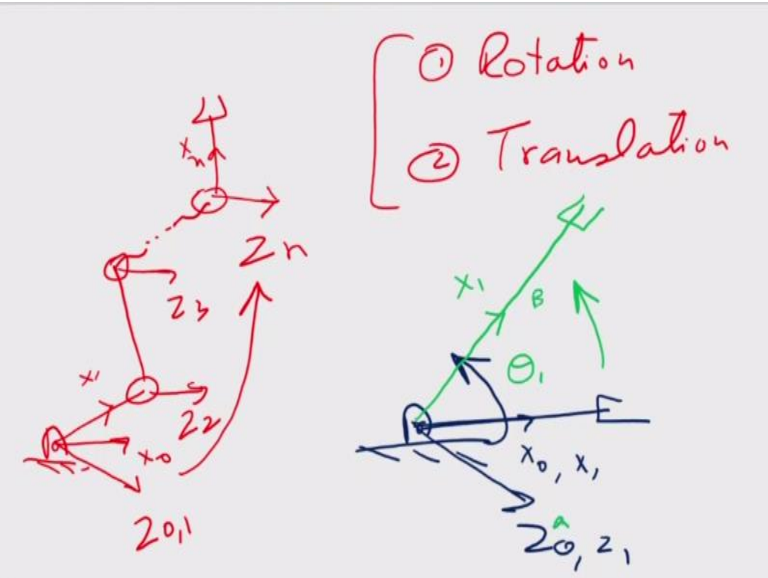

So today we look at transformations in a little bit more detail and I will very briefly go through what I had discussed in the last class before I proceed. So, in the last class, what we had looked at is the rotation transformation and we said that when we have a robotic manipulator like this having multiple links, we want to find the relation between the frame of one frame to the earlier frame and like that to the base frame.

So, we say that the base frame is always marked as 0, 1 and this is my x_0 this my x_1, I will come to the exact way of assigning frame a little bit later. And what we are trying to do is to find out the relationship between the n^{th} frame, so suppose this is the last frame and this is x_n

to that of the base frame why because the base frame does not move and when the robot manipulator is moving, we need to find the relation between all the moving frames with reference to the base frame.

So, we also saw in the last class that if you are seeing how did this n^{th} frame go here then there are two transformations that are involved. So, the first transformation is the rotation transformation and the second is the translation transformation. This means that the n^{th} frame has gone there by rotating and translating with respect to the base frame. So, what we can do is we can put Z_0 and Z_1 here, Z_2 on the next frame and proceed like this, this is Z_3 and suppose there is a break here.

So, there can be other links in there and then finally we go to the n^{th} frame. So, it basically shows that there are two transformations involved and if we know these two transformations, then you can go from any frame to any other frame in space, right. So, we started up in the last class by looking at the rotation transformation and I will revise it very quickly today before I proceed.

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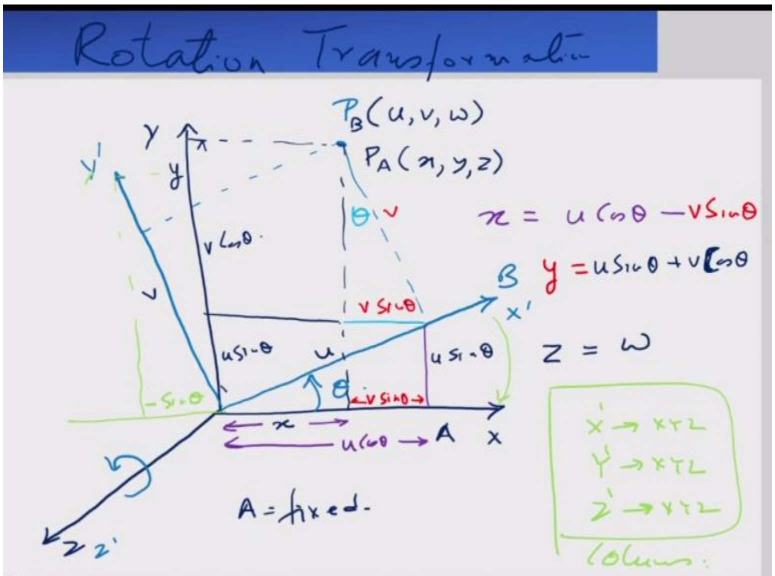

So, the rotation transformation is the first transformation that relates the rotation between two frames, so rotation operators or transformation. So, in the last class, we said the how do we define this is I have a frame which is fixed, so let me call this my frame A. So frame A is the fixed frame. So, this is my x, this is my y and this is my z and this is my frame A which is the fixed frame. So, frame A is the fixed frame.

Now, there is another frame B which has rotated by some angle, let me mark this like this. This is my frame B. So, this is x', this is y', z' is along here itself. Now, this is rotated by an angle θ in the anticlockwise direction. Now, if you have a point here called P, this point has coordinates in frame A and as well as frame B. The point is the same point, please note that. It has coordinates and the coordinates will be different in frame A and frame B.

Why because the x coordinate in frame B would be a projection here and on the y axis there will be a projection there, whereas the projection on the A axis would be like this, this is my x here and that is my y there. So, you can see very clearly the distances along the x and y directions are different. So, let me say this P has coordinates u, v and w with respect to B and P A that is the coordinate of the point P with respect to frame A is x, y, z.

I want to find the relation between x, y, z and u, v, w. Now, you can think upon this as a very simple case where we have put let us look at it like this I have one link system only just to understand. So, initially let us say there is one link like this and this is my z_0 axis and z_1 axis and my x_0 and x_1 are along that direction. Now, if this link moves anticlockwise by an angle θ what would happen is it will come to these locations, it will come here that is what has happened.

Now, you can see that z_1 is along here, but x_1 is along here now. So, you can see that there is an angle θ_1 we can say, which is the angle between the x_0 axis and the x_1 axis. So, this is rotated anticlockwise, right. So, exactly the same case is shown here where we have two frames, so frame A would be in this case the 0 frame and frame B would be the rotating frame. So, this is exactly what is shown here.

Now, what is u, v, w? The u, v, w is the projections of the point P along the B axis, so this is my u, that is my v, and w is along the z axis. What is x, y? The x, y, z is the projection x is this much on the A axis and y is that distance. So, I make a very small construction here. So, this construction is given here and this is my angle θ. Now, there also I am making this construction. So, now I want to find the relationship x, y, z and u, v, w.

But geometrically what is x, x is this distance that is my x distance. So, I can write x as x is equal to this full length, what is this full length? u cos θ is this full length. So, it can be written as u cos θ minus this small length which is let me write down, let me draw it in red colour this length, so that is v sin θ, x= u cos θ – v sin θ. Now v sine θ what is v? If this is v, then that is also v, so this is v sin θ, right. So, you get the value of x geometrically in terms of u and v.

