Robot Motion Planning
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Lecture – 5
Kinematics

Hello and welcome to this lecture number 5 of the course robot motion planning. In the last class, we had looked at the rotation matrix and we saw the various properties of the rotation matrix. And we also solve a few problems and then we moved on to the homogeneous transformation matrix that relates the coordinates of a frame, which is rotated and also translated with reference to the reference frame, right.

And then we today will move on to the next part that is how do you assign frames? And then after you assign frames, how do you find the relation between the end effector frame to the base coordinate frame. And then we will look at forward kinematics, inverse kinematics and also things like singularities. So today we continue with forward and inverse kinematics. But before we go to forward inverse kinematics, I will very briefly revise what we had done in the rotation matrix and the homogeneous transformation matrix.

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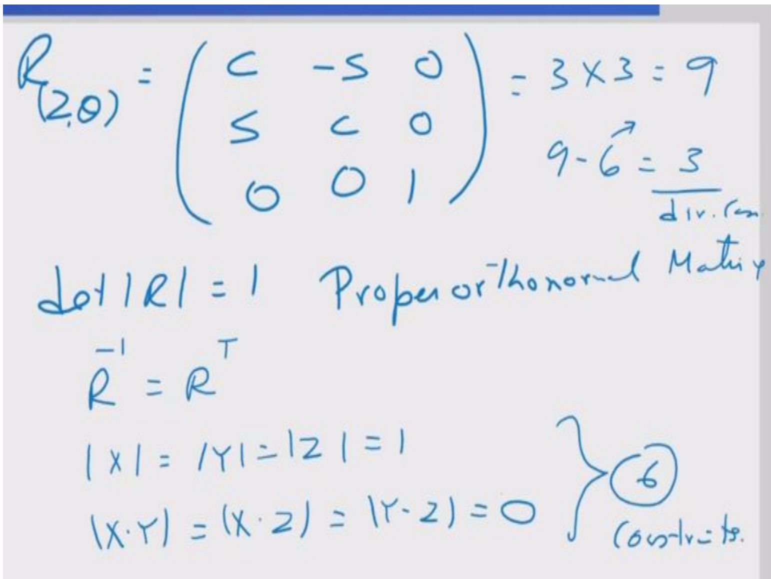

So what we have seen in the last class is that the rotation matrix consists of a 3 X 3 and we say rotation about R $R_{(z,\theta)} = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$. We also looked at the various properties of this matrix which means that the determinant of this matrix $|R| = 1$ and it is a proper orthonormal matrix. Now, because it is a proper orthonormal matrix, $R^{-1} = R^T$. Apart from this, it has the properties that $|X| = |Y| = |Z| = 1$.

And the $|X.Y| = |X.Z| = |Y.Z| = 1$. Now, because of this we have noticed that although this matrix is 3 X 3, it has 9 elements, but it does not require 9 parameters to specify the rotation of a body in space, you only need 9 – 6 = 3 parameters which are also called the direction cosines. So, the direction cosines. And the 6 are basically the constraints which are coming from this, these other 6 constraint equation. So this is just to revise about the rotation matrix.

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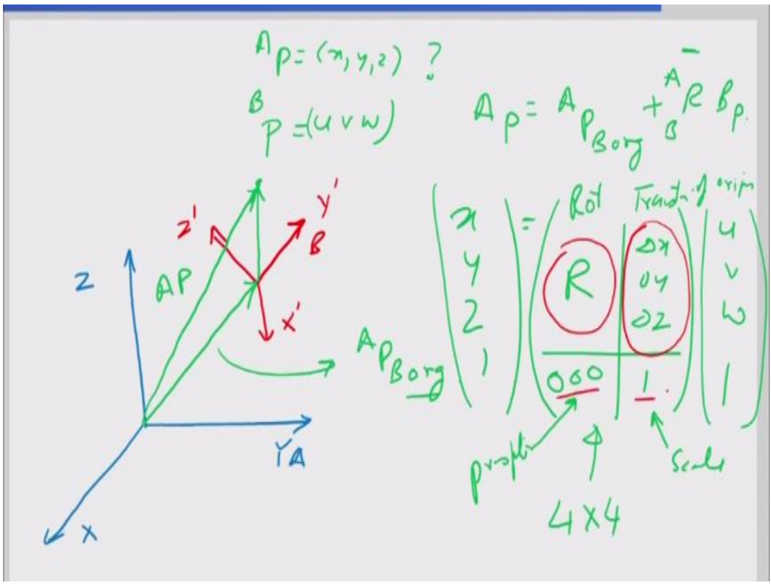

Then we moved on to the homogenous transformation matrix and we said that if you have a frame, this is my fixed frame. Let us call it frame A and this is x, y, z. Now, if you have another frame which is rotated and translated with reference to this, so I have another frame which is rotated and translated. So this is my frame B, and this is my x', y' and z'. And suppose I have the coordinates of a point in frame B.

So, the coordinate of a point in frame B is here. Let us call this point P and we expressing the coordinate frame B which is equal to u, v, w. And now I want to find what is this u, v, w in terms of the frame P? So I want to find what is A_P which is my x, y, z. This is something I want to find. Now, how do you find the homogeneous transformation matrix? So what we see is that I complete this vector diagram like this and then I write my closed loop equation, this is a $A_{P_{Borg}}$.

This is the org means the distance between the origins of two frames A and B, this is A_P and that is my B_P, right. So, I can write this as $A_P = A_{P_{Borg}} + {}^{A}_{B}RB_P$. So, we need a rotation matrix there why because otherwise the frames B and A are not aligned, there is rotation between them. Now, I can write this mathematical in the form of matrix as x, y, z, 1 is equal to the rotation part.

So, $\begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix} = \begin{pmatrix} & & & \Delta x \\ & & & \Delta y \\ & & & \Delta z \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} u \\ v \\ w \\ 1 \end{pmatrix}$. So, this matrix which is a 4 X 4 matrix is called the homogeneous transformation matrix and it has four parts. This is the rotation part, this is the translation part. Translation of what? Translation of origins. And this is called the scale parameter and this is called the perspective. So this matrix relates the coordinates of a point which is expressed in a rotated and translated frame to that of the reference frame.

Now, the next thing that we will need to see is we need to move on the assignment of frames. So, so far we have said that this is an axis system or this is a frame and this is x, y, z axis and this is rotating and things like that.

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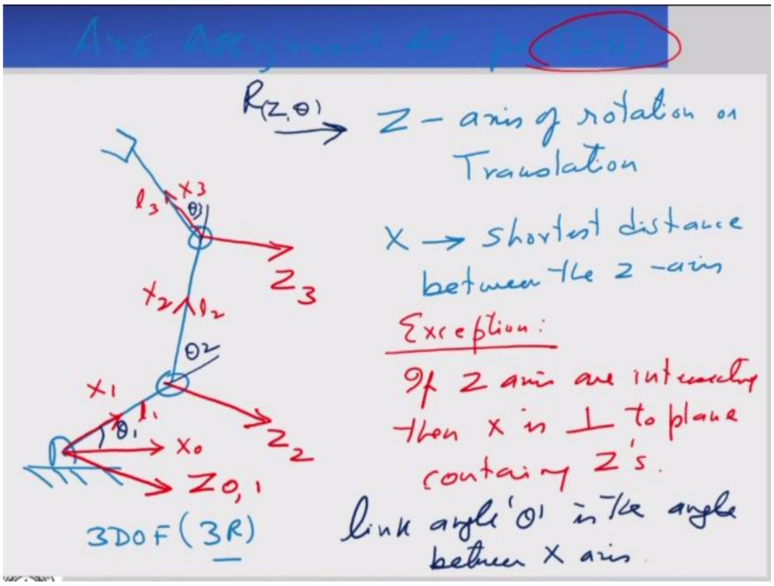

But then how do we assign this axis systematically in a robotic system and that is something which we will see now and that is what is called axis assignment as per DH notations. DH stands for Denavit–Hartenberg notations. Now, in this DH notations basically we are going to systematically assign the axis. So, let me say that this is robotic linkage. First of all understand this is 3 DOF, 0 of freedom, which has three revolute joints which is planar.

