Computational Fluid Dynamics

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Lecture No # 17

Important Consequences of Discretization of Time Dependent Diffusion Type Problems (Contd.) and Stability Analysis

We have been discussing on various issues, related to the consistency and stability of finite difference discretization schemes for time dependent diffusion type of problems.

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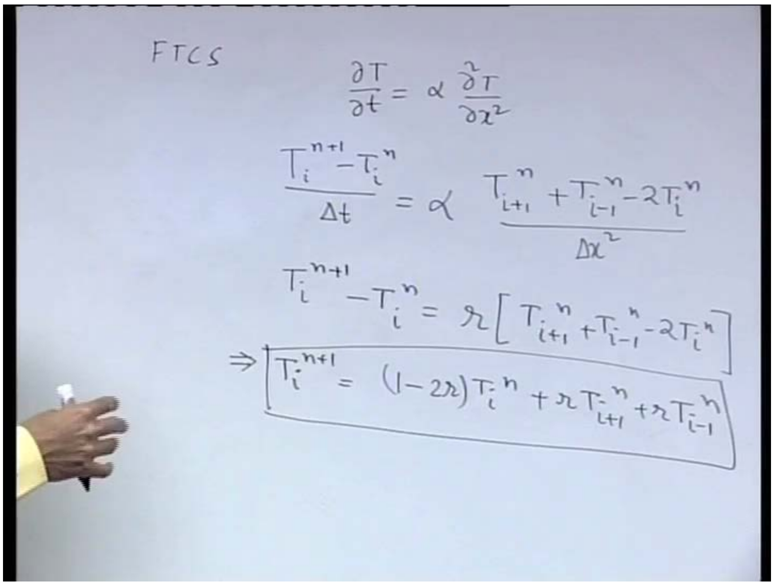

So, we were dealing with the F T C S scheme, remember that the governing differential equation that we were solving is like this. So, in the F T C S scheme we discretized the time in forward difference scheme. So, T i n plus 1 minus T i n by delta t is equal to and this term is in the central difference scheme.

We have already discussed about the consistency of this particular scheme, and we have shown that this scheme is consistent. Next what we will try to do is we will try to assess the stability of the scheme. So, as we have all ready discussed that what do we mean by stability, that there is some perturbation to the numerical scheme, because of may be round of errors and we want to see whether this perturbations get amplified with time, if they are getting amplified with time, then we say that it is an unstable scheme and if the perturbations die down, then we say that it is a stable scheme.

So, to assess whether it is stable or not let us work out the following steps. So, first let us write the discretization equation in a bit more compact form. So, this is the discretized equation. So, we have written T i n plus 1 in terms of T i n and T i plus 1 n and T i minus 1 n. So, it is what type of scheme if you compare these, see there is an interesting analogy of this with the finite volume time discretization, this is what explicit, implicit or crank Nicolson. You are able to express T i n plus 1 explicitly as a function of the temperature at all other neighboring point at the previous time steps. So, it is an explicit scheme. In fact, the equation of discretization of the explicit scheme that we have got through the finite volume method is exactly the same as this one if you do it and there is a reason behind these, see the methods are different in philosophy, but essentially they consider some kind of profile assumption. So, it is a linear profile assumption that if you if you consider that essentially gives rise to this differencing scheme and the profile assumption in the finite volume method for the explicit scheme itself suggest that the time differencing is also corresponding to this one that is a forward differencing in time and hence you get the same expression, what you get at the end after discretization of the equation using the finite volume method.

We will not discuss in that line any further just to keep in mind that for certain choices of the profile, what you are basically doing you are basically taking some polynomial approximation in the profile, if you think profile is a polynomial say, linear profile means it is a polynomial profile of order one. So, that means, that is also a way of choosing the terms in the form of or choosing the profile by considering a Taylor series by neglecting many higher order terms beyond the linear term. So, making a profile assumption has some linkage with considering a Taylor series where in which you neglect many high order terms and retain may be up to a linear term and that is where the correspondence between different methods, they creep up. Now, we will take it up from here, we have this as the equation that the discretized solution should satisfy. Now, let us say that you get an approximate solution; the approximate solution also should satisfy this difference equation.

