Robot Motion Planning
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Lecture – 09
Topology of C Space

Hello and welcome to Lecture number 9 of the course Robot Motion Planning. In the last class, we looked at C space mainly for, of the, for mobile robots. Today, we will look at C space for a serial arms and then we will move on to how to do path planning for serial arms in C space. And also, we will look at the topology of C space. So, we continue from where we left in the last class and briefly revise and then move on.

So, in the last class, we were looking at C space for mobile robots and we saw that the obstacles get enlarged by the amount of the by the radius of the mobile robot. Or, if it is a triangular robot then we look at the angle by which it is rotated and correspondingly increased the obstacle to form the C obstacle. Then we moved on to the C space for serial arms.

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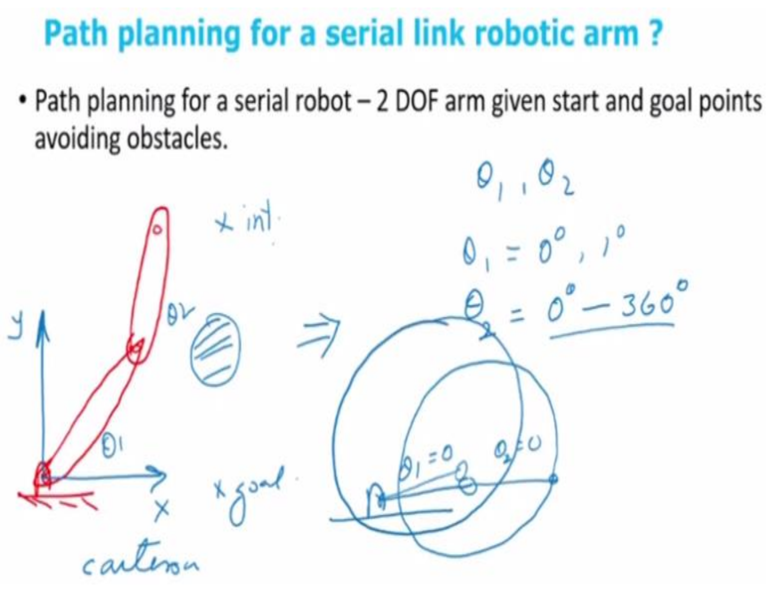

Today, we start off from there. Let me define the problem before we move on. So, we have a 2 link manipulator here. So, this is one link and there is another link. And basically, there is an obstacle here. So, this is my base and there is an obstacle here. Let me say that there is an obstacle here. Now, I want to go from some initial point. Let us say I have an initial point here and this my goal point. I want to go from the initial point to the goal point.

The objective is to find what are the various parts by which I can go from the initial to the goal point? And, if a path exists, how many paths exist? Or, if no path exist, then we also have the algorithm should also be able to say that no path exists. Now, this is basically Cartesian space. Why is it Cartesian space? Because this is my x axis, that is my y axis. So, this is my x y axis. Now, path planning is to be done in C space.

So, now, what we can do is we want to convert this into my C space. How do I do that? Basically, we look at I start off by saying that we have 2 variables here, θ_1 and θ_2. So, I have θ_1 and θ_2. So, we start off by saying $\theta_1 = 0^0$. That means my arm is completely stretched in the forward direction. That means this is my arm I am just running the link diagram.

This is my $\theta = 0$ and this is $\theta_1 = 0$ and this $\theta_2 = 0$. I start from there. Then, what I do is I start with $\theta_1 = 0^0$ and I move θ_2 from 0^0 to 360^0. That means I take 1 circle here like this. Then, I increment θ_1 by 1 degree. So, my θ_1 comes here now. And then, I do the same thing for θ_2. I go from 0 to 360^0. What would happen is I get another cycle. So, I keep doing this.

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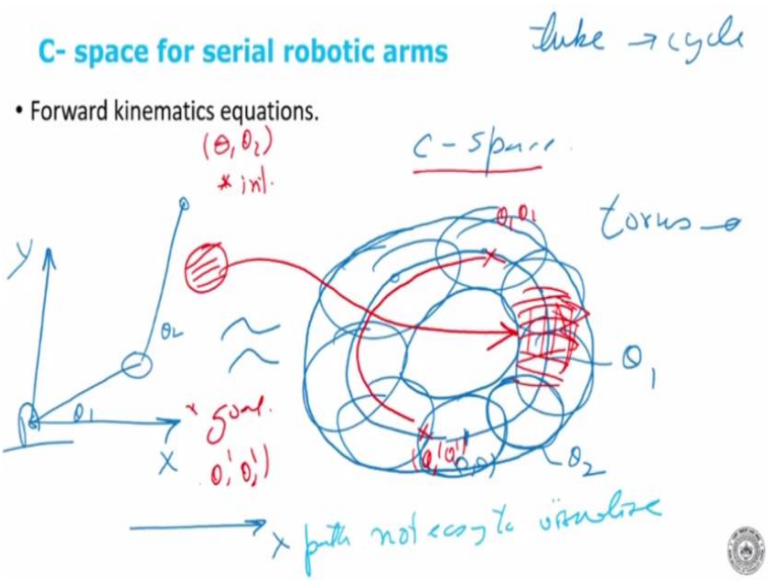

And, what we do get is, so, if I do that what I will get is if this is a 2 link manipulator I will be getting the set of circles which are moving like this. So, I will be getting this set of circles which are moving like this. And, this is the same. This is basically my θ_1 which is going round like that. So, this is my θ_1. And, for every starting from 0 0, this is my θ_2. Then, next is my θ_2 again. So, this is θ_2.

So, what we are seeing is that the shape of the C space that you are getting if you can imagine. This is what we call as torus. So, this is a torus shape. Now, the torus is something like the, in the tube of a bicycle. You must have seen a bicycle tube. So, when you inflate it, it forms a torus. Bicycle tube and then motorcycle tube, they all torus. So, this is my θ_1, this is my θ_2. Now, this is different from Cartesian space.

In Cartesian space, basically, we are talking about x and y. So, this is my Cartesian space. This is x and that is y. So, you can see that this space and this space are different now. This is my C space. Now, if there is an obstacle there, this obstacle here, then the obstacle will also appear somewhere here on the C space itself. That means on the torus. So, when I am doing my path planning say I want to go from this point to this point, this is my goal point, then essentially it is similar to saying that for I can find from inverse kinematics.

I can find what is θ_1 θ_2 corresponding to that point? And, I can find the $\theta_1^{'}$ $\theta_2^{'}$ corresponding to this point. So, it will be some θ_1 θ_2 which will come here and here, just for example. Suppose, this is my $\theta_1\theta_2^{'}$ and this is my $\theta_1\theta_2^{'}$. So, when I am finding a path I have to find a path which is going to go like this. And, if there is an obstacle, it cannot go from here.

So, the path planning is being done on the surface of a torus. And, the C shape C space of this serial arm is a torus. So, the object is going to become like this now. So, in the earlier case, it was very easy to say, what was the shape of the object in the case of the mobile robot? Because, the C space and the Cartesian space, they looked alike although they were different spaces. But, here, it is completely different. So, let us look at this.

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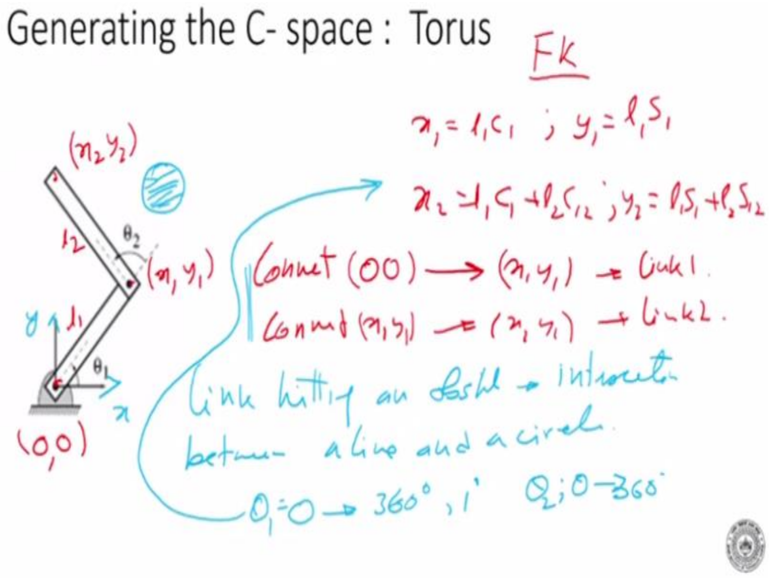

How do we generate the C space? Basically, I am taking this 2 link manipulator l_1 and l_2. The link lines θ_1 θ_2 are the, so, what I do is I get my coordinates 0 0 here and I get my coordinate (x_1,y_1) at this point and (x_2, y_2) at that point. So, what I am basically going to do is as I just explained I am going to vary θ_1 from 0^0 to 360^0 in units of 1^0 each.

And then, I am going to vary θ_2 from 0 to 360 for that corresponding θ_1. And, I will get the coordinates of this points (x_1 , y_1) (x_2, y_2). How do I get that? From my forward kinematics equations, I know that $x_1 = l_1 \cos\theta_1$ and $y_2 = l_1 \sin\theta_1$ and $x_2 = l_1 \cos\theta_1 + l_2 \cos(\theta_1 + \theta_2)$, $y_2 = l_1 \sin\theta_1 + l_2 \sin(\theta_1 + \theta_2)$. So, I have got these points the coordinates of the points the origin is 0 0.

So, if I connect 0 0 with (x_1 , y_1), this is actually my Link 1. And, if I connect (x_1 , y_1) with (x_2 , y_2), this is my Link 2. Now, if I am, if there is an obstacle there, let us say I have an obstacle here, like this, there is an obstacle. So, if I want to find intersection which is an obstacle I want to find the link hitting an obstacle. This basically means intersection between a line and a circle. So, I, this is basically an intersection between a line and a circle.

So, basically, what we are seeing is that this for different values of θ_1 and θ_2 as we are covering the full space θ_1 starting from 0 to 360^0 in increments of 1^0. And, for each θ_1, we are varying θ_2 to 0 to 360^0. And, by putting these values in this equation, we can get (x_1 , y_1), (x_2 ,y_2). Connect this by straight lines. Get the links. See if they are intersecting with that obstacle.

If they are intersecting with the obstacle, we know that the links are hitting the obstacle. I will come to the program a little bit more detail little bit later. But, this is basically how we find out how we go from the Cartesian space to the configuration space for a serial arm.

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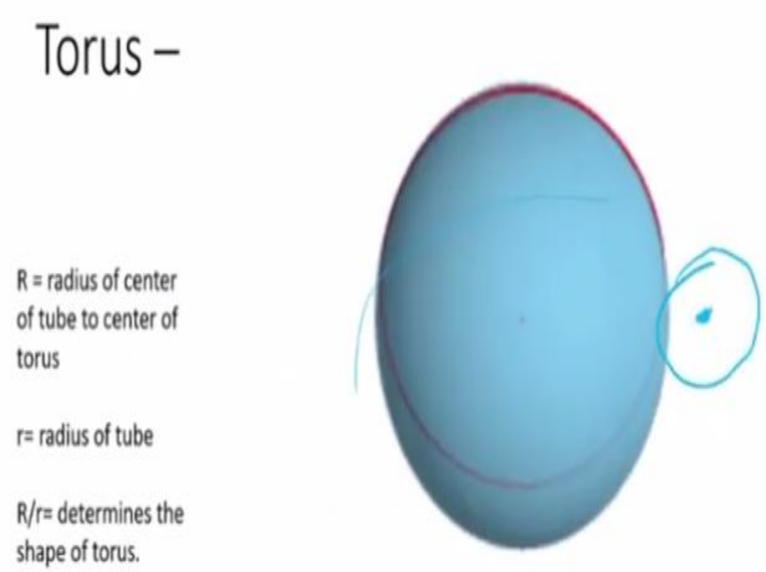

Now, this is something like this that as I just explained that θ_1 is at this point 0 0 and θ_2 will be a circle. So, let me go here again. So, for this 0 0, this is my θ_2. Now, θ_2 rotates for every θ_1 and what we get is a torus. That is how we get the shape of a torus. Now, if you are talking about a torus and we just said that the path planning has to be done on the surface of this torus, then it is not easy to visualize.