What it means is that the axis of this mechanism is pointing out of the part of the plane. So, that is the first thing that we see. Now as per DH notation the first thing is we assign the z axis as the axis of rotation for translation. So, this is the first thing you see. So, when you see a robotic mechanism where a link is like this, the first thing that you do is to look at where is the axis? What kind of mechanism is it? Is it a revolute joint, translating joint, spherical joint?

How many 0 of freedom? Is it planar? Once you understand these are the joints, then you assign the z axis. So let me assign the z axis is out of plane. So my first one is at 0, which is not moving and Z_1 is the next axis. So it basically means that I am assigning my axis 1 to the first link. So, this is my second Z_2, this is a Z_3. So, this is the origin which is assigned on to the link 3, link 2, and link 1 and Z_0 is the one that does not move.

So, the next thing that we need is to figure out where is the x axis. Now the X axis is the shortest distance between the Z axis. Now, in this particular case, we can see that the Z axis are parallel, so you can very easily find out the shortest distance. Also the first X_0 we can put horizontally, why? Just convention, we say that the x axis is horizontal that is the reason we put it there. The next one will go along the link which is X_1, then it is here X_2 and that is my X_3.

So we have assigned our X axis, Y axis and Z axis you can see as per DH notation. Now, we do not normally assign Y axis because the moment you fix X and Z, Y gets automatically fixed because of the right hand rule, right, so we do not assign y axis. We just assign the Z axis first, then we assign the x axis. And these are my linked lengths that is l_1, l_2 which is already given and that is l_3. So this is how we assign axis as per DH notation.

There is one exception. Now the exception is if Z axis, then x is perpendicular to plane containing Z's. So in this particular case, the Z axes are parallel to each other, they are not intersecting, but in case the Z axes are intersecting then there is no shortest distance between them right. So, in that case the X axis is perpendicular to plane containing the Z axis. So, let us have a look at this case. So, let me draw another robotic mechanism where this happens.

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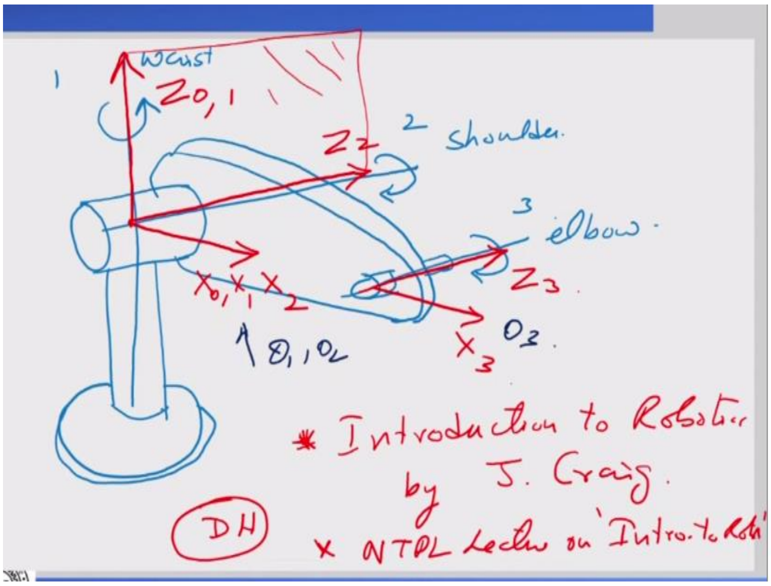

So if you can visualize, there is a link is like this and there is a link here. That is a link and this can rotate about this, so that is my base. And if you can imagine, then this is my axis that is the first axis so this can rotate like this, that is the second axis which can rotate like this. And there is a third axis which can rotate like this. These are axis 1, axis 2, axis 3. I hope we can imagine. So this is something similar to our waist.

So, the waist axis is vertical right, the shoulder axis is here and this can be the elbow axis similar to the human hand. So now let us assign our axis as per DH notation. So, our DH notation, the first one says that the first one is along the axis of rotation or translation, here there is a rotation axis, this is another rotation axis and this is my third rotation axis. So we have fixed our three rotation axis, let us call this Z_0 fixed at 1, this is at 2 and this is my Z_3.

Now what you see is that that Z_0 and Z_1 are the same axes. But Z_1 and Z_2 are perpendicular to each other that means they are intersecting now. So in that case, we find the plane of intersection, there is the plane of intersection, there is the plane where Z_1 and Z_2 lie, so the X_2 will be perpendicular to this plane, so my X_2 is here. So, I can put my X_0, X_1 and X_2 along the direction and this is my X_3.

So this showed the rule of exception where the Z axes are intersecting. Now I am doing it very quickly as just whatever basics is required for us to go ahead in this course. So those of you who would like to study this part of DH permit assignment for different kinds of robotic mechanisms, it can get tricky at times, so you can refer to the textbook Introduction to Robotics by John Craig. So, this textbook gives details.

So any robotics text book actually gives details on DH parameters. You can also refer to the NPTEL lecture on Intro to Robotics. So there is an introduction robotics course where this is done in much more detail. Here, I am just giving you the very basics just enough that is required. So, here we have seen that you assign your axis as per DH notation like this. And then after we have assigned our axis, we have to find what is the relation between the axes so that we can go from one frame to another frame. So let us look at that now.

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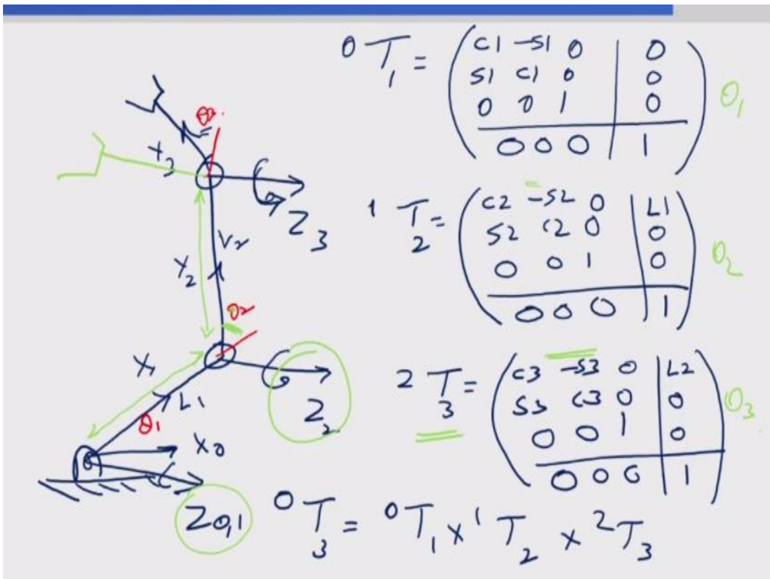

So, this is our manipulator where we have 3 links like this and this is my Z_0, Z_1, this is a Z_2, this is Z_3, and this is my X_0, X_1, X_2 and X_3 frame of segments, so assign my frames. Next I want to find what is θ here, something I missed out maybe here is what is the link angle? The link angle θ is the angle between the X axis. So in this particular case, this is my θ_1, that is my θ_2, and this is my θ_3. So, the only variable here is θ_1, θ_2, θ_3.

This is a rotating joint system, right. So here also, the angle is only between the X axis, so the θ's are there. So, θ_1 and θ_2 are there and θ_3 is here. So, the θ is the joint angle. So, in this case what is the joint angle? So in this case, the joint angle let me draw it in red. So, this is my θ_1, there is a θ_2 there and there is a θ_3 there. Let us go back quickly to the homogeneous transformation matrix that is the one I am going to use now.

So the homogeneous transformation matrix consists of four parts as shown here. This is the rotation matrix, the rotation part, rotation between what? Rotation between the two frames. This is the translation part, which is the distance between the origins of the two frames, this is always 0 0 0. This is always 1. So in this manipulator system, I am going to write the transformation matrix between the frames. So how many frames are there? There are three frames here.

So the first transformation matrix is $^{0}T_1 = \begin{pmatrix} \cos\theta_1 & -\sin\theta_1 & 0 & 0 \\ \sin\theta_1 & \cos\theta_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

Now, if you can connect the dots, so what I just said sometime back that in the DH notation, the Z axis is the axis of rotation. So you will always get $R_{z\theta}$, this axis will always be the same that simplifies matters. So this is the first one that relates the first frame to the 0 frame. The second one is relating the first frame to the second frame. Now have a look at the first frame and the second frame. This is 0 0 0, this is 1. Where is the second frame? It is here.