What is y? Now where is y, let us have a look at y, y is given here. Why is in A axis that is my y. So this total sum for y = u sine θ + v cos θ. So I hope you understand how we got x and y geometrically in terms of u and v. Now, what about z? The z = w because the rotation was about z.

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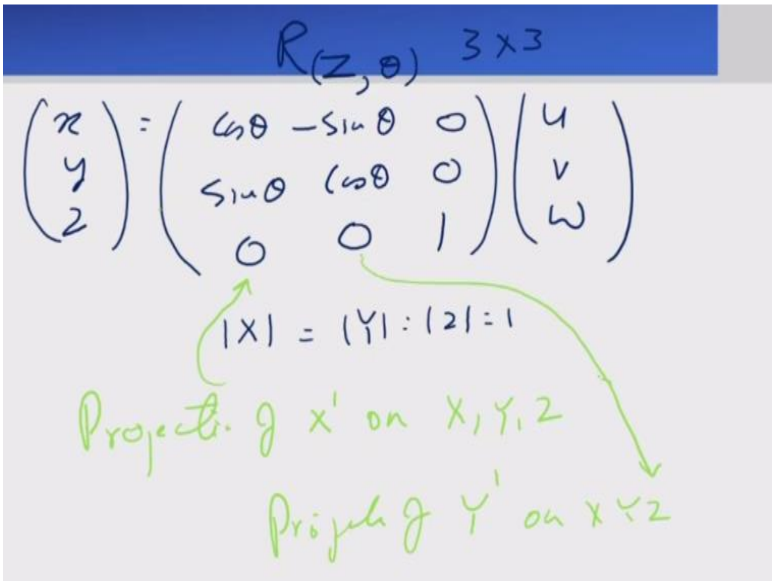

Now, if I write this in matrix form what we will get is $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix}$. This matrix is called the rotation matrix. And this is my rotation matrix, rotation about the z axis by an angle θ. Now, what we see is that this is a 3 X 3 matrix and it has some very interesting properties. The mod of each of the columns is equal to 1. So, now we are defining, how are we specifying the rotation matrix?

We are specifying the rotation matrix by a 3X 3 matrix of which each of these columns $|x| = |y| = |z|$ =1. Why is that happening? Let us go back here and see. Now, what, this basically means that the projection of the x axis, so this is my x axis, projection of the x axis of B on A is equal to cos θ. So, if I say what is the relation between frame B and frame A, then the x axis of frame B has been rotated by angle θ.

And the projection of the x' upon x is the cos θ and the projection of x' on the y is the sine θ. So, you can see that in this matrix what are we getting, we are getting cos θ and then again sin θ. So, that is the first column. So, it is like the projection of the x'. So; projection of x dash on x, y, and z that is the first column that is this one. Similarly, if I take the production of the y axis, so y axis of B, where the y axis of B? It is here.

So y'. If I take the projection on x what will happen is it will come this side now. So, you can see it is minus and it is minus sin. So, it is coming as a -sin θ that is the projection on the x axis, what is the projection the y axis is cos θ. So, what you can see is the cos of; so the projection of y' on x axis that is here is coming on the negative side, so it is - sin θ and what is the projection on the y axis it is cos θ.

So, what you are seeing is that here this is coming as a -sin θ and cos θ, there is no projection on z, so it is 0. So, this second term is basically a projection of y' on x, y, and z. So, we can also look upon this axis as because these are unit vectors as you see there are projection on the x axis, y axis, z axis that is why the mod is equal to 1. We started off by saying what is an axis. An axis is a reduction in space and they are unit vectors i, j, k in three orthogonal directions.

So here because they are unit vectors, the mods are also equal to 1. So we can also understand this rotation matrix as the projection of x, y and z of B or the projection of x, y and z on A. So x' on x, y, z; y' on x, y, z and z' on x, y, z that is what are the three columns. So, the three columns denote this.

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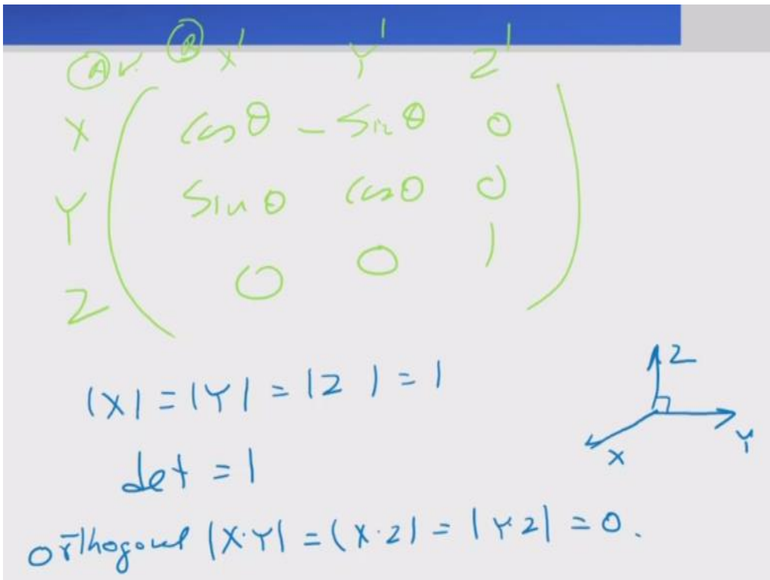

So, I can also write it like this $\begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$. So, this is the projection of x' on x, y and z projection of y' and projection of z', so that is projection of B on projection of A, so that is where this comes from. Now, if the columns are equal to 1, so if I say the columns $|x| = |y| = |z| = 1$, the determinant = 1. Why? Because just recall of the unit vectors and they are orthogonal.

So we see that these properties up there that the mod of each of the columns is 1 and the determinant = 1. Now, because they are orthogonal you know the axes we said that x, y, z are orthogonal to each other. So x, y, and z are orthogonal like this, right, this 90^0. So $|x \cdot y| = |x \cdot z| = |y \cdot z| = 0$. So, these are some of the properties of the rotation matrix that is a 3X3 matrix and the determinant is equal to 1, the mod of each of the columns is 1, and the dot product is equal to 0. Now, these properties are very important because this matrix is called a proper orthonormal matrix.