So, path is not easy to visualize here, unlike in the case of a mobile robot. So, what we do is that we basically take the torus.

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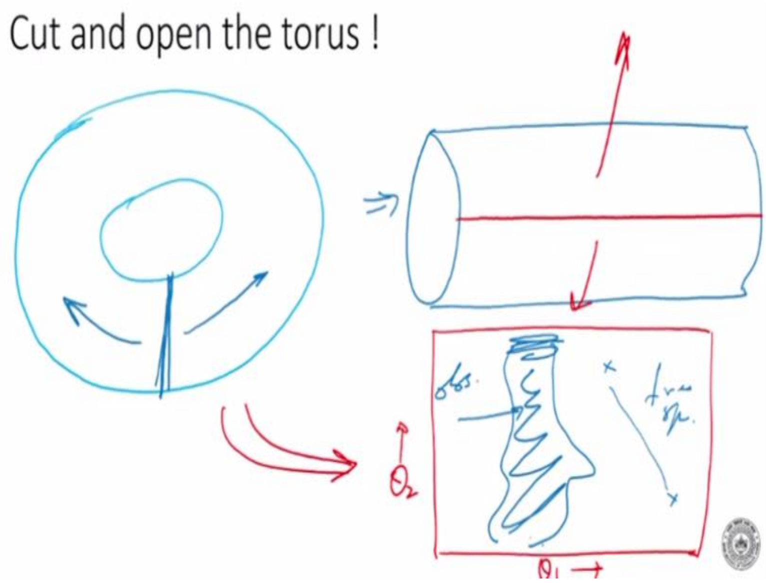

And, in order to be able to visualize it, let me draw the torus here. So, this is my torus. So, what we do is basically we cut the torus. So, if you can visualize this, what we do is I will cut the torus here. So, if I cut it, if you cut a bicycle tube, what will happen? This will become a cylinder like this, little exaggerated. But, I hope you understand. So, I cut it here and I stretch it out like this if you can imagine. So, it becomes a cylinder.

Then, what I do? In this cylindrical pipe, what I do is I cut it here again and I open it like this. So, if I open it, what will happen is this will become like this. Now, this is my θ_1 and this is my θ_2. So, from this shape, I have come to this shape. Again, this is my C space. It is not Cartesian space. But, it looks like Cartesian space. Here, it is easier to visualize. Say, for example, now, if you have the obstacle and the obstacle is something like this.

So, it has some funny shape, like this, this is my obstacle, then I can do path planning. I want to go from here to here. I can do path planning like this because the robot has become a point. And this, in my configuration space, this is becoming my obstacle now. So, this is obstacle space and this is free space. So, what we do is we form the torus. We write a program to form the torus.

And, we cut it open and display it like this. And then, do the path planning because it is much more easier to see and understand.

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So, something like this. So, you have a flat mat. If you rotate it, you get a cylinder. If you roll it up and then if you roll it again about this axis what you get is a torus. So, the reverse is if you cut it and open it, what will happen? It will be a cylinder. And then, if you open it, it will be a mat. So, it is like folding a mat and then again folding it in the other direction. So, this is the torus.

So, this is what I was meaning by saying that you cut the torus open it up, it becomes a cylinder. You cut along the axis of the cylinder and what would happen is and then open it up what you get is a rectangle which is similar to the Cartesian space. It is not Cartesian space but it looks like Cartesian space. Now, this is something we talked about. So, the shape of configuration space for mobile robots and serial arms is different.

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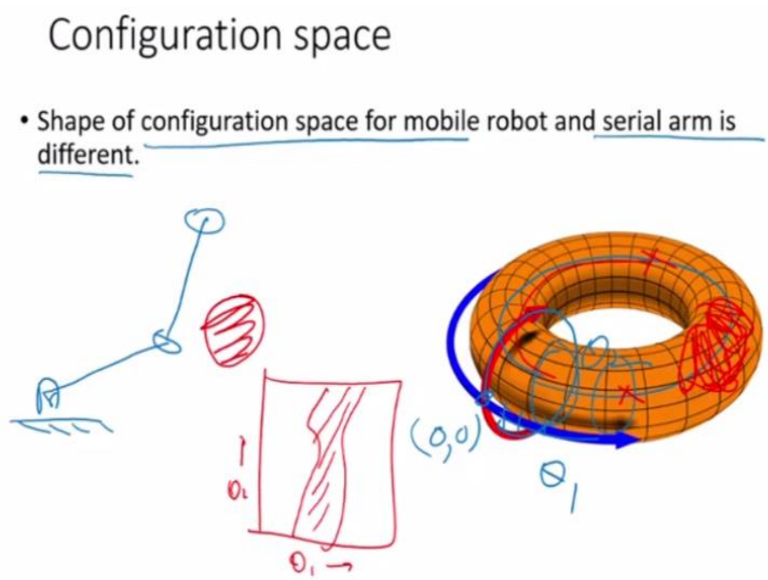

Now, in the case of a serial arm, it is a torus and what we do is we cut the torus in 2 locations. So, this is my, if you are starting from here 0 0, just for example, then this is my θ_1 and that is my θ_2. So, for every θ_1, there will be a θ_2. So, there will be a circle. So, all of it connected to together becomes a torus now. And, in order to visualize it, what we do is we cut it open.

So, this is my 2 link manipulator something like this. There is an obstacle. So, there is an obstacle there. So, the obstacle will come on this surface now. It will have some shape. And, if I want to do path planning from one point to another point, I have to do it on this surface now. So, I will cut it, open it, make it into a rectangle and then do path planning on that. So, this will be something like, this is my obstacle space.

So, as we go along, it will become more and more clearer how this is done. So, how to generate the C space and obstacle space in the form so, basically, what we do is we write a program.

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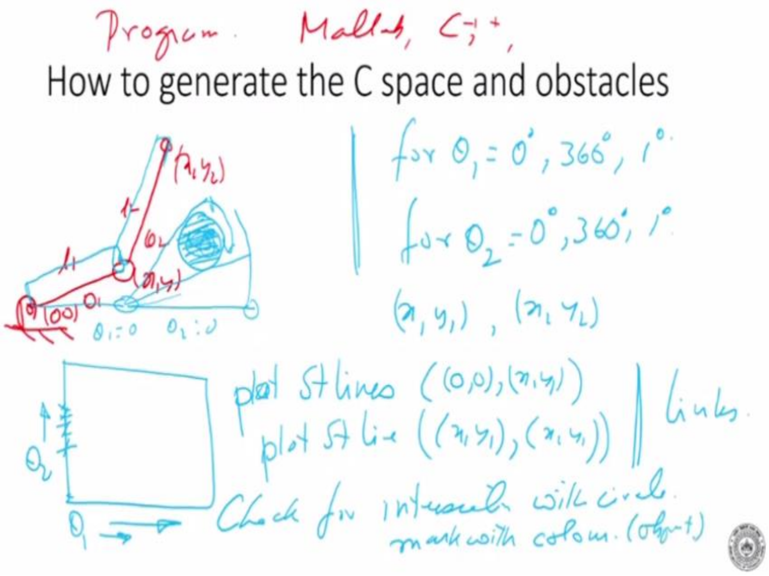

So, this is basically a program. You can write in MATLAB. You can write in C language or C++ or whichever language you want or Python. So, basically, what it is doing? What is the logic here is that we have this 2 link manipulator which has 2 links l_1 l_2. This is the point 0 0. This is (x_1 ,y_1). This is (x_2 , y_2). It is my θ_1. That is my θ_2. That means, and suppose, there is an obstacle. So, there is an obstacle here.

So, basically, what we are doing is basically we are running 2 for loops. So, I have one for loop here which is taking from θ_1 is going from 0^0 to 360^0 in intervals of 1^0. And, I have one more for loop which is taking θ_2 is going from 0^0 to 360^0 at intervals of 1^0. So, there are 2 for loops and I compute (x_1 ,y_1) from that equation then I compute (x_2 , y_2). I just explained how to get that values.

Then, I plot straight lines. So, there is the plot command in MATLAB for example. So, you connect 0 0 with (x_1 ,y_1). So, there is 1. And then, you can plot straight line again. You connect (x_1 ,y_1) with (x_2 ,y_2). So, we are connecting this by straight lines. Now, this is basically what is my links. Now, you need to check for intersection with circle. Now, you know the equation of the circle.

And, you also know that how to check for intersection if any part of this line is inside the circle or if the line is touching the circle that we can check from geometry. The other way is if the link is a rigid link for example. We say, this is a rigid link like this. That means it has some geometrical shape. That is also okay. So, what we can do is we can check for interference between the rigid geometrical rectangle and the circle.

So, in MATLAB, it is very easy, it can do the intersection very easily. So, what we do is we run these 2 loops. And, for every value of θ_1 and θ_2, we check if there is intersection. Now, if there is intersection between the circle, what we do? We put a particular color. Say, for example, this is blue color. Light blue color is the obstacle. So, this is my θ_1 is going in this direction. θ_2 is going in that direction. So, for θ_1, starting from 0.

Say, in this particular case, for example, if it is like that. This is just giving you an example. We start from $\theta_1 = 0$, $\theta_2 = 0$. Now, at $\theta_1 = 0$ as the link θ_2 is rotating after this angle, it will hit that object. You can see. Intersection will be there. So, from here to here, there is intersection. So, what would happen is I am going to mark it with this particular color. Same color as the obstacle and I proceed like that.

So, wherever there is intersection, I will mark with color of the object. So, what will happen is you will end up getting a figure like this.

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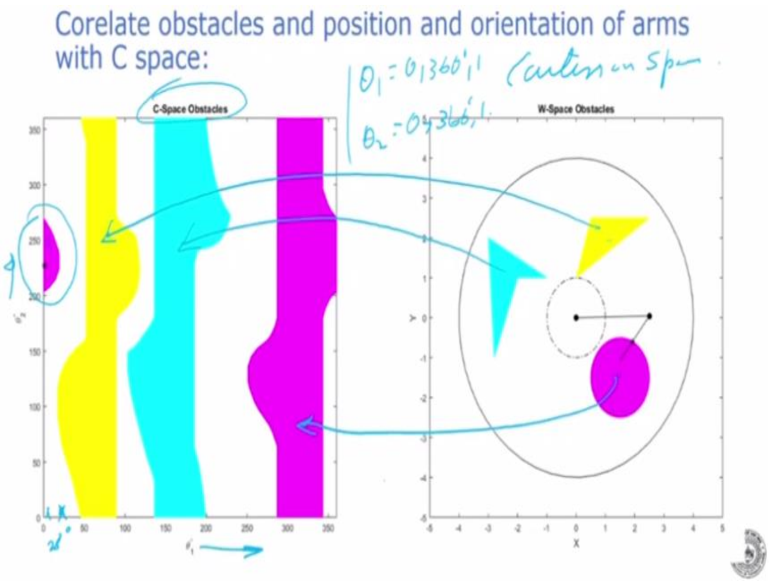

Now, let us look at this very clear very carefully and understand. On the right side is my Cartesian space. This is my Cartesian space. This is my C space or C obstacle space. So, we have a link. So, 0 0 is here. That is one the 0 0 in the Cartesian space. Then, we have a link which is 3 here. So, this is coming till 3. The link length is 3. And, there is another one which has the link length of about 2. So, this is the other link length.