So second frame is here, that is my second frame. The first frame is here. So what is the angle between the X axis again, so this is angle θ_2 that is my angle. So, this is going to become; the rotation is again Z, but the variable is θ_2. So, this is going to be

$$^{1}T_2 = \begin{pmatrix} \cos\theta_2 & -\sin\theta_2 & 0 & L_1 \\ \sin\theta_2 & \cos\theta_2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

So, this is how we write the homogeneous transformation matrix. The third one is for 2 to 3. Now, what is this matrix? This matrix is again 0 0 0 and 1. Now, what is the rotation? Rotation is corresponding to θ 3 and it is a rotation about Z, right, so the rotation is here.

So, it will become, $^{2}T_3 = \begin{pmatrix} \cos\theta_3 & -\sin\theta_3 & 0 & L_2 \\ \sin\theta_3 & \cos\theta_3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$.So now I have found the relationship between the third frame to the second frame, second to first and first to zero. So, if I want the relationship between the zero frame and the third frame, all I need to do is to multiply all of these three matrices together and I will get.

So, ${}^{0}T_3 = {}^{0}T_1 \times {}^{1}T_2 \times {}^{2}T_3$. So this is what we call frame assignment by DH parameters and then multiplying it to get the relationship between the third frame to the zero frame. Now, when the manipulator is moving in space because the 0 frame is not moving it is only the third frame which is moving, sorry the other frames can move, 0 frame is not moving. So we can find the relation between all the frames on 0 frame now.

In fact, we can find the relation between any frame and the 0 frame. So that is why we put the 0 frame as the reference frame which is not moving. So everything is found with reference to that 0 frame. Now, how can we find it? Simply by multiplying it like this. So, suppose θ_3 changes just for example, just for explaining, if θ_3 changes what will happen is this end effector will come here and that will be taken care of by this matrix.

So, the rotation takes place here, right. So if θ_2 changes, then the rotation will take place in the corresponding this matrix. So you can see that each of this has a variable which is θ_1, θ_2 and θ_3. So when I multiply them out, this is what I get. So next thing we need to do is to multiply this out, so let us multiply it out.

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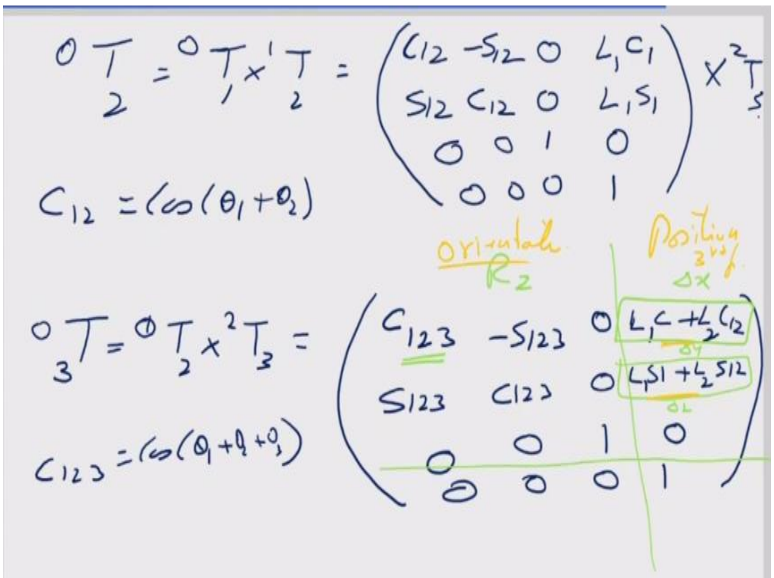

So, when we multiply all of them, $^{0}T_2 = {}^{0}T_1 \times {}^{1}T_2 = \begin{pmatrix} \cos\theta_{12} & -\sin\theta_{12} & 0 & L_1\cos\theta_1 \\ \sin\theta_{12} & \cos\theta_{12} & 0 & L_1\sin\theta_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$.

You can verify this. So $\cos\theta_{12} = \cos(\theta_1 + \theta_2)$. So similarly $\sin\theta_{12} = \sin(\theta_1 + \theta_2)$. I am writing it in short. Then we continue multiply this 0 to 3.

So, it becomes, $^{0}T_3 = {}^{0}T_2 \times {}^{2}T_3 = \begin{pmatrix} \cos\theta_{123} & -\sin\theta_{123} & 0 & L_1\cos\theta_1 + L_2\cos\theta_{12} \\ \sin\theta_{123} & \cos\theta_{123} & 0 & L_1\sin\theta_1 + L_2\sin\theta_{12} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

So, $\cos\theta_{123} = \cos(\theta_1 + \theta_2 + \theta_3)$. Now in this matrix that we have got multiplied out, see if it makes sense that is very important here. So this has four parts as we said. This is a rotation matrix. Now the rotation at cos-sin, sin cos, which is a rotation about Z axis and this is θ_1 + θ_2+ θ_3.

So the total rotation that this manipulator has had is the sum of θ_1 + θ_2+ θ_3 which you can see from here. So the total end effector rotation is the third frame. So, the total rotation of the third frame will be the sum of θ_1 + θ_2+ θ_3 and that is what we are getting here. And this is a rotation about Z, so it is correct. Now, what about this? This is Δx motion the x direction, this is Δy and this is Δz .

So, this is how much the origins are moving the x direction, this is how much the origins move in the y direction and this we can see from here, let us go back there again. So, if you see from here how much the origins have moved can be found from here. So, in the X direction is this much, Y direction is that and Z direction is this. X Y there are three, so it is going to be L_1 cos θ_1 is this distance, then the next distance is L_2 cos θ_1 + θ_2.

So, where has this fellow move if I say that is my X Y. Then in the x direction it has moved by L_1 cos θ_1 and then L_2 two cos θ_1 + cos θ_2 which we are getting here. In the Y direction it is moved by how much? In the Y direction it has moved this much that is L_1 sine θ_1 + this L_2

sine $\theta_1 + \theta_2$. So that is what we are getting in the Y direction. So, what you are seeing here is that this is the motion in the Y direction, that is in the X direction.

So this point here which is the origin of the third frame has moved in the X direction and Y direction by that much amount. plus the angle that the third axis is making X_3 is making with respect to X_0 is sum of $\theta_1 + \theta_2 + \theta_3$. So what we obtained here is correct. So it is very important for you to be able to correlate. It is extremely important robotics to be able to correlate the mathematics that we are doing plus the real system. If you cannot correlate, then there is no point.

So, we see that from here we get the coordinates of that point X and Y coordinates of that point plus the orientation of the; so this gives me the position of the third frame and this gives me the orientation. So now you understand how do we assign axes and how do we find the combined position orientation of the third frame or the last frame with reference to the zero frame. Now let us proceed further and look at forward kinematics and inverse kinematics.

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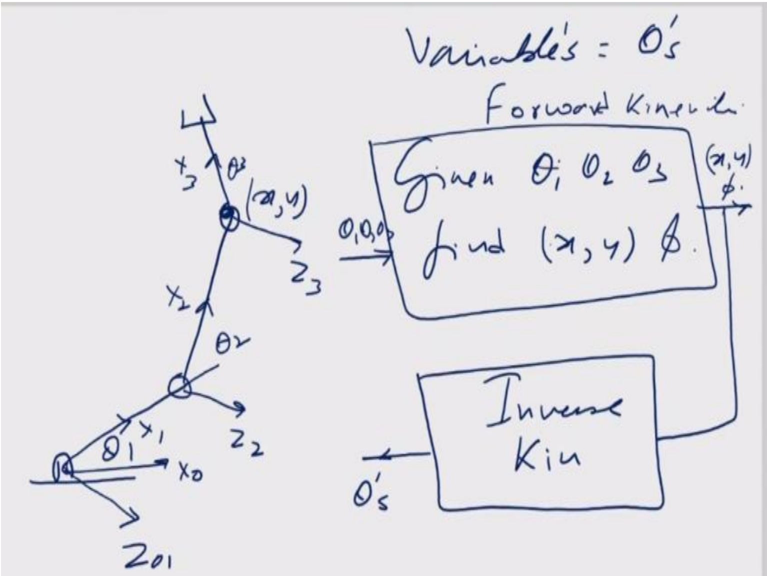

So what is that we need to do here is let us look at this example. So, suppose I am giving you this point x, y. There is a corresponding θ_1, θ_2 and θ_3. We are going to assign axes as Z_1, Z_2 that I have just explained Z_3. This is my X_1, this is X_0 and that is my X_2 and X_3. Now, what are the variables here? So, when a robot moves what changes? The θ's change, the other things are fixed, link lengths are fixed.