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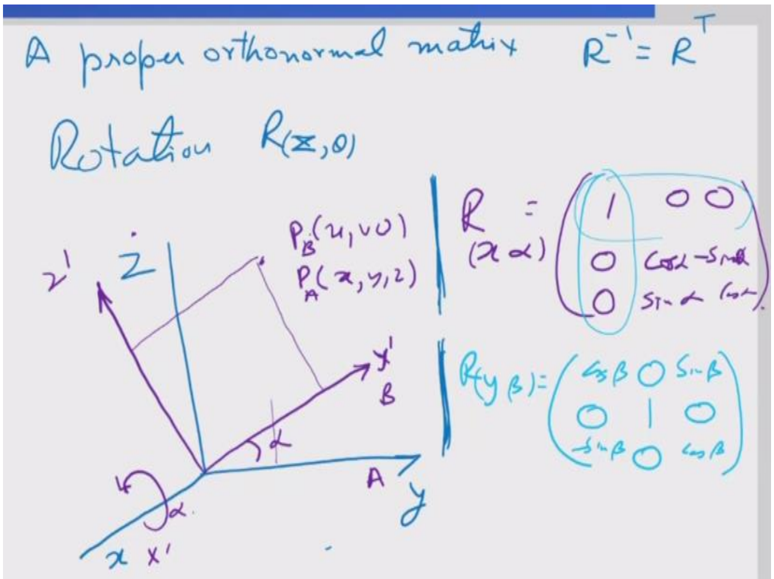

Now, the meaning of a proper orthogonal matrix is that $R^{-1} = R^T$. So, the inverse of a proper orthogonal matrix is equal to its transpose. So, if you want to find the inverse of this matrix, we do not do $\frac{adjA}{\det A}$ but we simply take its transpose. Now, this rotation matrix that we saw is a 3 X 3 matrix and we saw it as (z ,θ) the rotation of across it by an angle θ. You can similarly draw exactly the same thing.

Let me draw it this way now, say rotation over x, is y and there is z. Now, we have another axis which is here, this fellow in this case it is rotation about x. So, this might let us say α, this is my frame A, this is my frame B y', z' and this x' you might say, this is an angular form. Now, if I take a point P here which has coordinates u, v, w and I take a point P on the x frame which has x, y, z, sorry this is frame B, this frame A.

Now, I can find the relation between frame B and frame A again exactly the same way. Take the projection along the x y axis. Now the projections are along the y axis and the z axis because it is rotating about x, so there will be no projection there. So, now exactly the same way you can find what is this matrix. So, what you will see is that the matrix,

$$R_{(x\alpha)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{pmatrix}.$$

So, what you can notice also is that when we have a rotation about the z axis, this column does not change z = w. So, it basically means that there is no change on z axis. So, it is basically 0 0 1. So, it is basically rotating about the z axis. Now, when I was rotating about the x axis, what we found is that, this one is 1 now that means the axis about which it is rotating there will be no change there because the coordinate on that axis cannot change.

Similarly, I can write $R_{(x\beta)} = \begin{pmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ \sin\beta & 0 & \cos\beta \end{pmatrix}$. You can derive this, I would suggest that please derive it by yourself geometrically, it is very easy. First of all, start off by deriving this matrix which is very easy, this matrix how did we get the rotation matrix by means from geometry is very easy to write down this rotation matrix.

So, there is nothing to byheart here, just derive it by yourself once and it will be very clear how you are getting those terms. Now, in robotics we basically rotate about the z axis. So, it is enough this rotation matrix only, so it is co,s –sin, sin, cos, and the rest of it is 0 0 1, 0 0 1. And remember that it is 0 0 that the rotation about the axis that a particular axis does not change.

These are the properties of the rotation matrix. Now, it is interesting to note that this rotation matrix, similarly please do this by yourself as homework, you can do this and derive the rotation matrix for rotation about x axis and rotation about y axis geometrically and find out this corresponding matrix.

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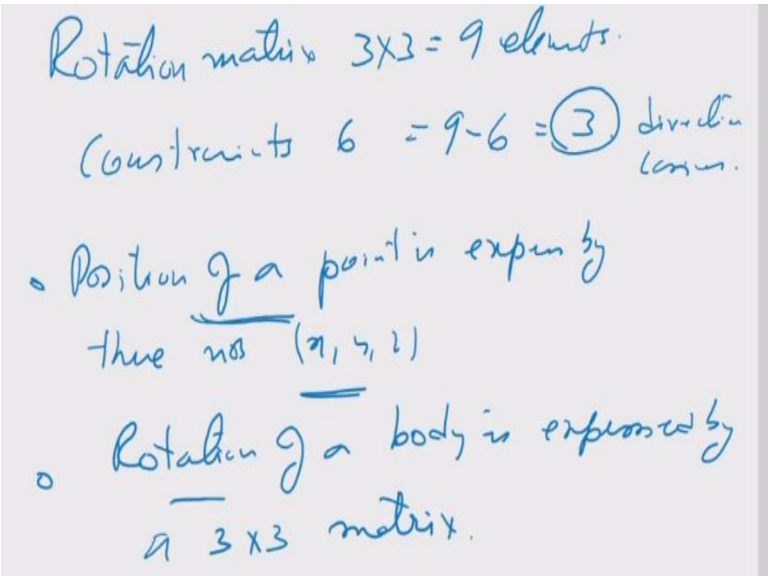

Now, it is interesting to note that the rotation matrix has 9 elements. It is a 3 X 3 matrix, so it has 9 elements. But you do not need 9 independent parameters to define the rotation of a body in space. So, to define the rotation of a body in space we only need 3 parameters, why is that? Because the constraints are 6, so this means 9 – 6 = 3. So, we need only three 3 parameters and these are basically what are called the direction cosines.

So, something to note here is that the position of a point is expressed in three coordinates. Position of a point of a point is expressed by three numbers that is the x, y, z whereas the rotation of a body is expressed by a 3 X 3 matrix. So, these things are important to know the very basics of robotics, as we move along you will see how these are applied. I hope you understand that the position of a point is by three coordinates or three numbers.

Rotation is by a rotation matrix and the angle between two bodies is given by we attach two frames as I just explained, so if I want to find the rotation of this link for example, so what I would do is I have a fixed link which is my z_0, z_1, z_0 is the fixed link. And this is my z axis and the body which is rotating like the link in this case on that link I put another frame and then I find the relative angle between these two frames.

When one frame is moving and one frame is fixed and that is how rotation of a body is expressed as a 3 X 3 matrix. Now, let us do a very simple example. Let us look at this very simple example as we go along.