Now, what we are doing is we will start from θ_1 = 0 to 360 intervals of 1 unit and θ_2 goes from 0 to 360 intervals of 1 unit. So, when θ_1 is 0, you can see. This is my θ_2 is going to rotate like this like. So, from here to here, this much range, it is hitting the red obstacle. So, θ_1 = 0 is here. Look at the left side figure. θ_1 = 0 is here.

So, at 200 ^{0}, so, we are measuring counter clockwise which is going like this still here. About 210 ^{0}, it is hitting. So, I am marking it with orange color. Why? Because this is, marking it with pink color because this obstacle is pinker just to know which obstacle it has hit because there are 3 obstacles here. So, there is a yellow obstacle, a blue obstacle and a pink obstacle. So, θ_1 = 0, it is hitting in this region which I am marking here.

Then, I increment by 1^0. So, what happens? It goes to the next one. So, this is incremented little bit. Again, we take one round. And, when it takes one round, Link 2 is going to hit this again. So, till here to here, it is hitting again. So, wherever it is hitting, I put a mark with that respective color. Then, I move further and we see that somewhere around here click on 25. So, it is about there. The Link 2 will start hitting the yellow obstacle.

Now, it hits the orange. It hits, it was hitting the pink one. But, now, it will start hitting the yellow one. So, it is somewhere here. So, we increase $\theta_1 = 25^0$ for example. And, we rotate θ_2 completely. So, we see that it starts hitting. Now, it is hitting the yellow obstacle. So, I put a yellow color. So, I do that until I go completely θ_1, 0 to 360 and θ_2, 0 to 360. That means I have covered the full space.

And, what I do get is θ_1 is on this axis, θ_2 is on that axis and this is my C space. Now, is there any correlation between the shape of the object in Cartesian space and the shape of the object in C space? It is very difficult to say. Now, you can also guess. Why we had put colors? Because, otherwise, it will become difficult to figure out which obstacle was which one because the shapes have completely changed.

So, what we are seeing here is that this obstacle is this one. This is this. And, the yellow one is there. And, very funny is that part of the pink one is here also. Now, if you can imagine what actually happened is that it actually formed a torus. And then, we cut the torus and opened it up. That is how we are getting the C space like this. So, if I were to join it now, if you can imagine, let us go here and imagine this.

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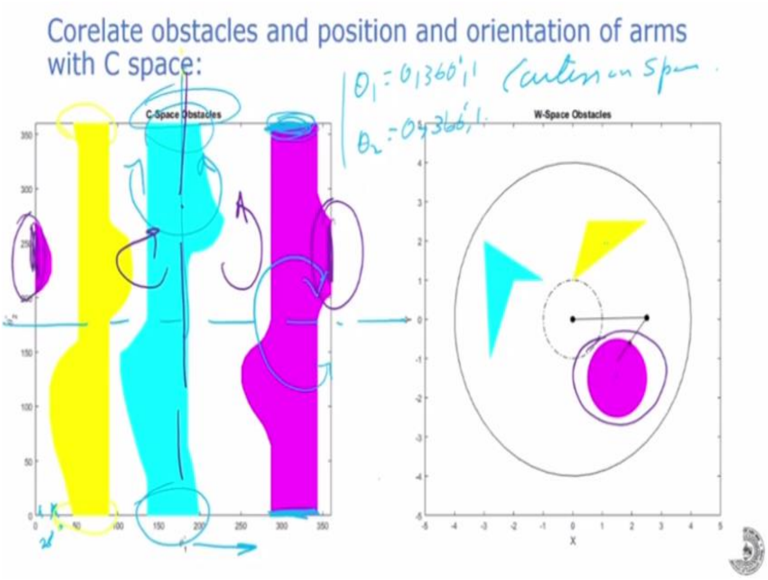

Suppose, I roll it about this, this is my axis, I roll it like this you will get a cylinder. And then, I roll it like this again, you will get a torus. So, please if you are imagining why it is cut here. It is not cut. It is actually continuous. So, when I roll it in this direction, this part will get joined with this part. The blue one, this part will get joined with that one. And, the yellow one, the yellow part, this part will get joined with this part.

So, it is actually not cut. It is continuous. And then, when I turn it when I rotate about the central axis. Now, let me rotate about the central axis about this axis and close it. What will happen is this part will get joined to that part. Why? Because this is a continuous object, it is not a broken object. So, it cannot be that one part is here another part is there and they are not connected.

They are connected because we have cut the torus it has come out like this. So, if you just go back a little bit and try and figure this out, so, this is what we showed.

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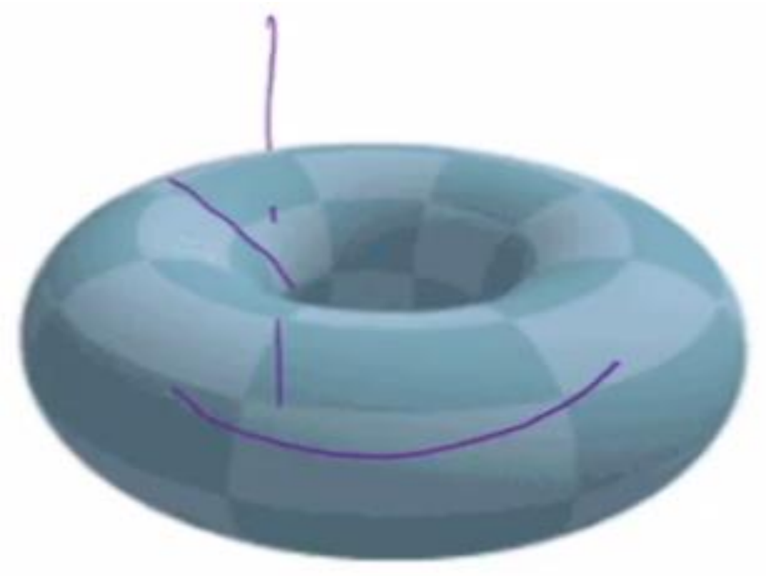

So, the torus, we cut it here and cut it along that axis. It is almost like saying your flat sheet you roll it like this. You roll it and then roll it again about the other axis. So, what will happen? It will become a torus. So, when we cut the torus, what will happen is we are going to get this kind of a C space.

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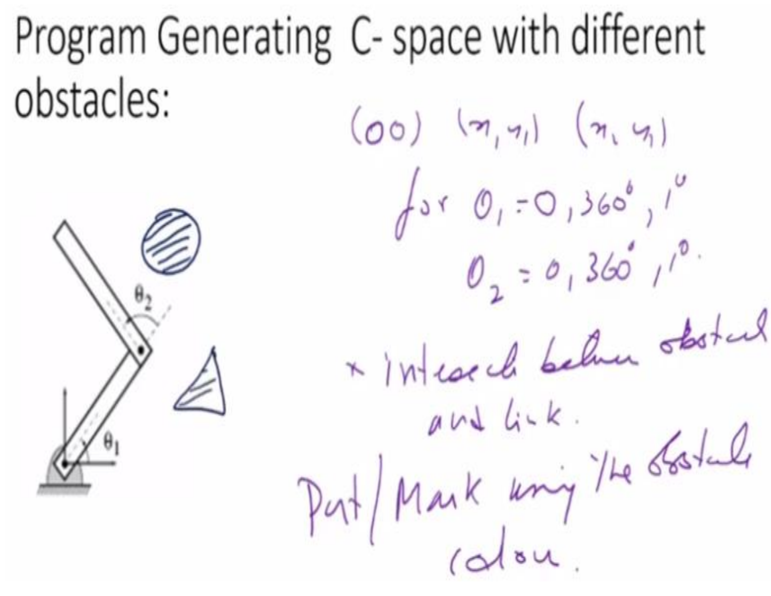

Now, program generating C space with different obstacles. Now, basically, what we do? We find the coordinates of 0 0. 0 0 is known (x_1, y_1) (x_2, y_2). Use 2 for loops, $\theta_1 = 0$ to 360^0 and $\theta_2 = 0$ to 360^0 intervals of 1^0 and plot. Find the intersection between obstacle and link. If there is intersection, put or mark using a color using the obstacle color. So, this will indicate that we are putting the obstacle color.

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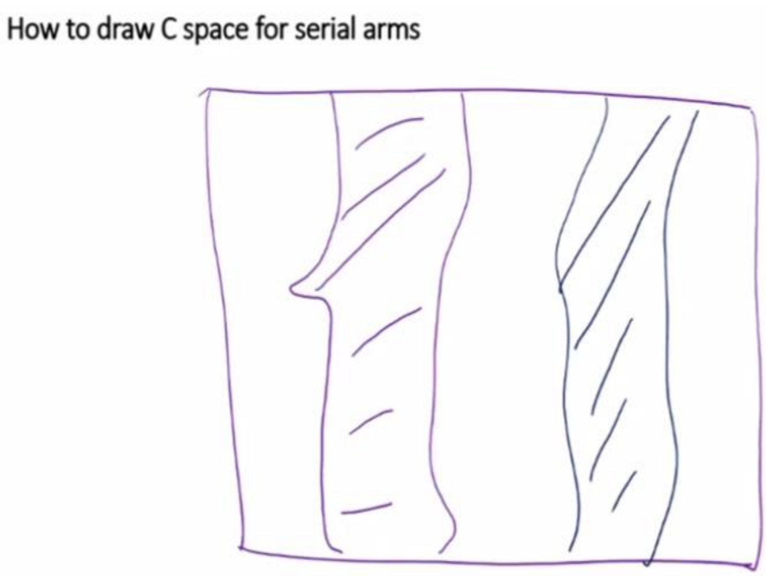

So, what we will end up getting is a torus. And, we cut and open the torus and what we get is say for example a shape like this and a shape like this, just for example. This may be one obstacle. Then, we could have another obstacle which is having some other shape like this, like this. Now, the colors will indicate which obstacle it was so that you are also clear as to which obstacle went there where because geometrically is difficult to correlate and say that okay this is exactly what happened or this obstacle is here.

So, this is how we write a program in MATLAB to find out or to convert from the Cartesian space to the C space. So, if you have obstacle, there could be an obstacle here, any obstacle in fact. Normally, we put different colors and then code it in color. So, now, let us look at the program. We have looked at the program.

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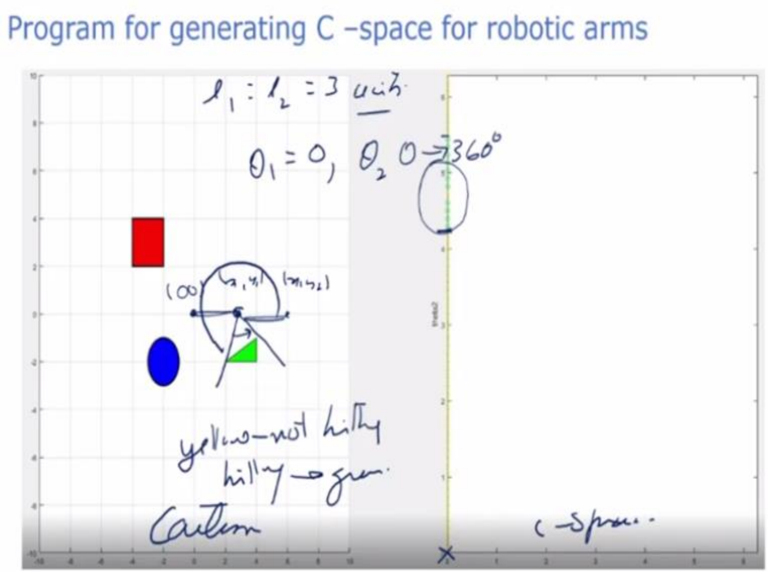

Now, let us look at another example of how the program works. So, remember that in this particular case as shown here, there are 3 obstacles. So, you can see there is a blue color circle and there is a green triangle and a red square and we have a manipulator which is starting from here to here. That is my manipulator. This is my 0 0. So, link lengths $l_1= l_2=3$ units.