So, we have two requirements in the forward direction and the reverse direction that is given θ 1, θ 2, θ three find x, y and Φ, Φ is the total rotation of the third axis. So, this is one and this is basically what we say is the forward kinematics. So, input to this can be θ_1, θ_2, θ_3 and the output of this is going to be the position x, y and Φ of the end effector.

So, this is what we call by forward kinematics. Now, inverse kinematics the input is this one. So, this is inverse kinematics in which case the input is the x and Φ, so x, y and Φ and the output is the θ. So, for example, let us take a numerical example which you understand better.

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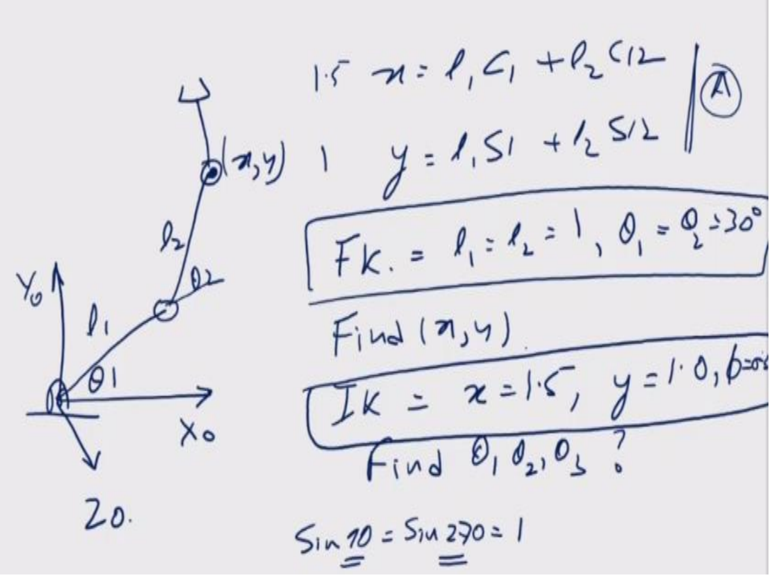

In this manner the same manipulator this is the point x, y it is actually X and Y_3. So, l_1, l_2, θ_1, θ_2. This is my X_0, you can imagine this is your Y_0, Z_0 is this side. So, what is x, y geometrically, $x = l_1 \cos \theta_1 + l_2 \cos (\theta_1 + \theta_2)$ geometrically, right. What is y. $y = l_1 \sin \theta_1 + l_2 \sin (\theta_1 + \theta_2)$, right. So, now suppose in forward kinematics I give you $l_1 = l_2 = 1$ unit and $\theta_1 = \theta_2 = 30^0$.

So now you can find x, y. Find x, y using forward kinematics. So I am giving you θ_1, θ_2 is 30^0 and 1_1, l_2 is 1, find what is the x, y for this corresponding position. How would you do that? Simply take the values of θ and put into this equation. Now inverse kinematics, I am going to give you the x, y position say suppose I say $x = 1.5$ and $y = 1.0$. Find θ_1, θ_2, and θ_3 and Φ is equal to something I will give it to you.

Now, you find θ_1, θ_2, and θ_3. This is the inverse kinematics problem. So, this is a statement of the inverse kinematics problem. So, I say $\Phi = 0.5$ just for example. So, now I hope you understand how to do that. You take x, y now, you take the values of x and y as I said it is 1.5 and 1 and then try and solve for θ_1 θ_2. So, this is the meaning of forward kinematics and inverse kinematics.

Now, obviously you understand that forward kinematics is much more simpler because given the values of θ_1, θ_2, θ_3, link lengths are of course given, you just put them in this equation A and then you solve it. But inverse kinematics is the one which is more tricky because you can see that although there are two equations, two unknowns, this appears in the form of sine and cos and sine and cos solving like this is not very easy.

Sometimes you may be able to solve, sometimes you may not be able to solve and there could be multiple solutions because you know that sine and cos always have two-two values for every values. For example, for sine 90 = sine 270 = 1. So, you are getting two values for θ here. Similarly, for cos also you will be getting two values of θ and that is what makes this solution difficult.

This is only two, but suppose you have a large manipulator like 6 degrees of freedom or 10 degrees, then there are 10 θs now and they will all come in these combinations. So, it is extremely difficult to solve. So, we are not going into detail here, again if you are interested in robotics in these particular aspects of kinematics and forward inverse, please look at the textbook I just mentioned or refer to the notes refer to the class on NPTEL on Introduction to Robotics.

Here let us proceed, you can just find out the relation. So this is basically forward kinematics and inverse kinematics. Now, let us see how do we do the inverse kinematics of this two-link manipulator system which is required for us. Very simple example again.

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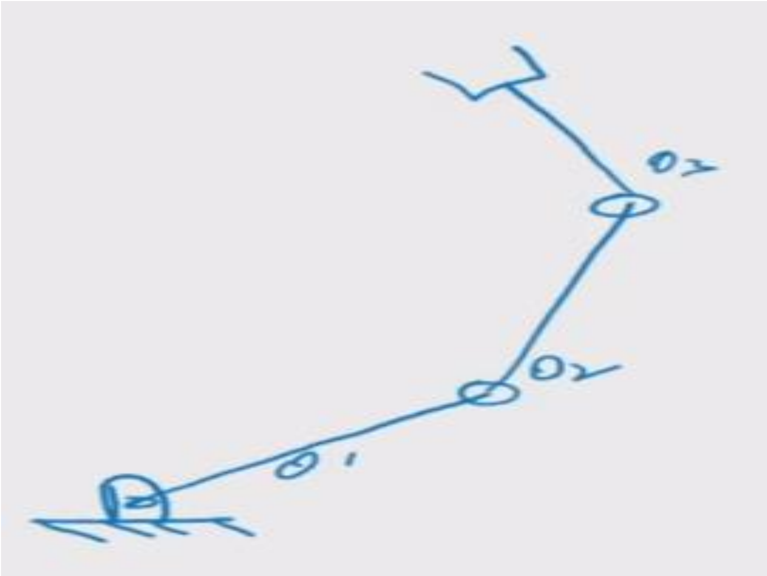

Let us look at this two link; I will further simplify this, so this is my two-link manipulator system. So, this is my θ_1, θ_2, θ_3. And my final matrix has come in the form, you remember what the final matrix was. So, in the final matrix what we got was from here, so this is what we got x is equal to that and y is equal that. So, I can solve θ_1, θ_2 from here and then θ_3 from here. So, what I do is I first write down x and y and try and solve for θ_1 and θ_2. So, I am going to write down x and y, let me rub this off.