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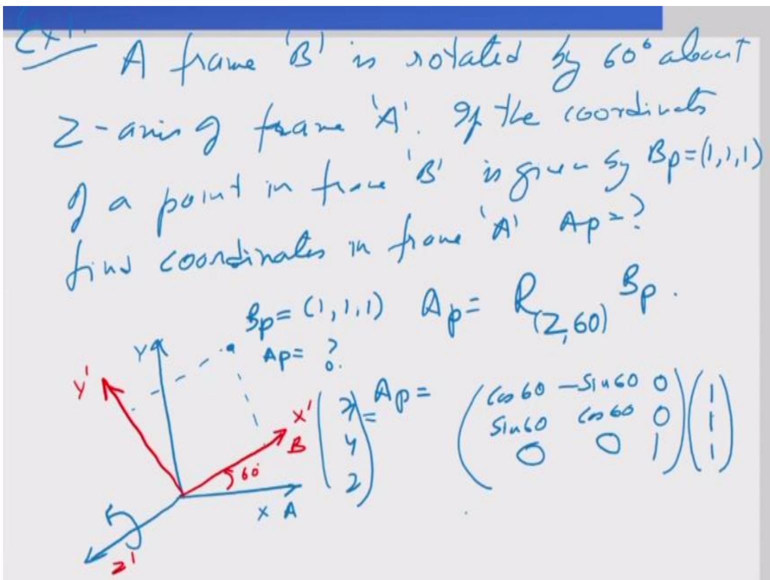

So problem number 1, example number 1; So a frame B is rotated by 60^0 about z axis of frame. We also have to say about which axis about, z axis of frame A. Now if the coordinates of a point in frame B is given by $B_P = 1$, just for example. Find point A_P. So in short form we write B_P and A_P, basically means the point is P and the frame is A or B.

So, in robotics we always have to write which frame we are referring to unlike in other geometry which you would have done is a coordinate of a point, you do not write which frame, but here there are different frames, so you have to specify which frame you are talking about. So, let us try to understand this. Frame B is rotated by 60^0. So, we have a frame. This is a fixed frame, which is my frame A and we have frame B P which has been rotated by 60^0.

So, this is my 60^0 about the z axis and it comes here. So, is this is my frame B. So, let us say this is x', y' and z' and whereas on the other side here we have our x, y and z here. Now, we have a point B_P has coordinates 1, 1, 1. That means projections of this point P on the x axis, y axis and z axis is 1, 1, 1. Now, we want to find what is A_P that is what we want to find, what is the corner of this point P in the frame A? How do we find that?

Basically, we write our equation A_P is equal to the rotation matrix z which is rotated by 60 0 into B P, B P is the corner of the point P. Now, what is $R_{(z,\theta)}$, we have just seen some time back, the rotation about the z axis is given by this matrix where is the z axis, it is given by this matrix. So, this is $R_{(z, \theta)}$, this matrix right, we derived it just some time back and so it is cos minus sine and sine cos, just four terms very easy to remember.

So, $A_P = \begin{pmatrix} \cos 60 & -\sin 60 & 0 \\ \sin 60 & \cos 60 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$. It is very simple. You just have to find the values of cos θ sine θ and multiply correspondingly and get the values of x, y, z. So this is actually showing how do we get the corresponding rotation matrix and how do we use this rotation matrix. You can do similarly rotation about x axis, about y axis.

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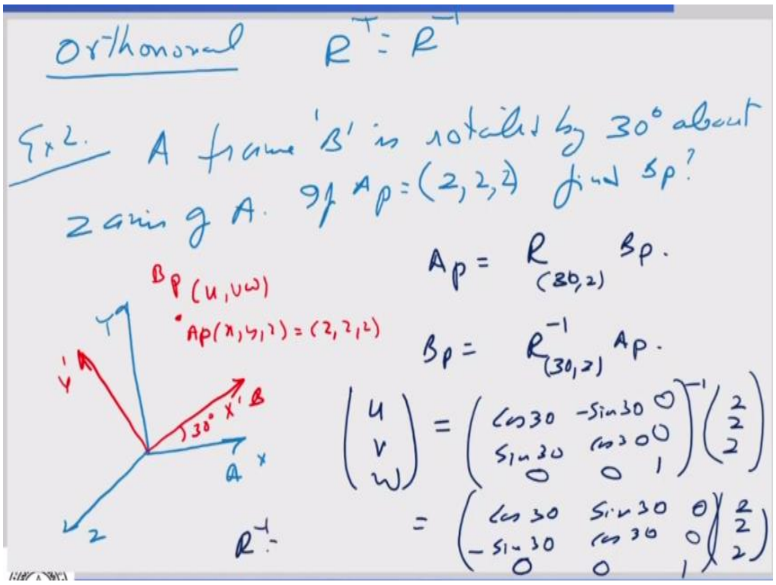

Now, we said that $R_{(z\ \theta)}$ is an orthonormal matrix. So it is an orthonormal matrix that means $R^T = R^{-1}$. So if I want to find let us look at this problem number 2. A frame B is rotated by let us say 30^0 about z axis of A. Let A_P is given by 2, 2, 2 find B_P? So in the previous case that we saw just now, we were given that B_P = 1, 1, 1 and you are asked to find A_P.

Now, in this case, in the second case, I am giving you A_P, I am asking to find what is B_P. The problem is similar. A frame B is rotated by 30^0, so we have a frame. This is my frame A, x, y and z and we have another frame which is my frame B. Now this is B, x', y'. So we have a point P there which has u, v, w in B and frame A to x, y, z. In the previous case we were given that B_P was 1, 1, 1. In this case, I am saying A_P is 2, 2, 2. Find B_P.

So the question remains the same, I hope you understand. Please understand this question very clearly. What was rotated? What is the rotation between the frames? So here the rotation is 30^0. Now let us solve this problem. How do we solve it? The equation remains the same $A_P = R_{(z\ 30)} B_p$. So it has rotation of 30^0 about z axis and this is my B_P. But here we are given A_P, I want to find B_P.

So what is B_P equal to? $B_P = R^{-1}_{(30,z)} A_P$. All I need to do is to write down the equation.

So $\begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} \cos 30 & -\sin 30 & 0 \\ \sin 30 & \cos 30 & 0 \\ 0 & 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}$.

So what is the inverse of this matrix is the transpose. $\begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} \cos 30 & \sin 30 & 0 \\ -\sin 30 & \cos 30 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}$

Now it is just a question of finding the values of sine 30 and cos 30 and then multiply. So, what we have seen as the inverse, I am making use of the property that, $R^{-1} = R^{T}$. So, these two examples show us very clearly that when there is a rotation between one frame over another frame, how do you relate the coordinates of a frame in the rotated system to that of the reference system.