Now, how does the program start? It starts by saying that $\theta_1 = 0$ and θ_2 is going from 0 to 360^0. And, it plots (x_1 ,y_1) (x_2 ,y_2) and checks for intersection between any of the obstacles. So, we start here on the right hand side. So, this is my Cartesian space. This is my C space. So, we start at (0, 0) here in my for loop. And then, wherever it is not hitting, it is yellow color. So, yellow color not hitting the obstacle.

So, you can see that it is not hitting not hitting. Only this, you can very well see this. So, when it is rotating like this, only at this angle, it will hit. Till here, it will hit. The second one second link for $\theta_1 = 0$. So, correspondingly, it is hitting from here to here and wherever it is hitting I am putting a green color. So, wherever it is hitting, the links are hitting the obstacle. I put a green color if it is hitting the green obstacle.

So, you can see there is green here. So, I can put any mark. I can put square. I can put rectangle. But, I can put circle. In this case, I am putting a circle.

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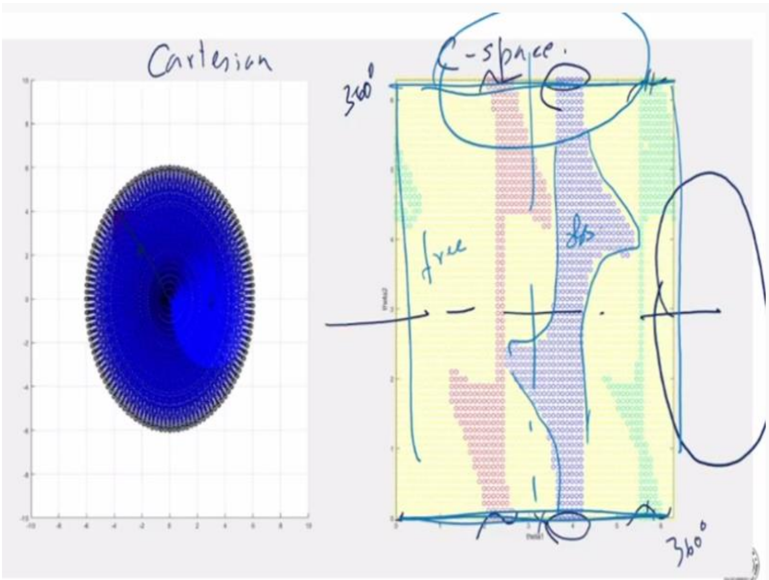

Now, let us see how the program works you can become clearer. So, how this program works is, so, this program is written in MATLAB. You can write in any language you want. So, let us start slowly and see how this is working. So, the first one is we have (0, 0) is here at this point. And, this is the second link. And, the second link is rotated one full circle. So, what we have got is a circle now. Now, wherever it is hitting, it is hitting there.

So, for θ equal to, I think it is a little bit more it is over $5^{0.}$ It starts from 0 1 2 3 4 like that. So, it is hitting here. So, we color it by green. So, this is how the program works. And, this is how we are going from Cartesian space. So, this is my Cartesian space and this is my C space. Now, let us see a little bit more how this is going. Let us increase that $\theta 1$ which has gone a little bit more. You can see how it is moving.

θ_1 is increasing and θ_2 is always taking a circle and wherever it is hitting look at the shape of the green object. So, it has been hitting. And then, now, it is not hitting. So, in this region, you can see that is not hitting. Then, we are going forward. So, θ_1 is increasing. Then, it will start hitting the red one. Now, it started hitting the red one. So, immediately, you can see that here it has started hitting the red one now.

So, θ_2 either of the links can hit or both can hit. Now, as we are going further, you can see that θ_{1} is increasing more and more. θ_1 has become 90. It has become more than 90. So, θ_1 is somewhere here now. θ_1 is about 90 more than 90^0 there now. And, θ_2 is taking this circle. So, it is hitting the red obstacle and we are marking it with red color on my C space. We further progress.

You can see that it is hitting the red one is going further. Now, it is hitting the red one, the, both the links are hitting the red one. So, you can see that the red color has extended right till the top. That means θ_2 completely θ_2 is hitting. It has almost gone like inside the object. You can see that. Here, this is my θ_1. So, θ_2 is almost gone inside the object. And, that is why right from 0 to 360^0. So, this end is 360^0.

Completely, it is hitting. It is completely hitting. So, from 0 degrees to 360, it is hitting that. Now, we are going further what is happening is, we are, now, we are start hitting the blue object. Till here, it is not touching the blue object. Now, we progress a little bit more. Now, it starts hitting the blue object. You can see that. Now, on my C space diagram which is hitting the blue one and the red one.

So, in this location, at this location of θ_1, it is hitting the blue one as well as the red one. So, both are being marked. Now, we move little bit further. It is going completely up to about 270 then comes to 360 and it is complete. Now, if you try and connect the shape of the obstacle and the shape in C space of the obstacle, C obstacle space, what is going to happen is the, there is very little correlation.

It is difficult to say which one is which one which was the circle which is the rectangle and that is the reason we coded by color. So, it is very easy to know that. This is the rectangle. This is the triangle. Now, in terms of continuity, I can rotate I can roll this sheet up. If I roll it, what will happen is this will get roll like that. I hope you can imagine that. So, this side will get joined with that side. This side will get joined with that side.

And, this will get joined with that. So, it is actually continuous. It is not broken. Then, if I rotate the sheet about rotate the cylinder about this axis, now, what will happen is this will close. So, now, what will happen is it will become a torus. So, on the surface of the torus, if you can imagine, this is going to be the obstacles now. This is my obstacle C obstacle. And, this is my free space. So, this is my free space and this is my obstacle space now.

They are connected. So, this side is connected to this side. And, this side is connected to this side because it is a torus actually. So, you can write this program and try and draw it and figure out how it works. So, this is showing it very quickly how we got this complete shape.

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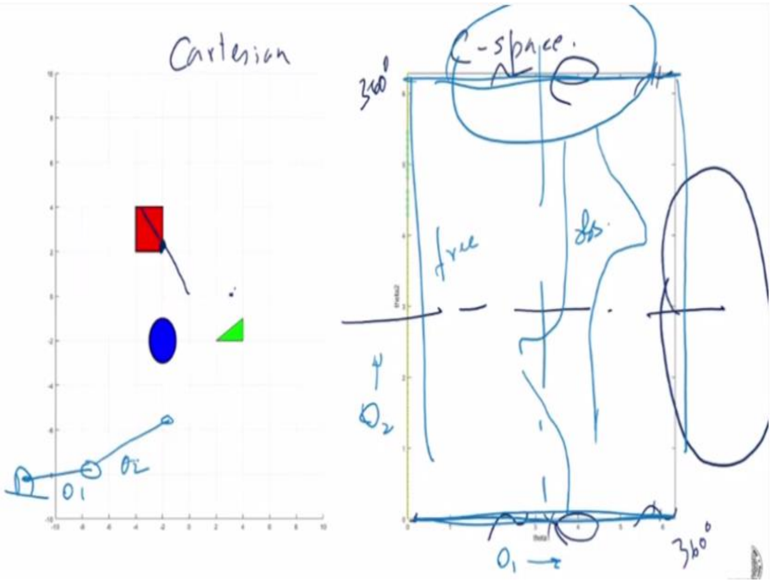

But, now, you will remember something that we talked only about 2 degrees of freedom. We only talked about 2 degrees of freedom. And, we have θ_1 and θ_2 because it was a 2 degree of freedom system. So, it was easy to visualize. So, when we converted, this is my θ_1 and this is my θ_2 because the serial arm actually had 2 degrees of freedom only. That is θ_1 θ_2. And, that is why you could convert and draw it easily like this.

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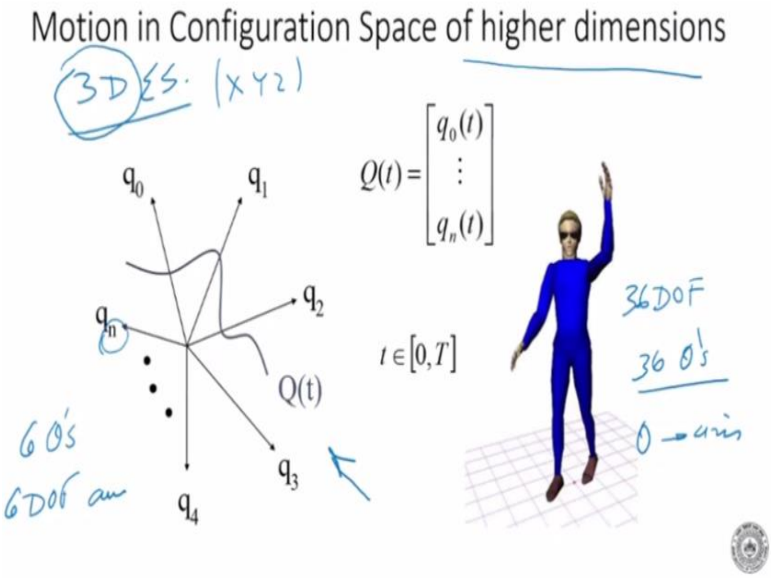

Now, suppose you are going for higher degree of freedom system, suppose I have a serial arm for example which has of 6 degrees of freedom. So, I have a 6 DOF arm. That means there are 6 θs. And now, if you have 6 θs, then you have to draw these 6 θs. How are you going to draw? How are you going to visualize? Now, there is a problem. Why? Because we are living in 3D Euclidean space and it is 3D.

So, we have at most 3 dimensions which is my x y z. So, if I have more than those dimensions, I find it difficult to draw. How do you draw? We do not have the degree of freedom in Euclidean space. For example, this humanoid robot or biped robot character, this can have like 36 degrees of freedom. This means there are 36 θs and each of this θ is an axis. How are you going to draw it? And, how are you going to represent obstacles?

How are you going to do the planning? Say, for example, this one is showing q_1 to q_n where n is a large number. So, this is actually planning in higher dimensional space which is very difficult to visualize also. 2 degree of freedom, we are lucky that 2 degree of freedom was very easy. We could visualize it very easily. And, that is the reason we can explain it very clearly that this is how it looks like. What about higher dimensions?

Higher dimension, there is a problem because you cannot draw it in Euclidean space and it will be very difficult to visualize where it is going. So, that brings us to the next topic that if you realize what we are doing here is we are basically doing the motion planning in our C space not in Cartesian space.