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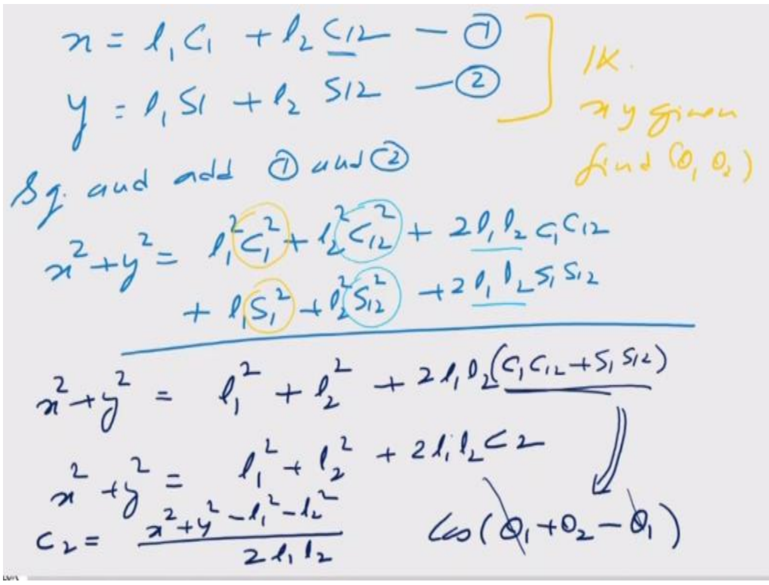

So, $x = l_1 \cos\theta_1 + l_2 \cos(\theta_1 + \theta_2)$ and $y = l_1 \sin\theta_1 + l_2 \sin(\theta_1 + \theta_2)$. Now, what you see is that there is equation 1 and equation 2; 2 equations, 2 unknowns, but the way it appears here you cannot solve so easily. So, there is a cos and a sine, so what we do is we square and add 1 and

2. So squaring and adding equations 1 and 2, what we get $x^2 + y^2 = 1_1^2 \cos^2\theta_1 + 1_2^2 \cos^2(\theta_1+\theta_2) + 2l_1l_2 \cos\theta_1 \cos(\theta_1+\theta_2)$ $+1_1^2 \sin^2\theta_1 + 1_2^2 \sin^2(\theta_1+\theta_2) + 2l_1l_2 \sin\theta_1 \sin(\theta_1+\theta_2)$ and then we add.

Now what you see here is that there is, $x^2 + y^2 = 1_1^2 + 1_2^2 + 2l_1l_2\left(\cos\theta_1 \cos(\theta_1+\theta_2) + \sin\theta_1 \sin(\theta_1+\theta_2)\right)$.

Now, if I further simplify $x^2 + y^2 = 1_1^2 + 1_2^2 + 2l_1l_2 \cos\theta_2$ because this one, $\cos\theta_1 \cos(\theta_1+\theta_2) + \sin\theta_1 \sin(\theta_1+\theta_2) = \cos(\theta_1+\theta_2-\theta_1)$.

So this and this will cancel, so you will be left only θ_2. Now, something important here is when I see this further, I can get $\cos\theta_2 = \frac{x^2 + y^2 - 1_1^2 - 1_2^2}{2l_1l_2}$. Now, we should not take a cos inverse here.

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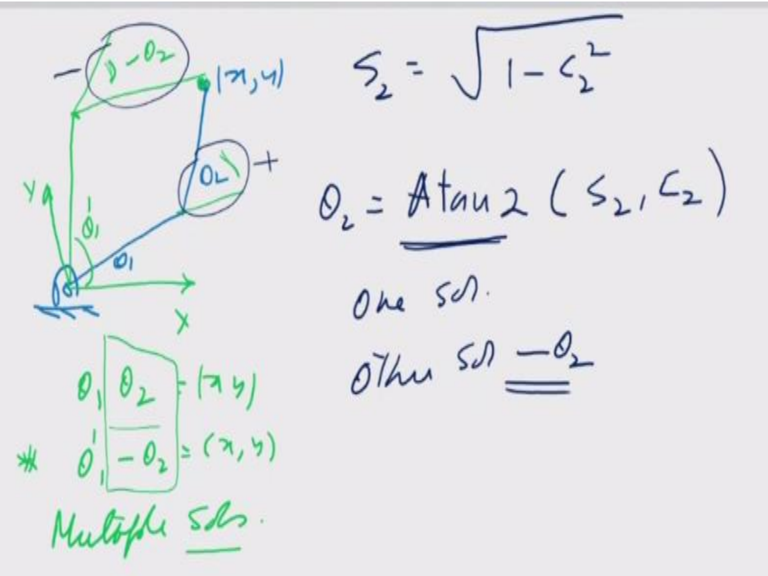

So, this is something which you need to understand carefully that we should not take a $\cos^{-1}$ to get cos 2 why because cos 2 has two values. What are the two values? For any x, y here. This is x, y, this is θ_1, this is θ_2. This is one value of θ_1, θ_2, what is the other value? So, for the same x, y you can have another configuration which is like this and the value is the same actually. I hope you can understand this. This is my x, that is my y.

So, for this value of θ_1, θ_2= (x y). But if you have the other value, this is my $\theta_1^{'} - \theta_2 = (x, y)$. Why is this happening? This is happening because the sine and cos have two-two terms. So, this shows you that there are two solutions actually and what we call multiple solutions. So, if you have higher degrees of freedom, we will have more number of solutions.

So because of this, we do not take the inverse there because you will get a plus and a minus, you will get this. Now the software if you are writing a computer program, it will not be able to distinguish between minus and plus. So this part is the one that causes the problem because $\cos(-\theta_2) = \cos\theta_2$. So as far as the computer is considered, it cannot distinguish. But as far as the real system, they are all different, you can see θ_2. - θ_2 is up there and θ_2 is here.

So, they are completely different angles. So what we do is we do not take a minus there, but we take this function, we convert it to $\sin\theta_2 = \sqrt{1-\cos^2\theta_2}$ and then we take $\theta_2 = A\tan\theta_2(\sin\theta_2, \cos\theta_2)$. So we use the special function called $A\tan\theta_2$. This function basically takes the positive $\sin\theta_2$ and the positive $\cos\theta_2$ and then it finds one solution.

So once you have found this solution, so one solution is this, what is the other solution? So other solution is -θ_2 because we have seen from here if this is your plus solution, that is the minus solution. So that is the minus solution. Next, we need to find what is θ_1.

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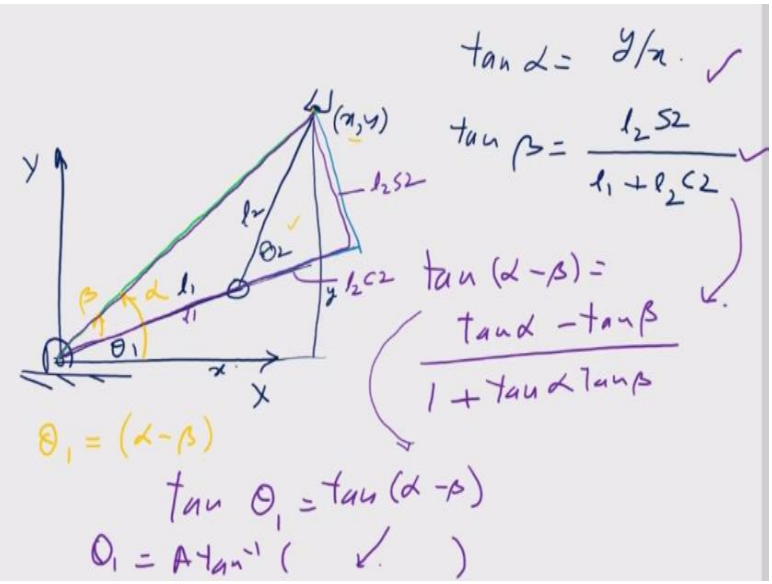

So for θ_1 again what we do is we use a geometrical method. You can use algebraic methods also. But for simplicity, we are using geometrical method here. So, we have our link is like this, this is very simple. So let us say this my point x, y. This is my X axis, this is my Y axis. So, this is θ_2, this is θ_1. Now, let me do this construction quickly. So, this construction I do this is a triangle here and there is another triangle there.

So this is one triangle here and there is one triangle there. So, what I do is I define two new angles. One angle is here, I call this angle α and I call this angle β. Now, from geometry we can see the $\theta_1 = \alpha - \beta$. So, if I can find α and β, then I can easily find what is θ_1, θ_2 is already known we just found and that x, y coordinates are given.

So now what we are saying is that in this equation that we just saw so here in the inverse kinematics solution x, y given, find θ_1, θ_2. Here, if you see from geometry again what we are saying is that what is tan α, tan α is this big triangle. So $\tan\alpha = \frac{y}{x}$ in that big triangle, so it has to go down straight like that. So, that is my tan α, this is y and that is x. So that is tan α, what is tan β? Tan β is this triangle, green and blue triangle.