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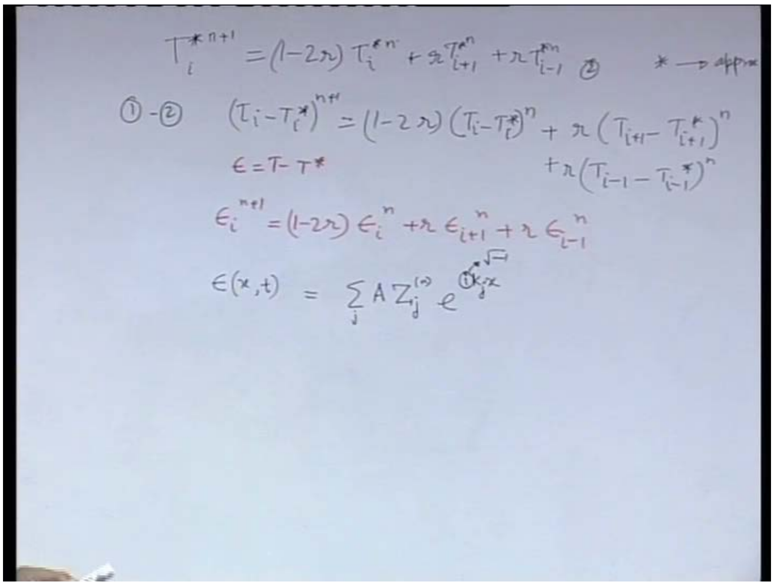

So, if you use asterix to indicate that it is an approximate solution, then we can say we can say that this is true where superscript asterix will mean that it is approximate solution.

Now, if you subtract the original discretization equation or the approximate solution from the original discretization equation, that is if this is 1 and this is 2 and if you write 1 minus 2, then what you get T i minus T i star both n plus 1 is equal to 1 minus 2 r T i minus T i star both n plus r T i plus 1 minus T i plus 1 star n and plus r T i minus 1 minus T i minus 1 star n, keep in mind this n are not indices these are just superscripts n n plus 1, n minus 1 like that . So, in each of these terms you have T minus T star that is the actual solution minus the approximate solution which is nothing but the error. So, we can write if we give a symbol epsilon as T minus T star. So, we can write epsilon i n plus 1 is equal to 1 minus 2 r epsilon i n plus r epsilon i plus 1 n plus r epsilon i minus 1 n. So, this is an equation that relates the errors at each grid point and corresponding to each time level. We can see that the errors satisfy exactly the same equation as the discretization equation, just as if you have replaced the value of the variable with the corresponding error and this has been possible, because of the linearity of the equation. So, you could just subtract the exact from the approximate to get a relation which is of the same form as that satisfied by either of the exact or the approximate 1.

Now, we have to see that whether this errors so, for stability analysis we require errors their estimates and whether they grow with time or not. So, we have to keep in mind that this epsilon is a function of x and T and we can write this in terms of a Fourier series. So, what we can do we can write this as sum summation of a, into some factor zee at the nth level, into e to the power of where k is called as wave number, we will discuss in an elaborate way that why do we choose such a form, there could be different types of forms which are possible, but there is something special or something advantageous of this form which we will try to utilize. So, this j is just a number on which this summation is there. So, it is just like a Fourier series where you can have the j ranging from minus infinity to plus infinity.

Now, the question is that if you choose a particular value of j, then what is the particular form of this one? This i remembers this i is square root of minus 1. Now, what can be the particular form of this zee, remember that we have considered it to be a separable function of position and time. So, here we are using a function or position which is this exponential form and this is a function of time.