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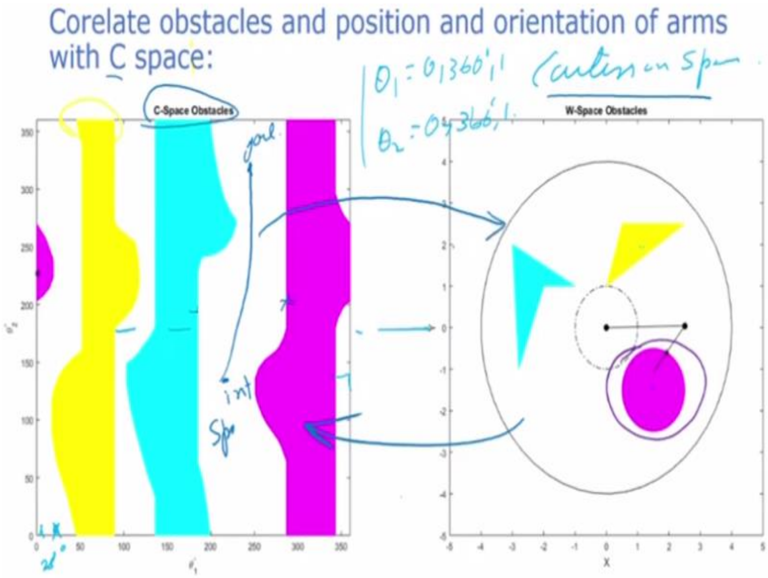

So, this is my Cartesian space. Here, we have Cartesian. This is my C space. This is C space. So, we are doing the motion planning in C space because the robot is a point now. And, everywhere here is free space. So, suppose I say I want to go from here, this is my initial point. I want to go to goal point. So, I can find a path like this because it is a point which is moving. As long as it does not hit the obstacle it is okay. That is a path.

Now, once you find the path, you have to go. So, from here, we came here, Cartesian space to C space. Now, from C space, we are finding the path and we are going back to the real world because the robot is moving in the real world. Isn't it? So, basically, it means that we are moving in between spaces now. Cartesian space to C space and C space back to Cartesian space.

So, this is some kind of mapping which is going on and that is where topology comes into picture that we have to look at the topology of spaces.

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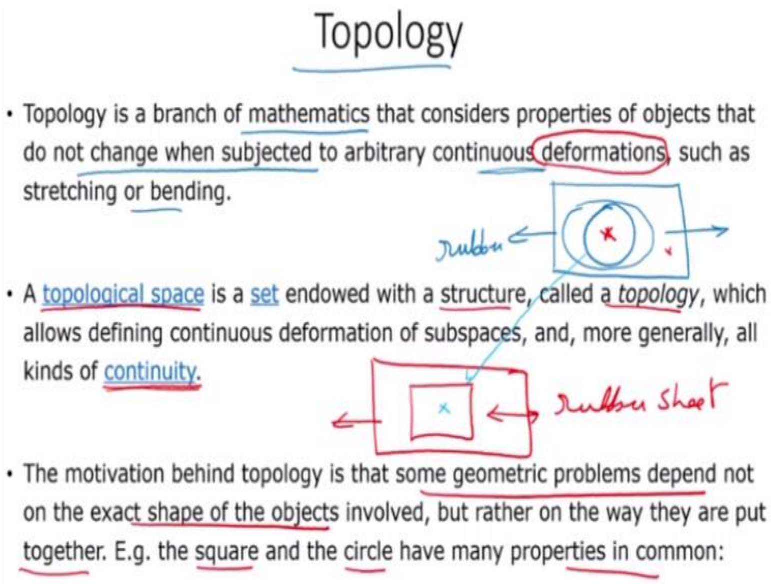

So, topology is a branch of mathematics that considers properties of objects that do not change when subjected to arbitrary continuous deformations such as stretching or bending. For example, if you have a rubber sheet, let us take this example. You have a rubber sheet. And I draw a circle in that. Suppose I stretch it, what is going to happen? The circle is going to become like this, probably ellipse.

But, the properties of the object do not change when subjected to continuous deformations. What property of this object? First of all, it is a closed curve so that property will remain a closed curve. So, circle can become an ellipse but it is still a closed curve. It cannot become a straight line for example. Points which are inside the circle will remain inside the circle. They cannot come outside.

So, the properties remain irrespective of the deformations that they are subjected to. And, this is basically what we study in topology. So, a topological space is a set endowed with a structure called topology which allows defining continuous deformations of spaces and more generally all kinds of continuity. So, topological space is a set endowed with a structure and this is basically called the topology of the space.

So, the motivation behind topology is that some geometrical properties or problems, like path planning is one of them depend not on the exact shape of the objects involved but rather on the way they are put together, example, the square in the circle have the same properties in common. For example, if I take this rubber sheet again and I draw a square and I subject it to deformations. I pull the rubber sheet. What will happen is a square is a closed curve.

So, it will remain a closed curve. Points which are inside will remain inside. They cannot come outside. So, in a sense, a square and a circle are the same as far as topology is considered. They are same. They are topologically similar. Now, this is useful in our case of path planning, we will see as we go along now.

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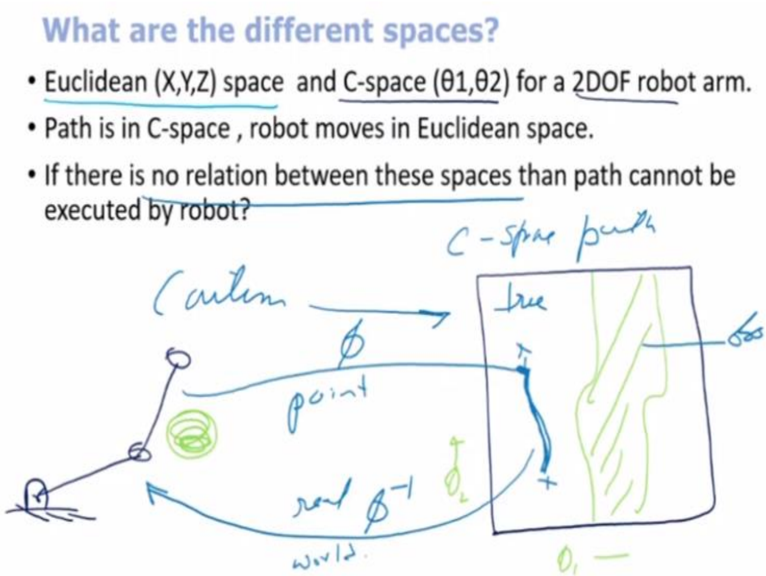

Now, what are the different spaces? We are talking about the topology of space. Now, Euclidean space is X Y Z 3 dimensional Euclidean space we are familiar with. We also call it Cartesian space. C space is θ_1 θ_2 for the 2 degree of freedom that we talked about. Now, the path is planned in C space. So, I have a link like this 2 degrees of freedom link and I drew the equivalent and with an obstacle maybe. So, there is an obstacle here.

So, I drew the corresponding obstacle space here. That is my obstacle. This is my θ_1. That is my θ_2. Now, I want to find the path from one point to another point. From this point to this point, I found a path. So, from Cartesian space, I went to C space. I found the path. Why? The robot has become a point. So, this has become a point now. And, this is my free space. This is my obstacle space.

So, I found a path now. Now, after having found the path in Cartesian space, I have to move back to the real world. So, that means I am going in one direction. And then, I am coming back again. So, this is my forward transform. This is my inverse transform between the spaces. So, basically I am going with one transform in the forward direction and finding the path and then coming backwards.

Now, if the inverse does not exist then again it is of no use because the path cannot be executed in the real world. So, this is what makes it that if there is no relation between these spaces, then the path cannot be executed by the robot. You cannot, you can go in the forward direction. You can come in the backward direction.

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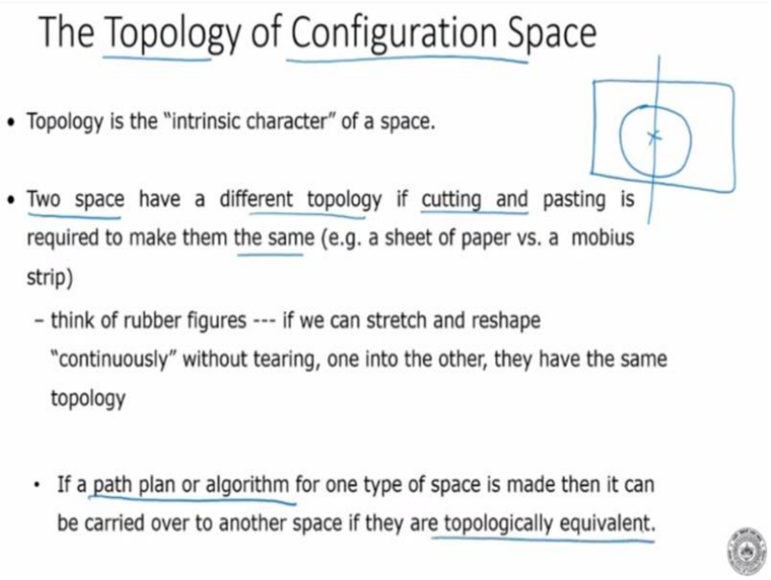

So, this, these are things which are studied in topology of configuration space. Now, topology is the intrinsic character of a space is something we have seen that topologically a circle and a square are the same because both of them have the same properties. They are closed curves. Points which are inside the curve cannot come outside when subjected to deformations. Now, 2 spaces have different topology if cutting and pasting is required to make them same.

Now, if you have to cut, if you want to take out the, here, suppose I have a circle and have points inside if I want to take this point outside I have to cut actually, then they can come out. So, 2 spaces are different topologically or they have different topology if cutting and pasting is required to make them same. Now, a path plan or an algorithm for one type of space is made then it can be carried over to another space if they are topologically equivalent. Otherwise, they cannot be transferred between spaces.

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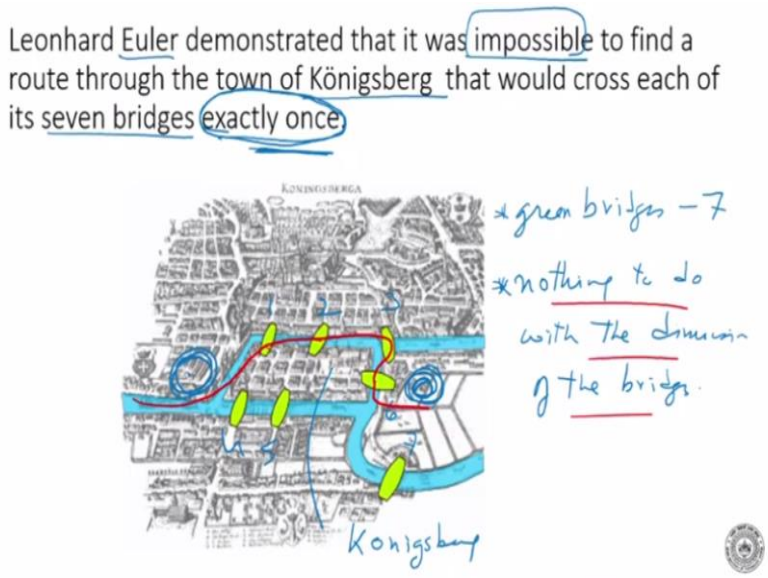

Now, where did this study work come from? Basically, it was started off by Euler and he demonstrated that it was impossible to find a route or a path through the town of Konigsberg that would cross each of the 7 bridges exactly once. So, this is my town of Konigsberg. Now, if you want to go from this side to that side, there are 7 bridges. So, this is 1 2 3 4 5 6 7 bridges. Now, he said that it is impossible to find a path to the town of Konigsberg that would cross each of the 7 bridges exactly once.

So, the catch is exactly once. So, you cannot go from one side to the other side by crossing each of the 7 bridges exactly once. Either you cross less number of times or you cross more than once. Now, there are these green bridges you can see. There are 7 bridges. Now, this problem has nothing to do with the dimensions of the bridges. This is something that you should note very clearly that there are 7 bridges here.