So, tan β is this triangle where it is $\tan\beta = \frac{l_2 \sin\theta_2}{l_1 + l_2 \cos\theta_2}$, let me draw it in some other colour, this triangle. So, I hope you understand this part is $l_2 \cos\theta_2$ and this part is $l_2 \sin\theta_2$ and this is my l_1. Now, we have got α and we have a β, now there is a formula $\tan(\alpha - \beta) = \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha \tan\beta}$

Now from here and here, we can put the values in there, we have got these values. So, now we can say that $\tan\theta_1 = \tan(\alpha - \beta)$ which we have found here. So this is something we are finding here, this is this one. So, I can take the tan inverse. I can take a tan of this and I can find it on $\theta_1 = A\tan^{-1}(\tan(\alpha - \beta))$. So, now we have been able to find θ_2, where we just found θ_2 and I am also able to find θ_1.

So, this is basically showing the inverse kinematics of two degree of freedom system. If you are interested in the higher degrees of freedom system, please refer to the textbook. For us this is enough for motion planning because motion planning of higher than this degree of freedom system becomes very complex that is something we will be seeing. So let us proceed now to the next topic which is singularities and energy consumption when the robot is moving.

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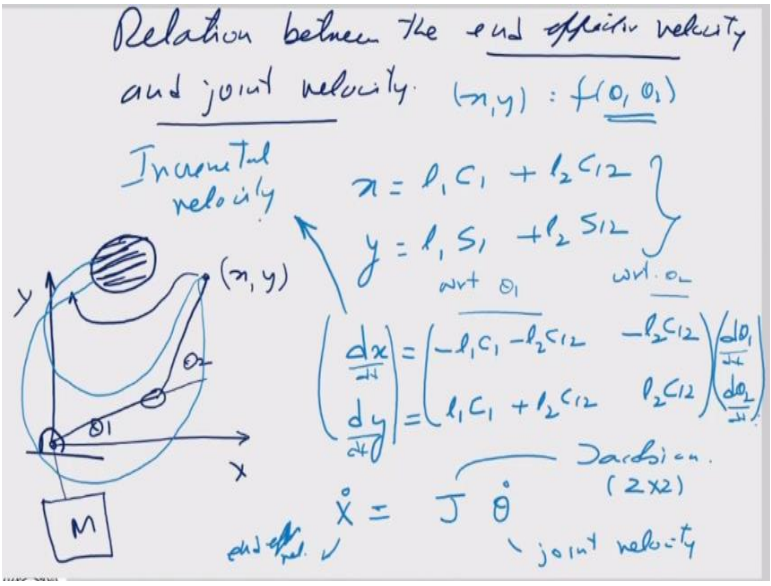

So let us look at this robot here, X axis, Y axis. Now this is my θ_1 and this my θ_2. Now how does this robot actually move. There is going to be a motor of the joint. So, there will be some motors attached to the joint side. The motor will rotate and the joint will rotate and then the end effector will rotate, so x, y. So basically, we want to find the relation between the end effector velocity and joint velocity.

Why because we are interested in moving the end effector in space, for example there is an obstacle here, I am just giving an example. And you want to move the manipulator such that it goes like this, does not hit the obstacle. In that case, you need to find the velocity, not only the position the path, but you need to find the velocities that will take the end effector along this path avoiding the obstacle.

So, you need to find the relation between end effector velocity and the joint velocity, right. Now, how do we get the end effector velocity and joint velocities? So, what we do is we write the position relationship, we know that $(x, y) = f(\theta_1, \theta_2)$ in this case and the relation also we know $x = l_1 \cos\theta_1 + l_2 \cos(\theta_1 + \theta_2)$ and $y = l_1 \sin\theta_1 + l_2 \sin(\theta_1 + \theta_2)$. Now we see that there are two variables, θ_1 and θ_2 and on the side we have x and y.

So if I take the partial derivative of the first equation and the second equation first with respect to θ_1, then with respect to θ_2, so what I am doing is I am differentiating equation 1 with respect to θ_1 and then with respect to θ_2 and then the second equation with respect to θ_1, then with respect to θ_2. So this I am getting,

$$\begin{pmatrix} dx \\ dy \end{pmatrix} = \begin{pmatrix} -l_1 \sin\theta_1 - l_2 \sin(\theta_1+\theta_2) & -l_2 \sin(\theta_1+\theta_2) \\ l_1 \cos\theta_1 + l_2 \cos(\theta_1+\theta_2) & l_2 \cos(\theta_1+\theta_2) \end{pmatrix} \begin{pmatrix} d\theta_1 \\ d\theta_2 \end{pmatrix}$$

I hope you understand the way I have written it. Now this you can write

$$\begin{pmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{pmatrix} = \begin{pmatrix} -l_1 \sin\theta_1 - l_2 \sin(\theta_1+\theta_2) & -l_2 \sin(\theta_1+\theta_2) \\ l_1 \cos\theta_1 + l_2 \cos(\theta_1+\theta_2) & l_2 \cos(\theta_1+\theta_2) \end{pmatrix} \begin{pmatrix} \frac{d\theta_1}{dt} \\ \frac{d\theta_2}{dt} \end{pmatrix}$$

So, first we are differentiating with respect to θ_1, then with respect to θ_2. So now in short, I can write this as $\dot{x} = J\,\dot{\theta}$.

Now, this is called the Jacobian that relates the joint velocities and the end effector velocity. So, this is my joint velocity, right. This is my joint velocity and this my end effector velocity. So, the Jacobian relates in this case the 2 X 2 matrix and this relates the joint velocity with end effector velocity. That means, when I am trying to follow a path or I am trying to go to a point, this is going to determine what the relationship is between the joint velocities and the end effector velocities.

And this is very important because for example to avoid this obstacle, you could do all kinds of things, you could go like this, you could go right from here and go like this and like this. So, which path are you going to take? Well, you might say you will take the shortest path. Well, shortest path may not be the optimal energy path. What we normally try to do is we try to minimize energy. So, for that this relationship comes into effect. Let us look at the relationship a little bit more detail because it is very interesting and very important.

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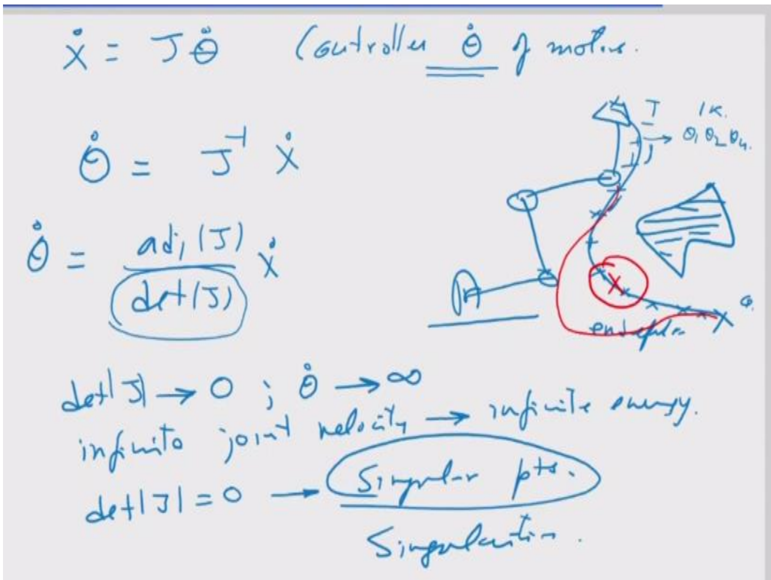

So $\dot{x} = J\dot{\theta}$. Now from the controller point of view, the controller would control $\dot{\theta}$ of motors that is the only controls. So, for a given end effector velocity, the controller has to control the motor velocities. So, what is more important is $\dot{\theta} = J^{-1}\dot{x}$. For example, I have a manipulator here and there is an obstacle here. As you are doing motion planning, this is a forward motion planning, I want to go from this point to this point, this is my initial point I and this is my goal point J.

I want to go from there to there. Now, there can be infinite paths, there could be no paths. So, now suppose I find a path like this, like this, like this, I am finding the path at the end effector level, not at the joint. Now, for every point here to take the end effector there, the motor controller must be able to give a particular joint velocity. So, for this I will be able to get corresponding angles which is θ_1, θ_2, how many θs are? θ 4's are there.