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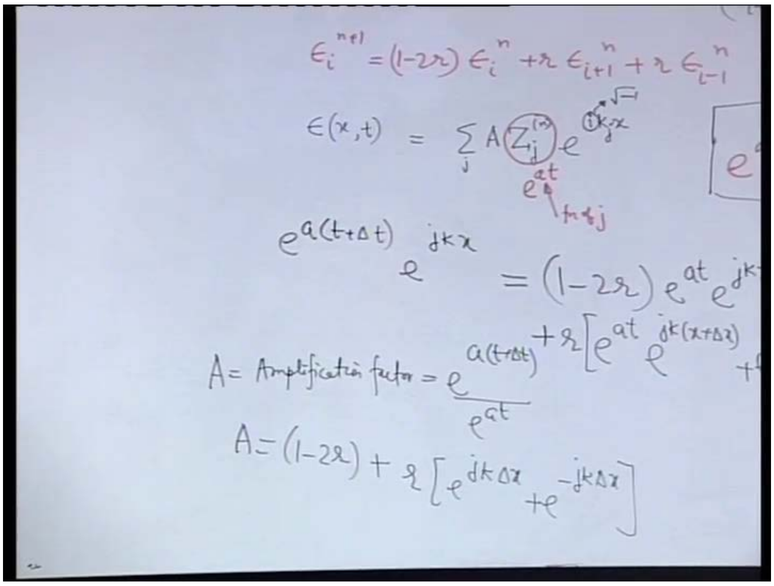

So, a convenient form of this is to choose it as, e to the power at where a is a function of j. So, depending on value of j you can have different a and e to the power a t, why e to the power a t? The reason is that exponential behavior in a way it can help you to assess whether there is a growth or there is a decay. So, it depends on what how e to the power a at T plus delta t behaves as compared to e to the power a t. So, if e to the power a at T plus delta t is greater than e to the power a t. So, we say that it is an amplification, on the other hand if e to the power a into T plus delta t is less than e to the power a at T then we say that it is a decay. So, this allows you to assess whether there is any exponential growth or decay. Now, here when you say e to the power i k x here you are not you are not really assessing whether this there is a exponential growth or decay with respect to space, with respect to space there is a periodicity or there is a repeatability and this repeatability comes from this wave number typically that wavelength over which the behavior repeats is the length or the size of each control volume. So, over which you have a periodic behavior of the structure of the discretization equation, because the same discretization equation you are using for each cell.

So, that means, for each cell whatever is the length it is a delta x that is the period over which the same behavior of the discretized equation is repeated and therefore, the corresponding wavelength of this 1, if this is the wave number wave number is basically number of waves that you have over a time period. So, you if you if you consider that the wavelength of the corresponding exponential behavior which is depicted by this term then the corresponding wavelength is nothing but the cell length. Now, why you are considering e to the power i k x? The reason is you do not allow this to be exponentially growing or decaying. So, remember e to the power I theta is Cos theta plus i sine theta and if you consider Cos of something plus i of sine that something, then what is the amplitude of or modulus of that one, root over Cos square theta plus sine square theta. So, that means, you are not by putting this exponential square root of minus 1, you are constraining it in a way that it cannot amplify beyond 1. So, the entire responsibility of the amplification is the time dependent is been carried by the time dependent term. So, that is these are some of the rationally behind choosing the term in this particular form.

Now, because the governing equations are linear it may be possible to make an assessment of the stability of these types of discretized equation by considering just one term in the Fourier series, because the resultant solution is a combination of all the terms in the Fourier series, if one of the terms is what is assessed here then the same is true for each and every individual term and the same is true for the resultant also, if it is a linear it is a linear super position of all the solutions that is the final solution. So, you can take just one representative term from the series. So, you can just assess it by considering that this epsilon is of the form e to the power a t into e to the power i k x, just to avoid a confusion between this i and this index i what we can do is we can use may be here j where we call it just like people in electrical engineering do, we call that as minus 1 because in electrical engineering i is preserved for (( )). So, that is why people many times use j, because i is such a common symbol here also where in that one dimensional discretization i is so common for a discretization of space. So, we use just a different symbol j for square root of minus 1.