And, this problem has nothing to do with the dimension of the bridges. The bridge is small or big it does not matter. And, this is essentially what topology is talking about that this has nothing to do with the dimension of the bridges. It essentially has to do with the topology of the space we are talking about. So, if you want to go from one side to the other side for example, you go like this.

Start from here, go like this, like this, like this, like this and go to the other side. You have crossed only 1 2 3 4 bridges. But, you cannot cross all the 7 bridges at least once and go from one sector on the side. And, this has nothing to do with the dimension of the bridges. This is something that is important here that this is where the study of topology of spaces came up.

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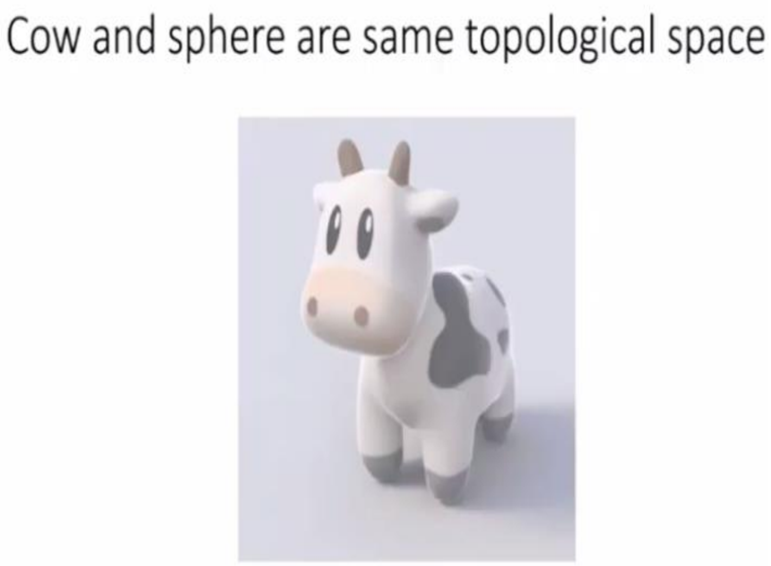

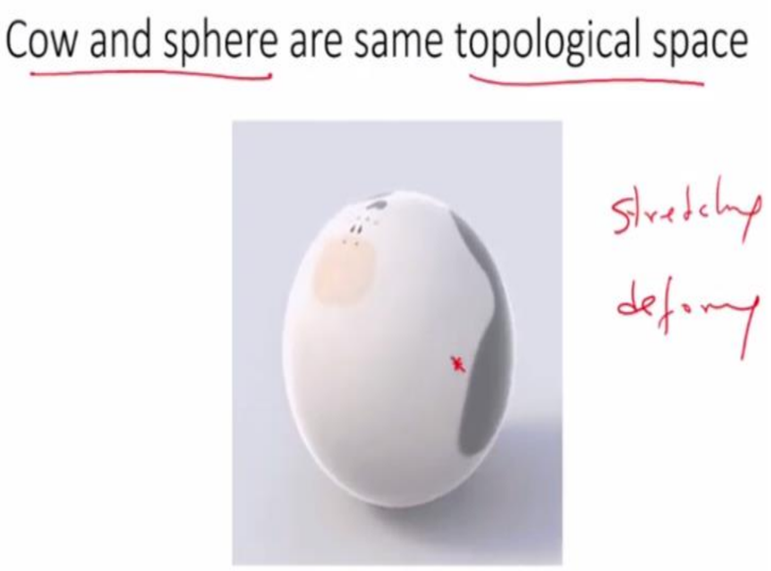

Now, if you look at in terms of topology the cow and the sphere they are the same space because simply by deforming we can go from one space to another simply by stretching or deforming. We are going from one to the other. Now, any object with any point which is on the surface of the cow is still on the surface of the sphere. It cannot go inside the sphere.

Similarly, any point which is inside the sphere cannot come outside the sphere, similarly, that for the cow. Any point which is inside the cow cannot come outside. So, topologically, this is the same. They are the same topological space something like the circle and square we talked about.

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Now, there are many examples, the donut and the mug. So, this is a donut something like a torus and this mug. They are topologically the same space because simply by deforming we are going from one space to the other space. And hence, they are topologically the same. So, it is very easy to figure out that any point which is inside the donut will remain inside the donut.

Any point which is on the surface of the donut will remain on the surface of the cup also. It cannot go inside.

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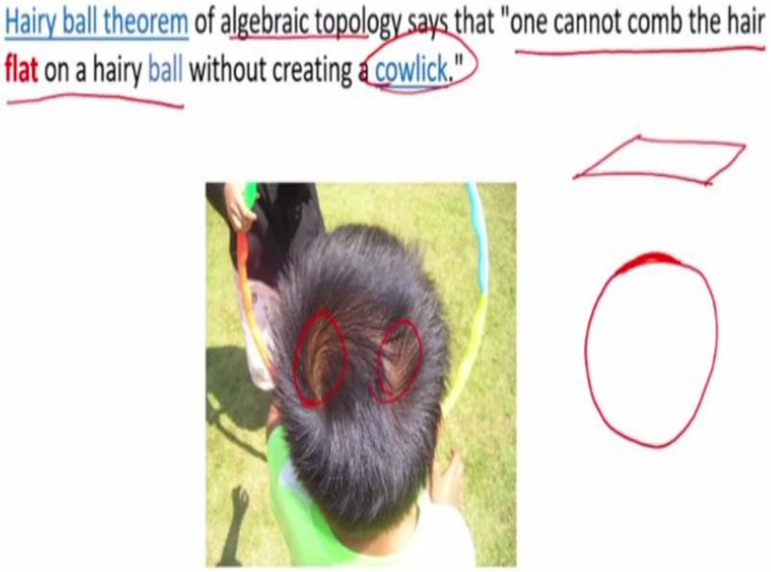

So, this is something that is interesting in terms of topology. There is also an interesting theorem called the hairy ball theorem of algebraic topology which says that one cannot comb the hair flat on a hairy ball without creating a cowlick. Now, you know that when you are combing your hair what happens is you are trying to make it flat. This is a flat surface. But your head is circular is spherical. So, you are trying to make this flat surface sit on a sphere.

So, you can do that only locally. But, you cannot do it globally. We will come to local and global as we go along. Now, if you try to do that what would happen is you will end up with a space like this which is called a cowlick either 1 or 2 of them. So, this is supposed to be the hairy ball of algebraic topology.

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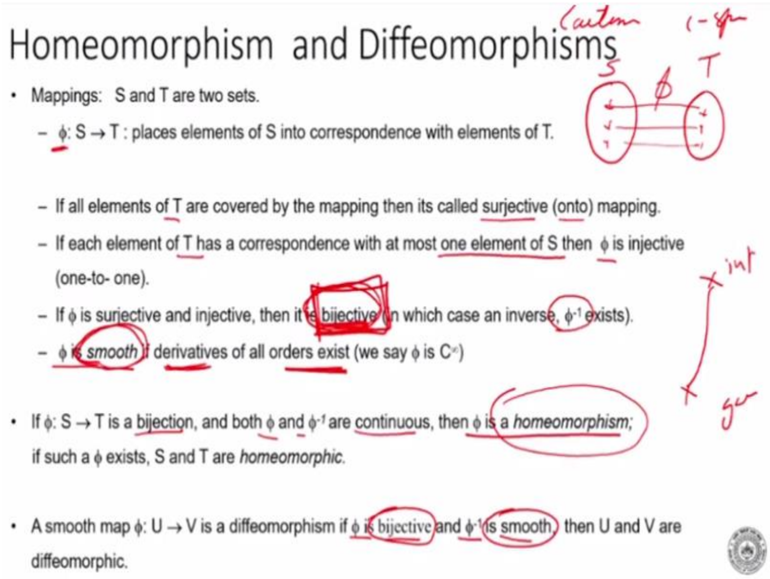

Now, we have to look more, further into this in terms of the mapping actually. So, let us say, we have 2 spaces. We have S and we have T. These are 2 spaces. We can talk about them as Cartesian space and C space. Now, there is a mapping Φ which is going taking elements from C. These are different elements of C. 2 elements of T. Maybe this is my mapping which is Φ.

So, this mapping Φ places elements of S into correspondence with elements of T. S and T are 2 different sets. Now, if all elements of T are covered by the mapping then it is called a surjective mapping or onto mapping. Now, if each element of T has correspondence with at most 1 element of S then Φ is injective or one is to one mapping. Now, if Φ is surjective and injective then it is bijective in which case an inverse exists.

So, we talked about going from one space to another space and then coming back. That if you find the path in cut in configuration space you need to come back to the Cartesian space. Otherwise, it is of no use which means the inverse has to exist. And, the inverse will exist if this Φ mapping is bijective. That means it is surjective. That means all the elements are covered and there is a one is to one correspondence.

So, please note the importance of this bijective mapping. The other point is Φ is smooth if derivatives of all orders exist. We said that we have a path from initial point to goal point. Here, this is my point. Now, this path has to be continuous. So, continuous means it is smooth if derivatives of all orders exist. So, it has to be bijective and it should be smooth. So, Φ: S to T is a bijection and both are continuous. Then, Φ is a homeomorphism.

But, one step ahead of that a smooth map U to V is diffeomorphic if Φ is bijective and Φ inverse is smooth. This is even more important. Then, U and V are diffeomorphic. So, we are looking at 2 terms, what is homeomorphism and diffeomorphism? And, it basically has to do with bijective mapping and which is smooth. So, if it is bijective, inverse exists. If it is smooth, it is continuous. And, both of these are important when we are going from one space to another space.

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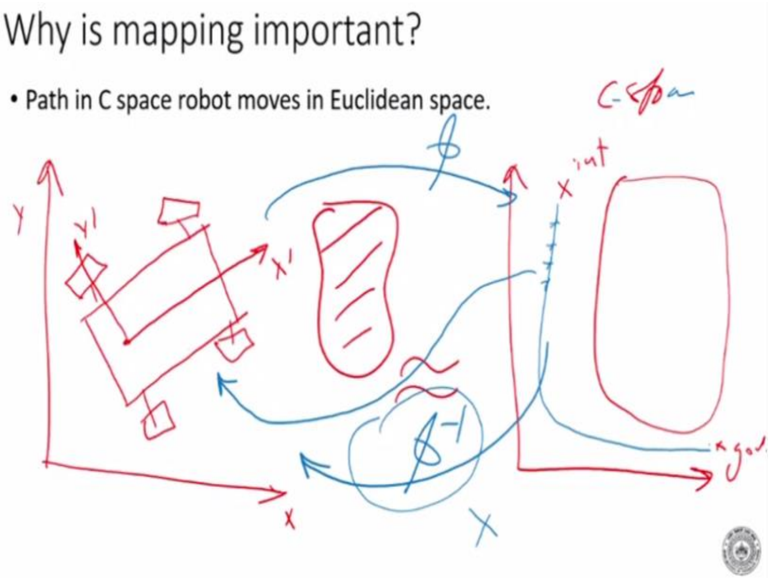

Now, why is mapping important? Now, it is very easy to understand why mapping is important because in the case of the serial arm, I have just given example of the mobile robot also. So, we have a mobile robot which is here. And, we have there is a steering wheel. Let us say, there is a steering wheel here. So, these are my steering wheels. This is my steering wheel at a particular angle. And, this is my local axis $x'y'$.

And, this is my x y. Now, there is an obstacle here, let us say. I want to avoid this obstacle. So, basically, what we are doing is we are going from here to the C space which is my C space here. And, we have a particular shape. And then, we are finding a path from the initial points to the goal point maybe this is my goal. So, I find a path which is taking me from the initial point to the goal point like this.