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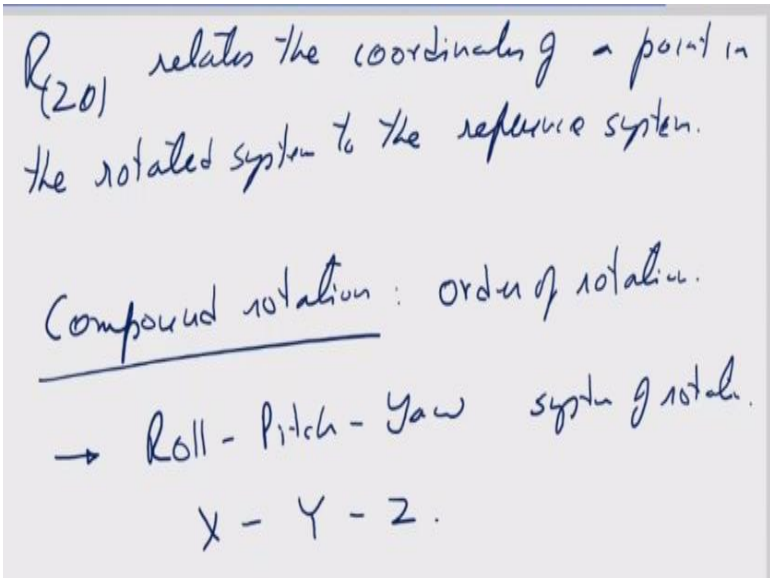

So, what is the definition of rotation matrix? So, rotation matrix relates the coordinates of a point in the rotated system to the reference system that is the definition of a rotation matrix. Now, you can have compound rotations. Now, in terms of compound rotations, we have more than one rotation. Till now, we had only one rotation about z axis. Now, the robot link can move in different directions and different links can move in different ways.

So, you can have compound, rotations that makes it rotated about the x axis first, then about y axis, then about z axis or it could be that it is rotating about some other axis in some order. So, we can have compound rotations about any axis. Something to remember is that this rotation matrix it is a matrix. So, A X B ≠ B X A. So, the order of rotation is very important. Compound rotation and the order of rotation are very important.

So, in this particular case, what we are seeing is that there are different methods, one which we call the roll-pitch-yaw system of rotation, but this is also called x, y, z system of rotation. So, first you rotate about x axis, then you can rotate about y axis, then you can rotate about z axis in that order. So, what does this mean? Let us look at our initial.

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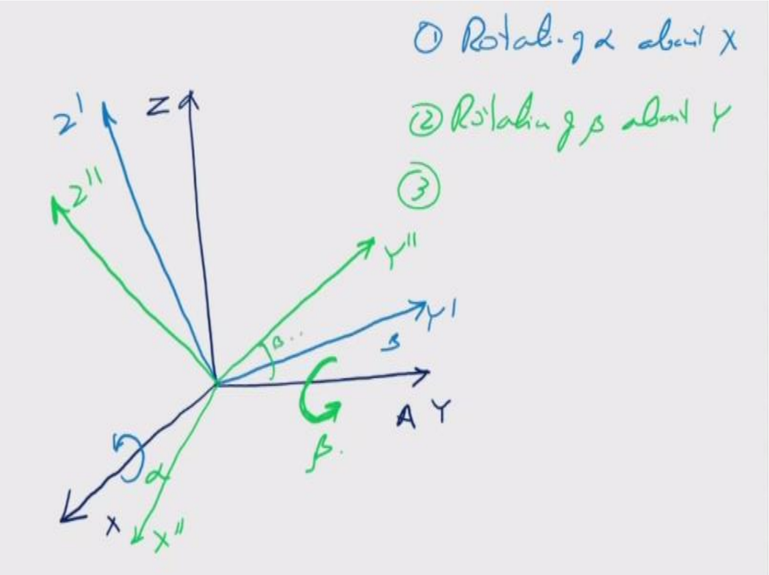

This is our figure. So frame A, this is x, let me call it x because here y is there, z is there on frame A. So, first of all we have a rotation about the x axis. So, this is my x axis rotation. So, frame B goes there, so y' B goes here, z' of frame B. So, first is rotation α, let me call this rotation α about x axis. Number two is a rotation of β about y axis. So, this was my α. Now, you see in the first case there was only one option, you could rotate about x, y or z.

But once the system has been rotated, I have an option of rotating about the fixed system or about the rotated system. So, I could have rotations about the fixed y or I could have rotation about the rotated y also because that option is there. So, here we are saying in the roll-pitch-yaw system which is also called the fixed angle rotation system, rotations are always about the fixed axis. So, this is also called a fixed angle rotation which means that I am going to rotate about the fixed axis.

So, my β rotation is about the fixed one, not about the rotated one. So, the second one is rotation of β about y. And then finally so if you rotate about y what will happen? Can you imagine what will happen? So, what will happen B will go this side, z' will come this side, now it is double dash, y'' and x has come off now x'' that is the rotation of β, so this is my angle β about the y axis.

Now the third one is the rotation about z axis. So, let me put number 3 here, this is rotation γ about z, again fixed angles. So now when it is rotating over the fixed angles, it is going like this. So, you can imagine what will happen, this will go there y'''. It is a triple dash and x'''. So we have 3 rotations, rotation of α about x, rotation of β about y, and rotation of γ about z.

Now, I am saying that we have a point here P, the final rotation, let us call it P_B. the same what is this coordinate in the B? Fixed system now. Now you can see that has been rotated three times. So what is A_P and A_B. A_P is the compound rotation matrix into B_P. This will have three matrices now because there are three rotations. If there are two rotations, there will be two. If there is more than two, it will be that many.

So, this will be a compound matrix of that many rotations. So you basically find each independent rotation about x, y and z and multiply them together and get this $R_{(x, y, z)}$. But the important question here is that matrix multiplication does not commute, so what is the order of rotation? That is an important question here. Now, what is the meaning of this? This basically means when we said that, so we were looking at this figure, when I am saying what is the relation between $A_P = R_{z\theta} B_P$?

What is the relation of B_P to A_P that is what I am saying that is what this statement says what is the relation of the coordinate of a point in frame B, which has been rotated to that other reference frame? How do you find that? We basically multiply with this matrix. So, what we are doing is we are bringing this back, we have rotated, how did we rotate? We rotated like this by θ angle.

Now, suppose I rotate it back by θ angle, then it will come back and become the same axis, then this matrix will become entity matrix, right. So, it basically means that in order to relate the rotated frame to the reference frame, you have to rotate it backwards now. So, in this case, in our case where we have three rotations, the first one was α rotation then it was the β. So, first rotation was α, then was β, then was γ.

But if you want to bring it back, then first you have to rotate the last one, so it will have to become γ. So R_γ, R_β and then R_α, please think of this very carefully. This is y and that is x.

This is my compound rotation matrix x, y, z. So what does this mean that I rotated α first, α about x, yeah, and then I rotated β about y, and then I wrote the γ about z.

So if I want to bring it back, I have to rotate the γ first, then the red one will come here and become the green one. Then I have to rotate the green one, then it will become the blue one. Then I will rotate the blue one to become the black one. So, in going from this to this, again this is number 1, this is number 2 and that is number 3 in their order. We are now coming back you see I have reversed my order.