So, for every point here it has to do inverse kinematics and find what is the corresponding joint, then the difference between this and this is an incremental velocity, yeah it is something I forgot to mention here that this is an incremental velocity. So, these are incremental velocities and these are not average velocities. like 2 m/s, this is an incremental velocity. So, this also as we saw here the difference between these two points are incremental velocity.

So, if you want to follow a path, we must do inverse kinematics at every point on the path and then correspondingly go back to the joint velocities that is one way of controlling it if you want to control the end effectors. So, what is very important for the controller is to be able to control the joint velocities now and the joint velocity is given as $\dot{\theta} = J^{-1}\dot{x}$. So, this is $\dot{\theta} = \frac{adj(J)}{\det(J)}\dot{x}$.

Now, in terms of energy you see that the relationship is governed by the determinant of J, the amount of J comes in the denominator. So, if det J $\rightarrow 0$, then $\dot{\theta} \rightarrow \infty$. Now, this basically means that infinite joint velocity means infinite energy, right. So, to produce infinite velocity you need infinite energy in the motor that is not possible, right. And such points where det J= 0 these points are called singular points, singular points of singularities.

So, a robotic mechanism cannot go into a singularity, it cannot go anywhere near a singularity because the energy will suddenly become very high. Just for example, if this was the path and we found that there was a singularity here, then it cannot go anywhere near that, it was go off in some other direction. The question of singularity is very important here.

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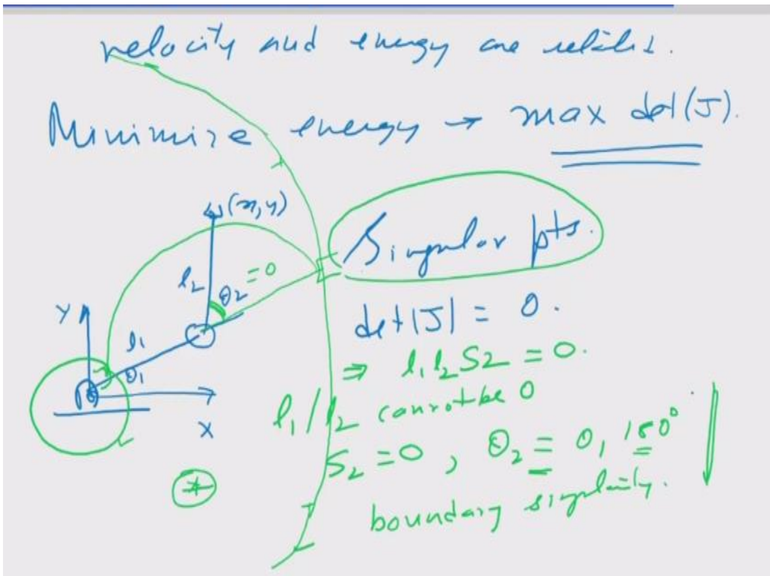

The second is this also gives us an indicator of energy because you know that velocity and energy are related because if you want more velocity you have to give more energy, so there is a relationship there between the motor velocity and the energy. So, if you want to minimize energy, then you must maximize determinant J, this is very clear from here from the previous relationship. So, in this relationship if you want to minimize the joint velocities, then you must maximize J to minimize the joint velocities.

In the reverse also if you want to maximize joint velocity you have to minimize J. So, to get minimum joint velocity you have to maximize J and to get maximum joint velocity you must minimize J. So, to minimize energy, you must maximize determinant of J. So, let us look at these two examples singularity and maximum manipulation ability or what we say the best way of doing a task. So, let us look at this two-link manipulator.

This is a two-link manipulator here and I have assigned my axis. This is my X, that is my Y. So, this is my θ_1, this is l_1, this is θ_2, this is l_2. The end effector point is x, y that is the point. Now I want to find what are the singular positions for this? What are the singular points? Now the singular points would mean that det J = 0. So let us find the det J. Where is the determinant gone? This is the determinant.

So if you multiply it out, what you will get is det J = 0, $l_1 l_2 \sin\theta_2 = 0$. It is interesting that θ_1 is not in the picture at all. Because of the sine and cos what will happen is θ_1 will cancel. Please find the determinant and actually find. Now $l_1 l_2 \sin\theta_2 = 0$. Now, l_1, $l_2 \neq 0$ because they are link lengths. So, the only option is $\theta_2 = 0$. Now, if $\theta_2 = 0$, then sine $\theta_2 = 0$ then $\theta_2 = 0$ and 180^0 that is only two possibilities.

That means when θ_2 is here, so it is fully stretched like that then $\theta_2 = 0$ and that is the single position. θ_1 is not in the picture. So, it basically means that it will be a curve like this. So, this is called the outside singularity, the boundary singularity. This is called the boundary singularity and in case that is corresponding to $\theta_2 = 0$. Now, if it is fully folded like this that the other end, so that is also a similarity where it is 180^0.

So, if the link lengths are not equal then inside of you will get a region like that. So, outside region and inside region and this is my boundary singularity, this is my inside singularity. Now, the two degree of freedom manipulator does not have inside workspace singularities, it does not have, other manipulators can have, 6 degrees of freedom, 7 degrees of freedom, they can have inside the workspace, but this 2 degrees of freedom does not have.

So, we have seen where are the singular positions for this two-link manipulator having 2 degrees of freedom here, two revolute joints and we have answered that where are the singular points. Now, what about the next part? This is a very important question especially because we are doing motion planning.

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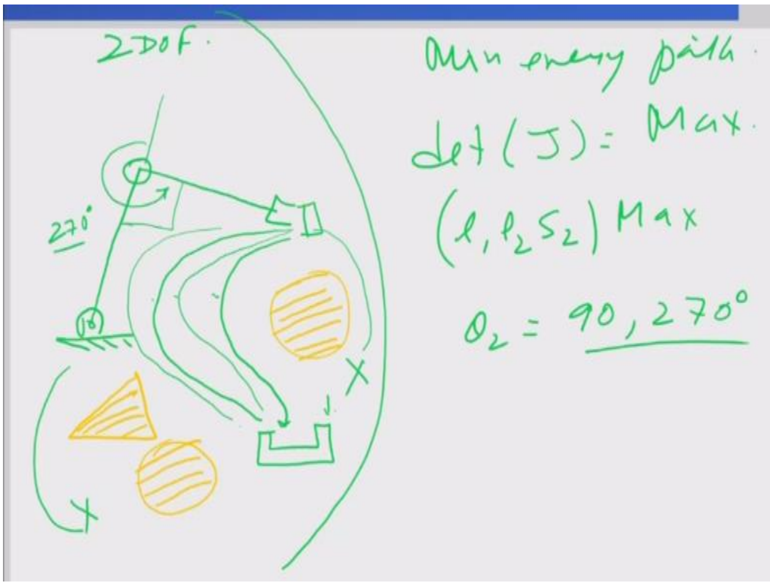

We have a link like this, 2 degrees of freedom, 2 DoF. It has a work volume, the inner work volume will be something like this. Now, you can perform the task anywhere, right. Suppose you want to assemble two parts like this. Assemble one part like this, inside another part like that. You can perform the task anywhere in the workspace. So, for example you can perform the task to here. So, we are going to perform the task in such a way that the energy is going to get minimized. This is one example I have given.

Now, there can be other examples. For example, this is an obstacle, this is another obstacle, and there is a third obstacle here. We are doing motion planning, this is very relevant for us. Now, suppose I want to take an object from here. I want to take this object from here, pick it and insert it there. Now how many paths can there be? There can be no path, there can be many paths. Can you go this way? Probably no. Can you go this way? Probably no. But going this way you can have many paths.

So, if you have many paths in going like this, then which path you will take? You will take the minimum energy paths. So minimum energy path. So, if you take a minimum energy path, how do you find the minimum energy path? That is essentially by finding the determinant of the Jacobian again. When I want to do a task, where would I do the task? I will do the task in a place where the Jacobian is maximized, so this should be max. So, if you want to maximize the Jacobian; we have seen $l_1 l_2 \sin\theta_2$ is the Jacobian.