Now, once we have found the path in the C space, we have to come back to the real world now. That means at every point here I have to find out what are the velocities which will actually take me that from doing inverse kinematics. So, this is the forward map. And, this is the inverse map. And, both have to exist. And, the path has to be continuous. We will see why.

So, this is the importance of what we call a diffeomorphism which is bijective and Φ inverse is smooth when we are mapping between 2 spaces. Now, you can also appreciate that if this Φ does not exist Φ inverse does not exist then there is actually no point, the, because the path cannot be transferred to the real world. That is one. The second is if the spaces are same for this mobile robot.

If the spaces are same in which it is moving in which we are talking about the path then we can transfer the path between different spaces also. Provided this mapping exists.

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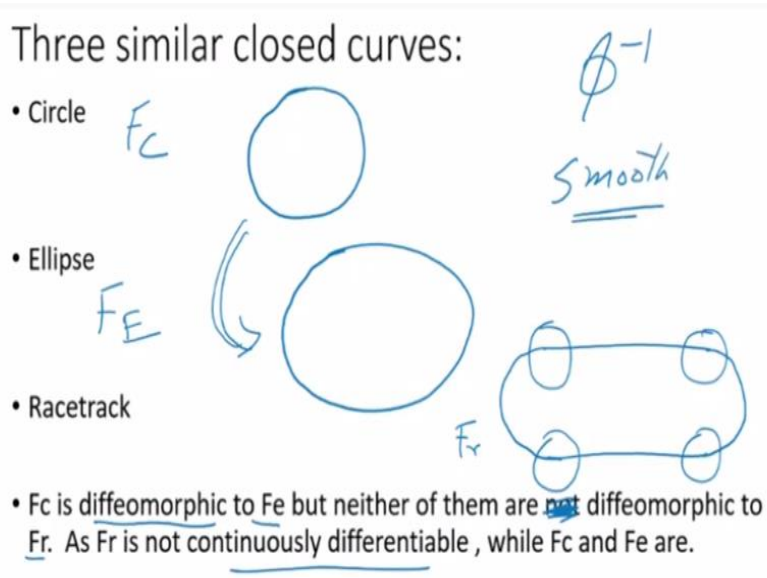

So, now, for example, cases where the mapping may not exist the inverse may not exist or it need not be continuous is for example you have a circle. So, we can call this F_c. So, it is a circle for example. The ends have to meet. So, this is a circle. And, this is an ellipse. Now, in the ellipse, let us call it F_e. And, we have a race track. Race track is 2 straight lines joined by 2 semicircular points.

Now, F_c is diffeomorphic to F_e. F $_c$ is diffeomorphic to this one. But, neither of them are diffeomorphic to F_r, the racetrack which is F_r. Why? Because F_r is not continuously differentiable, while F_c and F_e are, that means at these points they are not continuously differentiable where the semicircle is meeting the straight line. So, and hence, it is not diffeomorphic.

So, F_c and F_e are not diffeomorphic to F_c because F_r is not continuously differentiable. So, we need to look at this one and smooth. So, for smooth, it has to be continuously differentiable because you need the velocities.

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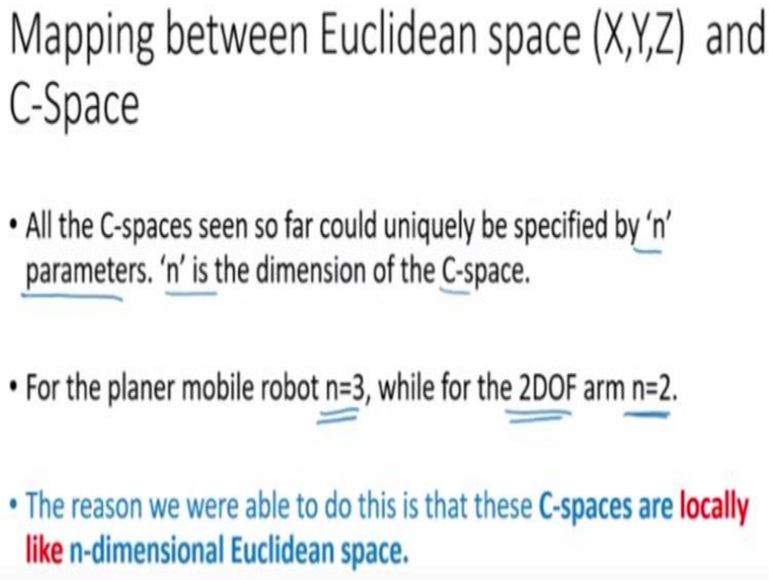

Now, mapping between Euclidean space and C space, all the C spaces seen so far could uniquely be specified by n parameters. n is the dimension of the C space. For the planar mobile robot, n = 3. n is the dimension of the configuration space. While for 2 degree of freedom arm, n =2. That we have seen. Now, the reason we are able to do this is that these C spaces are locally like n dimensional Euclidean spaces.

Now, in this case, we are able to map. Why are we able to map? Because we are able to map, because, the C spaces are locally like n dimensional Euclidean spaces. That is the reason why we are able to map. Now, to be able to map to go to the Cartesian space from Cartesian space to C space and then to come back to the Cartesian space again we need to have a mapping which has the inverse and that also has that is also smooth. Now, let us see a little bit further into this mapping business.

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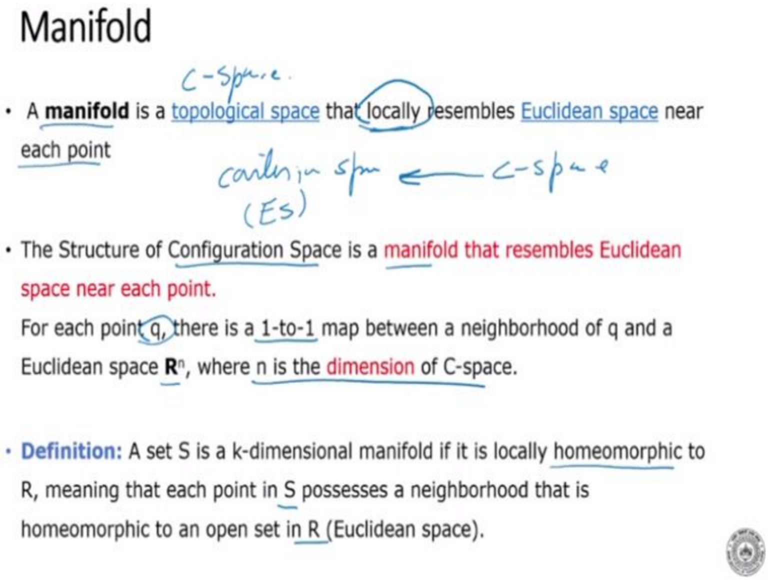

So, next, we have looked at this terms. These are some terms that you need to understand very clearly. One is this question of topological space. We just talked about topological space, the properties that is bijection and being smooth. Now, we have another term which is called a manifold. So, a manifold is a topological space that locally resembles Euclidean space near each point.

It is locally not globally. What is this local and global is that all throughout it may not represent but at local points it does represent the Euclidean space. For example, Cartesian space and a C space in configuration space. So, this configuration space Cartesian space is Euclidean space 3D Euclidean space. Now, C space resembles Cartesian space locally. And, that is one of the reasons we were able to do this mapping because locally it does represent and configuration space is a manifold.

So, manifold is a topological space. For example, in our case, this C space that locally resembles Euclidean space near each point. The structure of configuration space is a manifold that resembles Euclidean space near each point. So, what this basically also means that at each point q, there is a one to one map between a neighbourhood of q and a Euclidean space R raised to n where n is the dimension of the C space. Now, this is a definition.

A set k is a k -dimensional manifold. If it is locally homeomorphic to R meaning that each point in S possesses a neighbourhood that is homeomorphic to an open set in R. This is may be a little confusing but as we go along it will become clear. So, it basically means that you can map from C space to Cartesian space because both these spaces the configuration space resembles Cartesian space locally. And, configuration space is a manifold. That is the reason why it is called a manifold.

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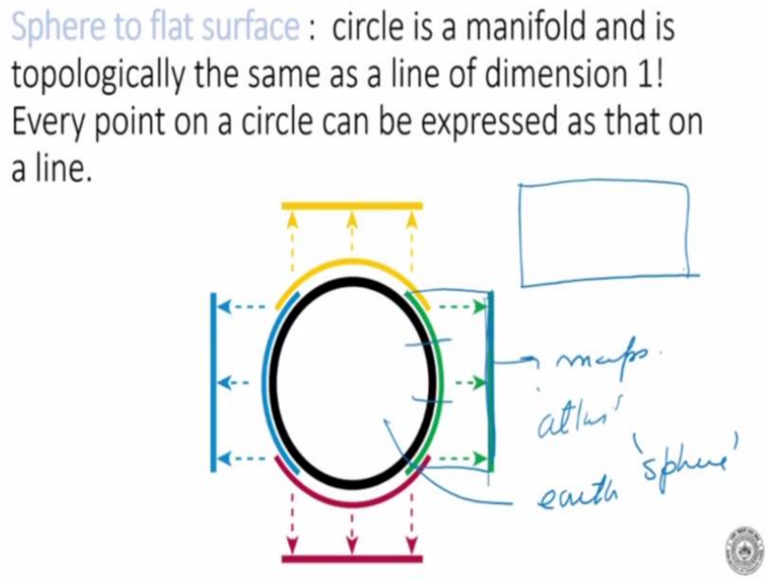

Now, what is this local resembling space locally? So, if you look at this example, it becomes very clear that what we are talking about. So, let us look at this case where we have a sphere and we have a flat surface. This is similar to what we are talking about in the case of the combing hair. So, when we are combing hair we saw that we are trying to make the hair sit flatly on a sphere.

Here, it is the reverse where we are trying to represent a sphere by means of flat surfaces where can you imagine this. Say, for example, this is the earth and these are maps. Now, you see that maps are flat. A map is a flat surface like this whereas the earth is a sphere. So, we are trying to represent the sphere by means of a map. So, locally, a sphere can be represented by as a flat surface locally not globally.

So, this is an example of, saying, that this spherical surface can be represented by a flat surface but locally only. Only locally in this region and that is how we have these maps and you are familiar with maps.

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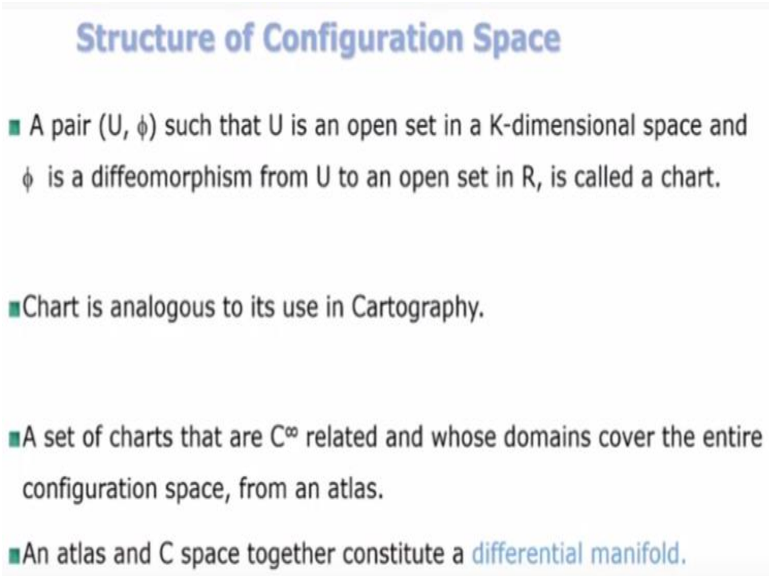

Now, let us talk a little bit more about structure of configuration space.