So, this can become 3 dash, this is 2 dash, this is 1 dash. So, you reverse the order. Please note this and understand very carefully why. So basically, when we are finding the relation between compound rotations with reference to the reference frame in the fixed angle system, we say we pre multiply the matrices. Please note that we are pre multiplying matrices here. So first was α, then β, then γ.

When we are multiplying first is γ, then β, then α, why? If you look at this figure, you will understand very clearly that first we rotated by α, then by β, then by γ. Now, when I bring it back, I have to first undo the change made by γ, then undo the change made by β and then undo the change made by α.

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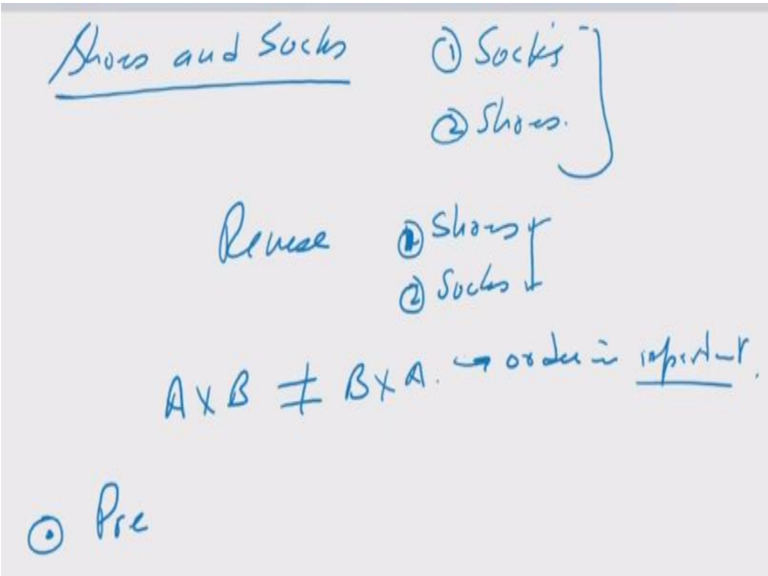

Now, there is a very simple way of understanding this. I call it shoes and socks. So, you look at the shoes and socks. So, first what do you do you put on your socks, and then you put on your shoes that is the process of putting on your shoes. Now if you inverse the process, you reverse it. Then what will happen is first you take out your shoes and then you have to take out your socks, is not it? You are reversing the process.

So when you are undoing the process, then you are doing the reverse of what you did. So first you are putting on your socks, then you put on your shoes, but when you are opening your shoes, you have to first open your shoes first and then you have to remove your socks that means you reverse the order now. And why did you reverse the order? It is essentially because of this same thing here that these rotation matrices in space do not commute, they are matrices.

So A X B ≠ B X A, because these are matrices. So, we always have to pre multiply, so the order is important here. Please remember this. If you remember shoes and socks, you never forget that if there are two matrices and you have to find the combined rotation matrix, then you just have to pre multiply. Let us do a very simple example before we proceed.

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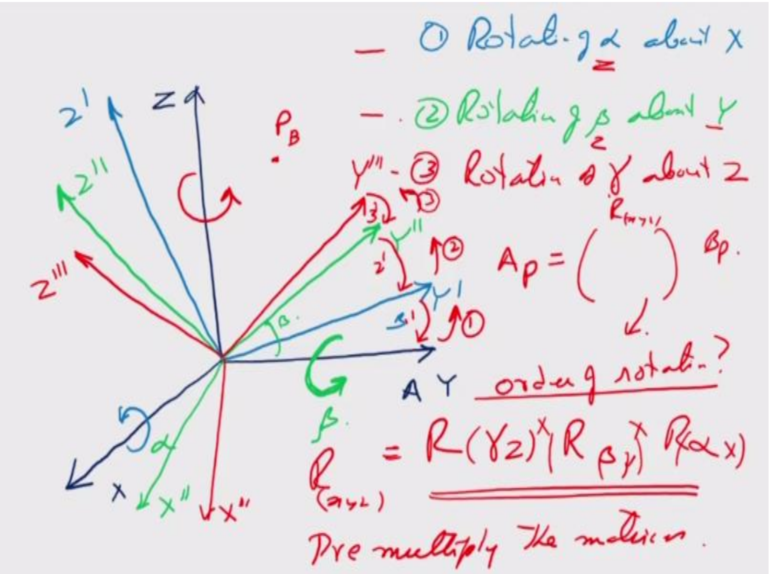

So, frame B is rotated by 30^0 about axis of A and then rotated by 60^0 about z axis of A. So if B_P that means a point P in the frame B has coordinates 1, 1, 1 find A_P. I hope the question is clear. Now, let me draw it and explain. So, we have the fixed frame, frame A. So, this is my x axis, y axis, z axis of frame A. now, the frame B is rotated by 30^0 about x axis. So, the frame B has two rotations.

The first rotation is like this, so it comes here 30^0, this goes there frame B, y' , z' . And then it is rotated by 60^0 about z axis and then the second rotation is it is rotated by 60^0 about z axis.

So if it is here, where is my z axis? z is there. So if it is like this 60^0, what will happen is B x will come out like this, so that is my y'', that is my z'', that is my x'' as we know.

Now I am given a point B_P having coordinates 1, 1, 1, find what is A_P? Now, this will make it very clear that the first rotation was 30^0 about x axis, number two was 60^0 about z axis. Now if you want to bring it back, what you have to do is so it went from here to here that is one, then from there to there that is two. If you want to bring it back, you have to reverse the order. So, the first thing that will happen is this will become 1 and then this will become 2.

It is very clear from this figure, right? So if you want to make the green or you want to align the green axis with the blue axis that is as good as saying you want to make the green axis into the reference axis, then you need two operations. The first thing you need to do is to do one and then do two. So you are inverting the process here. So, basically, $A_P = R_{(z,60)} \times R_{(x,60)} \times B_P$.

Please note that this order has been reversed, it was first 30^0 and then 60^0, but when we undoing it first is 60, then is 30. Why? It is the shoes and socks theory again and if you look at this figure it is very clear. So, I can write the values of this matrices, $A_P = \begin{pmatrix} \cos 60 & -\sin 60 & 0 \\ \sin 60 & \cos 60 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos 30 & -\sin 30 \\ 0 & \sin 30 & \cos 30 \end{pmatrix} B_P$. So just multiply the matrices and that is it.

So A_P is equal to whatever the matrix comes into B_P, which is 1, 1, 1. So this basically shows us that when you are rotating about the fixed axis, the order of rotation is very important here because it is a matrix. So, if you change the order of rotation, it will be wrong. And in the vertex, we normally use the concept of fixed angle rotations where we always rotate about the fixed axis, not about the rotated axis, but you can also rotate about the rotated axis, it is not that you cannot.