And if you want to maximize this, maximize this, then $\theta_2 = 90^0$ and 270^0. So, now we know where θ_2 is maximized, where your energy requirement will become minimum. So, you should do the task such that this θ_2, where is θ_2, θ_2 is here, this is 270, that is 270, so somewhere in that region. So the closer the arm, the closer this is to 90^0, the more energy is going to get minimized. So, you should try and perform task as keeping that angle close to 90.

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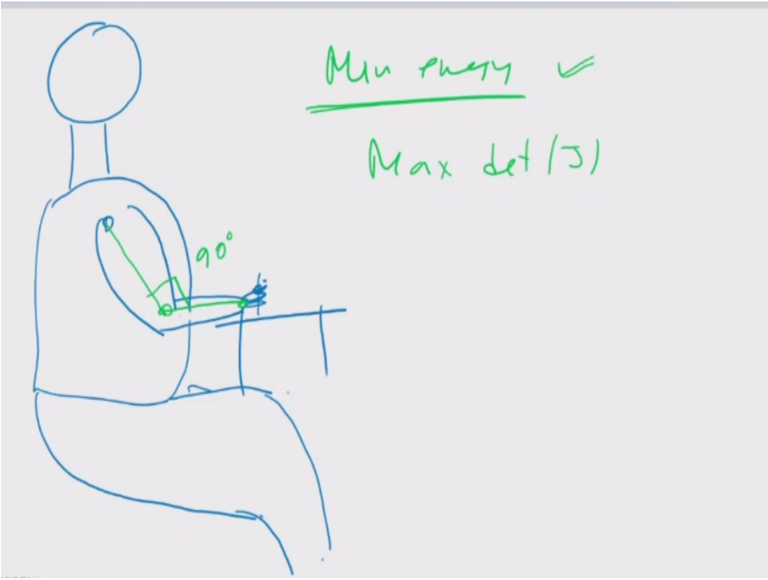

Now, it is very interesting that if you look at human beings, even for us, so we looking at a human being, maybe next time we do this, please note. So, when you are doing a task, for example you are writing, sitting and writing. Let us say you are sitting on the chair and then writing. You are writing on that. Now, our arm is something like a two-link mechanism. This is one linkage, this is another linkage. It is almost like two links, just the two links I have drawn.

So, in this two-link system, we are going to write such that, it is not correctly drawn. So, in this two-link system also we are going to do a task such that our energy is minimized and we all do it like this. So, when we are writing you have multiple solutions, you can write in different ways, you can write in different parts of the workspace. Similarly, when you are doing a task, you are doing it in different parts of the workspace.

So, if you can imagine the robot can do the task here, the robot can do the task here, here, here, here. Where should the robot do the task? This is saying that it should do the task such that this angle is 90^0 and that region is somewhere here. So, this is the optimal region. Now, in terms of writing for example, a very interesting example, you see that our link, our hands are having 2 degrees of freedom and it is like this, we can approximate it.

So, in that case your minimum energy would be in a case where your angle this angle is 90^0. So next time you write or you do some task, you please note that the angle at your elbow is around 90. And in fact, we all do the task like that. Why? Because that is the minimum energy condition. So, it is not only that we are studying robotics and we are talking about mechanisms and things like that, but this is valid even for us because in our legs, hands, arms, body, we have also linkages, right?

So whatever we are studying is valid for us also. So, minimum energy we have seen. You can ensure minimum energy by maximizing det J that is very interesting and very important for us because we are always interested in finding paths where your energy gets minimized. So now, today we have looked at homogeneous transformation matrix. Then we looked at how do you find, how do we assign frames and how do you go from one frame to another frame?

And how do you find the relationship between one axis, the end effector axis to the 0^{th} axis or the base frame such that when any of the joints move, we always find its relation with respect to the base frame, which is not moving. We also looked at singularities and we also looked at this concept of energy minimization. So, this is for linkage system.

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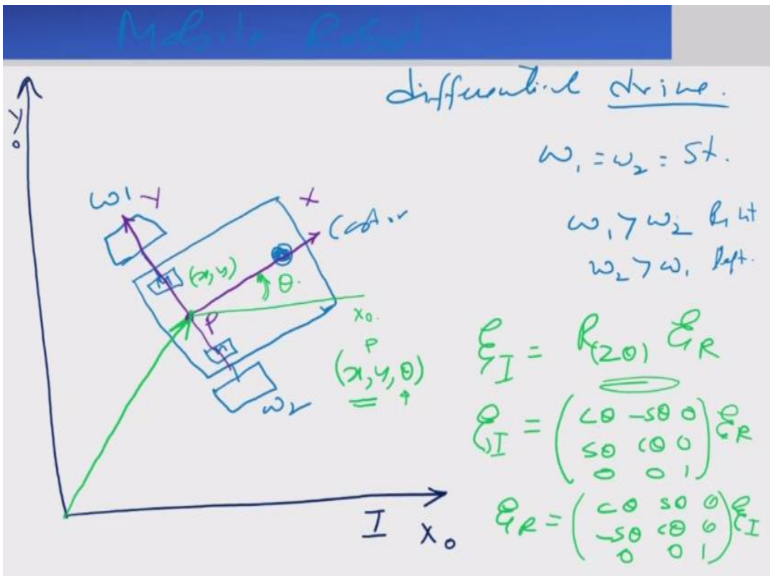

Now, let us move on to a mobile robot. Let us say in the case of mobile robot, what is the relationship? So mobile robot. So in this course, we will be dealing with mobile robots and also will be dealing with serial arms. So, this is a mobile robot, a differential drive mobile robot which means that there are two wheels which has ω_1 velocity and this one has another velocity ω_2 and there is a castor wheel there, which basically supports differential drive with castor wheels.

So, there is a castor wheel there. So, there are motors connected here. There is one motor here and there is a motor here. So, if the $\omega_1 = \omega_2$ it will go straight. If $\omega_1 > \omega_2$, it will turn on the right side. And if $\omega_2 > \omega_1$, then it will turn on the left side if ω_2 is greater; yeah this will be right, so right turn ω_1 is greater. ω_2 is greater it is left turn that is how it moves. Now, the first thing we need to do here is to assign frames. So, let us write our global frame.

In the case of mobile robots, we assign a global frame which is not moving. Let us call this the I frame. This my X axis, Y axis which is not moving. I can call it X_0, Y_0 also the frame which is not moving. Then I assign a frame onto the robot. So, what I do is I connect these two with a straight line and the centre point there I call it point P and then I take this in the forward direction as my X axis and this one as my Y axis.

So, I hope you understood P point is the centre point between the two wheels. And now what we are doing is this is making an angle θ with the X_0, this my X_0. So, now you can see that there is ω_1, ω_2, and θ is there. So, now what is the relationship between the frames or what is the relationship between the coordinates? We have a coordinate x, y here. So, we have x, y and x, y and θ.

So, to specify the position of the robot, we specify x, y of that point P and θ is the angle the X axis of the robot is making with the fixed frame. Now, what is the rotation matrix? Now, Z is pointing out of the plane because it is a rotation about Z again. So, what is ζ? Let us call this $\zeta_I = R_{(z\theta)}\zeta_R$. Now, you know the $\zeta_I = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}\zeta_R$.

So, when this robot is moving and rotating, it is rotating basically, then we exactly know what is the relationship between the rotated system and the reference system. Now, if I take the derivative of this, then if I take the inverse of this $\zeta_R = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \zeta_I$. So if I want to find the velocity, I just take the derivative of this and that, so I am getting the joint velocity.

So, the velocity of the robot is also governed by this relationship only. So, now we have seen that when the mobile robot is moving, we are defining its position by this position x and y and it is making an angle θ with the X_0 and the relationship is also governed by this rotation matrix which is $R_{(z\ \theta)}$ which we have just studied is for a mobile robot also. This is my position x, y. So, today we looked at homogeneous transformation matrices.

Then we looked at how to find forward kinematics and inverse kinematics. And then further we went on to look at the question of singularities and I explained the importance of looking at minimizing energy in terms of performing a task, finding a path in robot motion planning. So, from next class we will move on to the various algorithms in motion planning and we will start off with the bug algorithm.

And then we will move on to other kinds of algorithms which are there for serial arms, mobile robots. We also will look at topology which is required basically because path planning is not done in Cartesian space, it is done in configuration space. So, all these terms we will be seeing as we go along. So, we stop today. Thank you.