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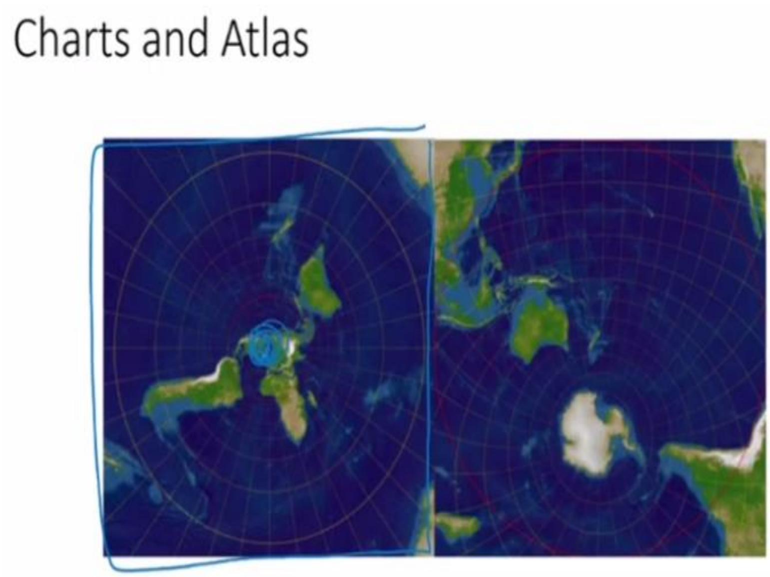

And then, we will move on. So, this is what I was talking about. So, when we are talking about maps, maps represent a spherical earth. But, maps are flat. For example, this one you can see is flat. And, how is it that we are representing a flat spherical surface on a flat surface? Because, locally, they are, you can map them. So, this is a manifold in a way. Now, let us look a little bit more.

So, structure of configuration space, a pair (U, Φ) such that U is an open set in a k dimensional space and Φ is a diffeomorphism from U to an open set R is called a chart. Now, chart is analogous to its use in cartography or maps. This, you would have heard about maps. You would have heard of atlas. You might have used this in schools, maps and atlas. So, essentially, a pair U Φ, Φ is the mapping such that U is an open set in k dimensional space and Φ is a diffeomorphism from U to an open set R is called a chart.

And a set of charts that are C infinity related and whose domains cover the entire configuration space from an atlas. So, an atlas and C space together constitute a differentiable manifold. Just to know that configuration space is a manifold because locally it is similar to Euclidean space. That you understand. Now, because it is locally we can make it represent the Cartesian space or Euclidean space.

That is why you can represent the earth in the form of maps. And, if you put all these maps together basically we formed an atlas. That is basically the definition of a chart. Each one is a chart and putting them all together, you get an atlas.

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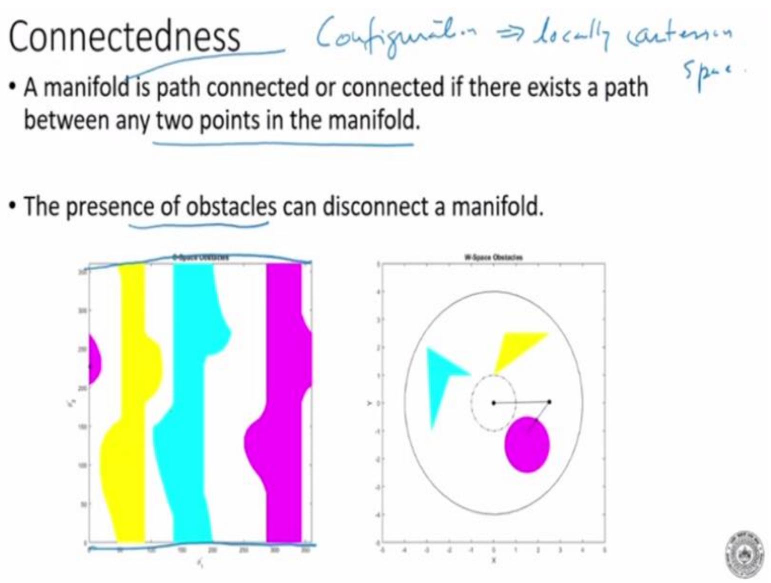

Now, connectedness is something that we have to be very careful here. So, a manifold is path connected or connected if there exists a path between any 2 points in the manifold. What is a manifold? Manifold just remember is a configuration space. Why is it a manifold? Because it is locally like Cartesian space, that is why, it is called a manifold. Now, a manifold is path connected or connected if there exists a path between 2 points in the manifold.

Now, the presence of obstacles can disconnect the manifold. Now, let us look at this example we were talking about some time back. Now, we have the 3 obstacles which are shown here. Now, this side is connected to this side. That means this part is connected. And, this part is connected. This part is also connected. However, this part is not connected to that part. They are not connected.

So, when we are making the path, the path has to be connected. Of course, it cannot be a disconnected path. It should be continuous and connected. So, you cannot join 2 parts in a manifold which are not connected. So, let us look at the case of path planning for example.

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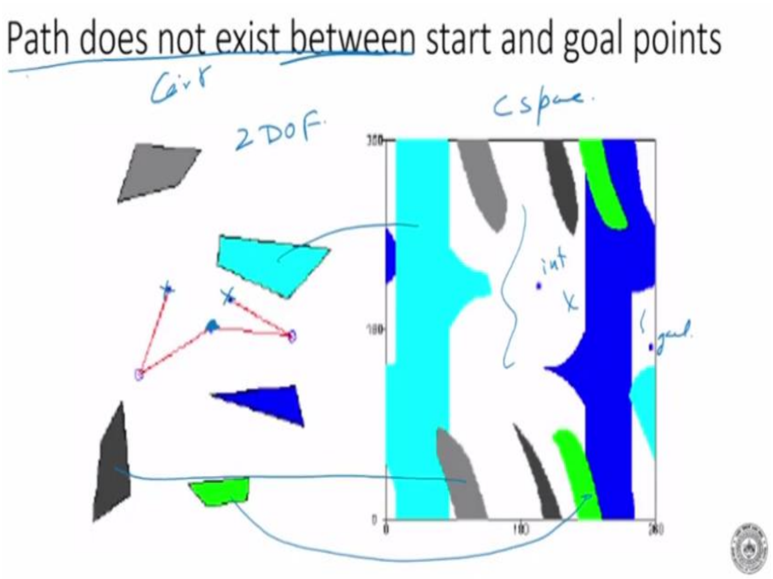

Let us look at this example here. Now, in this case, this is an example of 2 link manipulator again, 2 DOF manipulator. Please look at this. There is one link here another link there. Now, I want to go from this point to that point. Each of the obstacles are shown there. This is my C space. This is my Cartesian space. So, this one is here. This one is there. And, this one is here and the green one is there.

Now, I have an initial point here. I have a goal point here. Now, this is not connected because they are not connected. You can see that. So, this part is not connected to that side. So, no path exists. So, this says that part does not exist because the manifold parts are not connected.

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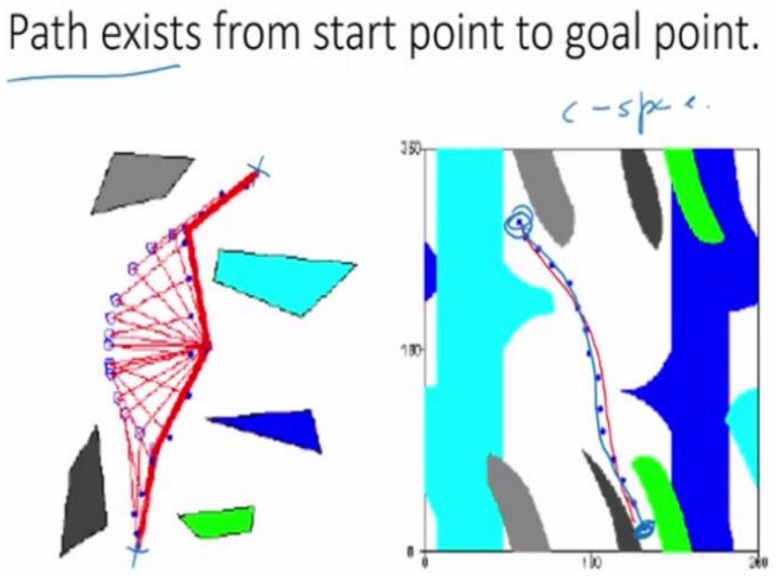

Second is, in this case, we have again 2 degree of freedom manipulator. This is one degree of freedom. This is one link another link. Now, in this case, I want to go from this point to this point. This is my C space. So, this is my start point. That is my end point. But, in this case, I can go because this is connected. All of them are in the white space in free space. So, this is showing that a path exists from start point to the goal point.

Now, you can understand why we need to do the planning in C space because, otherwise, it will be extremely difficult in this particular case to even visualize, to try, to figure out doing inverse kinematics whether it can go from one point from the initial to the goal point.

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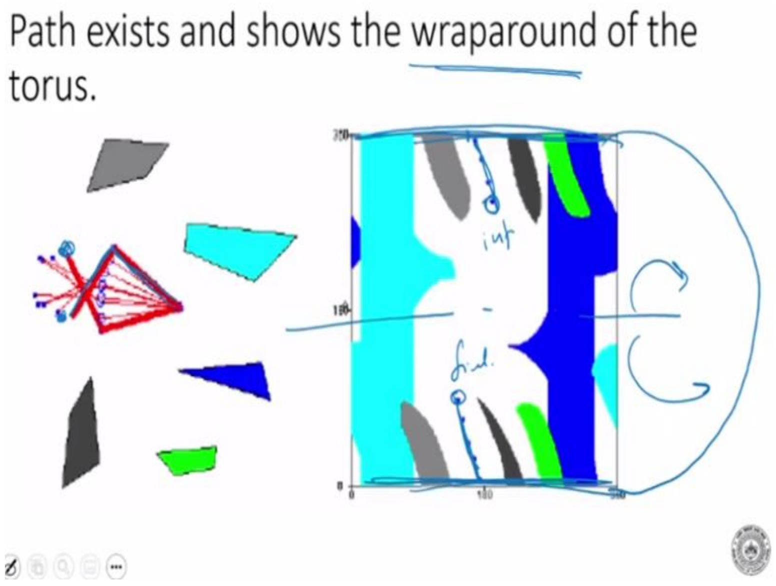

Now, let us look at this case. Now, in this case, this is my 2 link manipulator again. It has to go from this point to this point. Now, in this case, it is from here to there. This is an initial point. This is final point or goal point. Now, because, this side and this side are connected, this part when we connect it, when I roll it about this axis like this and like this, when I roll it, this will join like.

So, this side and this side are joined which means it can go like this. Come out from there and go there because it is path connected. So, you can go from here to here because it is path connected because of this wraparound of the torus. So, this is something that we looked at in today’s class. We talked about the ways of drawing the Cartesian space.

So, today’s class we talked about how to draw the Cartesian space for a serial arm robot. And, after you make the Cartesian space for the serial arm robot if you have to find a path it has to be a path connected which means that in the manifold there should not be breaks due to the obstacle. And, the initial point and the final point should be connected in the manifold itself. We also talked about topology of configuration space.

We said that configuration space is a manifold because locally it is like Cartesian space. So, we will stop here today. And, in the next class, we will continue with our discussion of how to find paths which are going from one point to another point. So, let us stop today, thank you.