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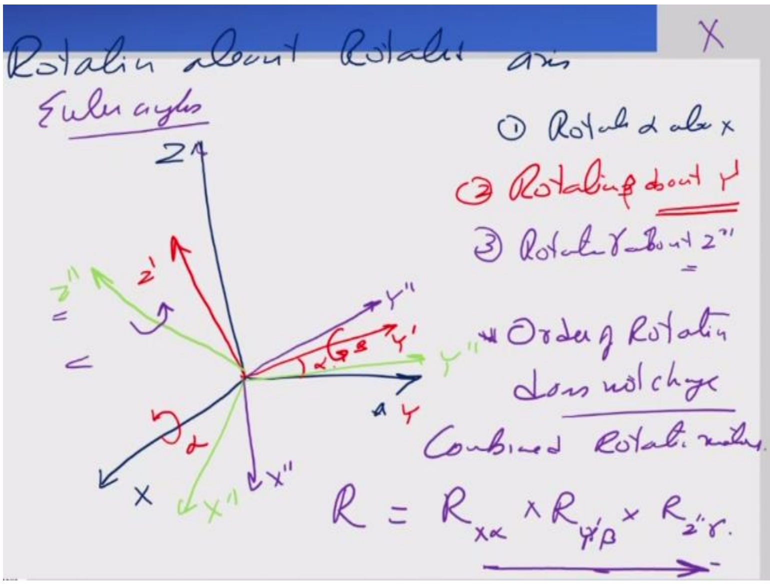

So rotations about rotated axis now. So now this is my first frame x, y and z. So first I am going to rotate about x. So, what would happen? This is my frame A. So, what would happen is that about x by α, so this will come here now, that goes there. That is my B axis, that is my y, so B axis, so y', z'. So at first rotation by how much was it? By an angle let us call it α.

Next I am going to rotate about the rotated axis. So rotation of β about the rotated y dash, please note this very carefully. The previous case we are talking about this axis, now I will not rotate about this axis, I rotate about that axis now. So, what will happen is everything will move around and this will come; z will come here, so it becomes a double dash, this will become here x'' and this will go there just y''.

Now, the third one is a rotation about let us call this the rotation about of γ about z''. So, it is a rotation about this now, so it is rotating like this. So, what is happening is so gone here, here and here. So, this is y'', this is my x''. So, now I hope you understand that when you are rotating about the rotated system, what is happening is you are sitting on the frame and you are rotating about that frame.

So, the reference frame is moving now. So, you can see here that I did the rotation about of α vertex because there was one change, then I did a β about y, so β about y is what we did is the y'' change. And then I do the rotation about z axis, so my second rotation was this one. So, that is my second rotation. Third rotation is about this axis. So, what will happen is your x axis will change, x'' has gone here, y'' has gone there.

And we can also say that double dash has moved because with reference to the reference frame. So now it is not clear. It is confusing whether you are sitting on the rotated frame and you are watching the reference that is the confusing part. This is called Euler angle. But in this particular case, the order of rotation does not change, so please note this. The order of rotation of the combined rotation matrix does not change.

So, when we are making the combined rotation matrix, then the combined rotation matrix will be $R = R_{x\alpha} \times R_{y'\beta} \times R_{z''\gamma}$. So, rotation about α, β, γ remain in the same order, we do not change it. So, something to note about when you are doing the rotation about the rotated axis.

Because you are sitting on the rotated axis and you are looking at the reference axis because of which the order of rotation does not change and the combined rotation matrix remains, the order remains the same. We do not normally use this in robotics because it is very confusing because you are sitting on the rotated system now and with reference to you the reference is moving now. But in the previous case you are sitting on the reference system and that is never moving.

So in robotics we basically use this system only, the fixed angle rotation. So fixed angle rotation is always used in robotics. So this is my fixed angle. So, we have looked at the rotation matrix, we looked at the property of the rotation matrices and then we also looked at what happens when we have compound rotation matrices. In the case of fixed angle rotation, they reverse in order. And in case of where the rotation is about the rotated frame, then the order does not change, it remains the same order.

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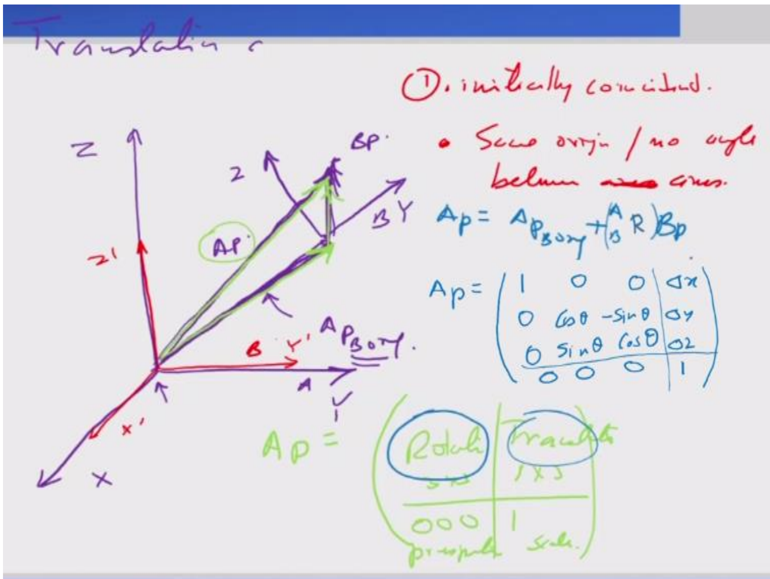

Now, let us look at in robotics I said that we always use the fixed angle rotation and it is always rotation about z. So, life is simple if you just know the rotation about z matrix and it is rotated about the z axis and it is a fixed angle rotation, then that is enough you do not need to know more. Now, let us look at this translation also. When looking at translation, suppose I have one axis here, this is one axis and there is another axis which is initially coincident.

So, this is the other axis initially they are coincident, coincident means same origin and no angle between frames between the axis. So, these two things are satisfied. So, initially they are coincident that means they have the same origin and there is no angle between axes. That means, this is my B frame and that one is my A frame, then the origins are the same and there is not angle between the x, y or z axis.

Now, suppose I move, I rotate and I translate, this frame B, I rotate it like this by some angle θ, and I translate it here. So, I have rotated it an angle θ and now I have translated it. So this frame is gone here now. So this frame has now gone here. Now, if you have a point B P there, so this my y and this is my z. Now if you have a point B_P, this vector on frame B, this is my frame B, this is my frame A.

What is B_P with respect B_A or what is B_P in frame A? What we are trying to answer is what is A_P now when there is a translation and there is a rotation also between two frames B and A. So let me draw the vector diagram, we will close this. Now, what we see is that if I close the loop, then the sum of this one plus this is equal to that. So this gives me some way of finding what is A_P, this is A_P.

So what is A_P equal to? $A_P = A_{P_{Borg}} + B_P$. So now if I write in matrix form, A P will be equal to this homogeneous matrix consists of two parts, there is a rotation matrix. This is my rotation matrix, which is 3 X 3. There is a translation operator part here which is 1 X 3. This is perspective, it is always 0 0 0, this is always 1, perspective and this is scale.

So now if I write it in matrix form, what will it look like it? So, it will look something like this. Let me write this little bit higher. $A_P = A_{P_{Borg}} + B_P$. Now, is this equation correct? Well, this equation is not fully correct why because B P and A P are not in the same frame. So, this is something in robotics that we have to talk about frames. So, if the frames are not correct then you cannot add them like this.

So, what we can do is we can convert the frame of B into frame of A. So $A_P = A_{P_{Borg}} + {}_B^A R B_P$. So, now the frames are aligned, now you can add vectors. So, if I write in the matrix $A_P = \begin{pmatrix} 1 & 0 & 0 & \Delta x \\ 0 & \cos\theta & -\sin\theta & \Delta y \\ 0 & \sin\theta & \cos\theta & \Delta z \\ 0 & 0 & 0 & 1 \end{pmatrix}$. So, this is the rotation part, this is the translation part perspective is here 0 0 0 and this is 1. Now, this is something that we are seeing in terms of our homogeneous transformation matrix.

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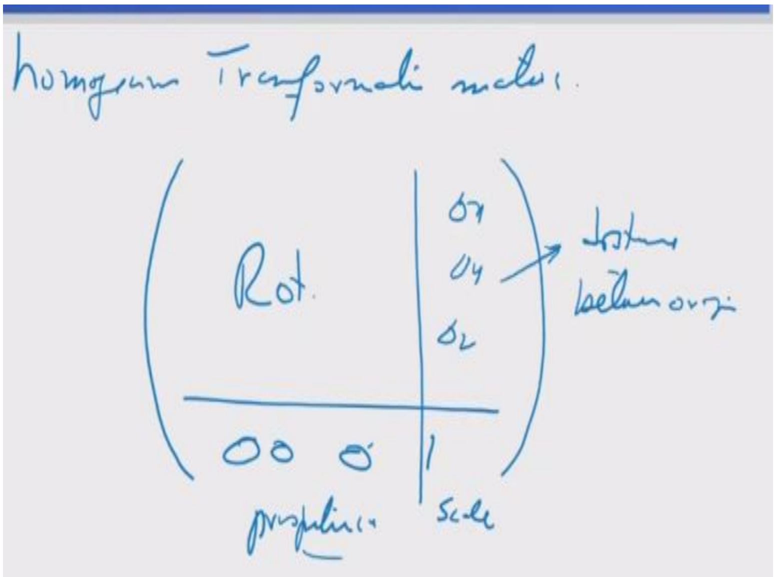

So, let me write the homogeneous transformation matrix. So, there was rotation about x. So, this matrix will change and the matrix structure is something like this, whatever is the rotation matrix it will come here. The distance between origins Δx, Δy, Δz is equal to distance, origin, this is perspective, this is always 1. So, this is something that gives us a rotation and a translation at the same time. So, we can do a very small problem, you can try it by yourself, we will solve in the next class.

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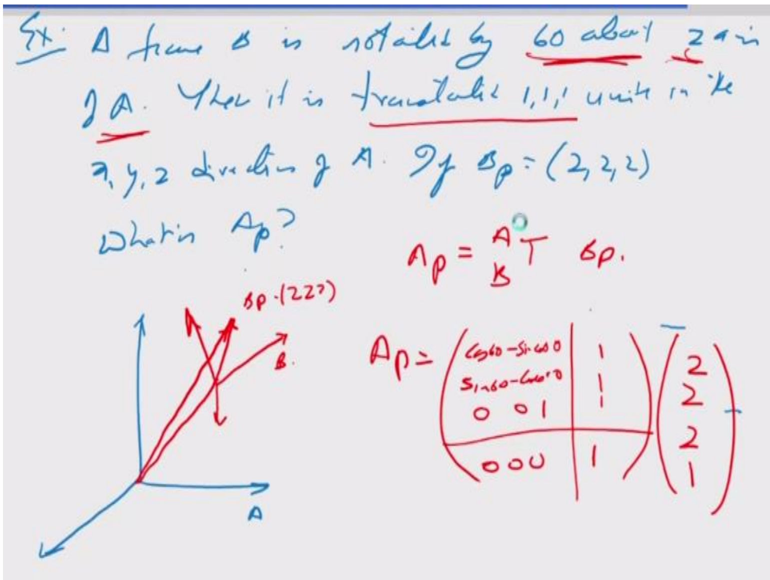

A frame B is rotated by 60^0 about z axis of A. Then it is translated by 1, 1, 1 units in the x, y, z directions. So, if B_p = (2, 2, 2) what is A_P? So, this is translation and rotation. So, let us do it very quickly. So, this is a frame, frame is A and then there is another frame which has been rotated and translated which has gone there now, it is rotated and translated. So, let us call this frame B. There is a point B_P there.

Now this B_P has coordinates (2, 2, 2). What is this coordinate in frame A?. So, we do exactly the same what we have done. So, I close my vector loop back and I can write $A_P = {}^{A}_{B}T \times B_P$.

Now, this T matrix consists of four parts and A_p is equal to what are the four parts? B_p was 2, 2, 2. Wat are the four parts? The first part is the rotation matrix. So, it is saying it is what is by 60^0 about z axis that is z.

So, it is $A_P = \begin{pmatrix} \cos 60 & \sin 60 & 0 & 1 \\ \sin 60 & -\cos 60 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 2 \\ 2 \\ 1 \end{pmatrix}$.

Now, you can simply multiply both of these matrices and get the answer. So, this is a very simple example to show translation and rotations together. So, today in class what we did is we looked at the rotation operator.

And after we looked at the rotation operator, we basically looked at rotation and translation in terms of the homogeneous transformation matrix. So, using these two matrices, we can go from any frame to any other frame in space. So, when the robot is moving, we can relate the rotation of the frame and the translation of the origin of the frame with reference to the base frame always. So, we can do our computations very easily and everything is with reference to the base frame which is not moving.

So you exactly know where the robot is with respect to this frame. So, in the next class we will move on to homogeneous transformation matrix, which considers rotations and translations together. I would request that you please look at the textbook also. There are some problems, I am solving problems in class, but there are more problems in the textbook. So, you can solve those and see also how problems are put and how it is relevant to robotics. So, let us stop today here and we will continue in the next class.