Now with this background, let us try to see that if you substitute this a error expression in the discretized equation for error what you get. So, you get in place of epsilon i that is e to the power a n plus 1 means time T plus delta t. So, e to the power a T plus delta t into e to the power j k what is x? At i in x equal to x. So, e to the power j k x is equal to 1 minus 2 r epsilon i n. So, e to the power a t, because time is n space is x. So, e to the power j k x plus r, e to the power a t into e to the power j k x plus delta x plus e to the power a t into e to the power j k x minus delta x. So, if we consider an amplification factor, which we call as A, that is e to the power a t plus delta t divided by e to the power a t. So, what we get a is equal to 1 minus 2 r plus r e to the power j k delta x plus e to the power minus j k delta x. Our job will be to see that whether we can ensure that A is in magnitude, that is mode of A is always less than 1, if mode of A is always less than 1 irrespective of the value of r we say that it is unconditionally stable, but if there are certain conditions for only or certain restrictions on based on which you have mode A less than 1, then we say that it is conditionally stable if there is no such case for which a is mode of A is less than 1, we say it is unconditionally unstable. So, we will make an assessment which of these cases it belongs to just for writing convenience, let us consider k into delta x equal to theta.

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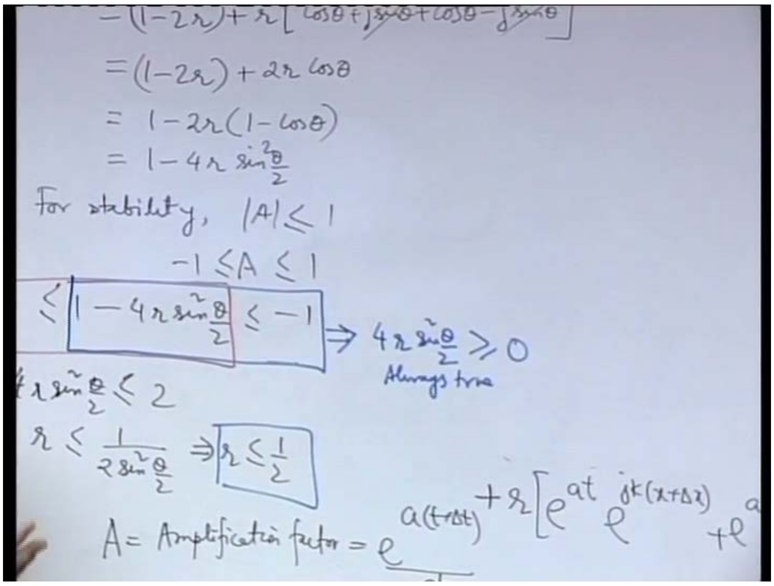

So, A is equal to 1 minus 2 r plus r e to the power j theta plus e to the power minus j theta. So, 1 minus 2 r plus r Cos theta plus j sine theta plus Cos theta minus j sine theta. We are using the de mover's theorem e to the power I theta is Cos theta plus i sine theta in place of I we are writing j. So, this becomes 1 minus 2 r plus 2 r Cos theta.

So, that is equal to 1 minus 2 r into 1 minus Cos theta, 1 minus Cos theta is 2 sine square theta by 2, now you must have for stability mode of A less than 1 or less than equal to 1, equal to 1 is a limiting case obviously; that means, A should lie between minus 1 to plus 1. So, 1 minus 4 r sine square theta by 2 less than equal to it should lie between minus 1 to 1. Let us consider the left hand inequality that is this 1. So, you have 4 r sine square theta by 2 less than equal to 2. So, r less than equal to 1 by 2 sine square theta by 2. What is the conservative estimate on r you can make out of it. So, you can have sine theta by 2 ranging sine sine square theta by 2 ranging from 0 to 1. So, if it is 0 very small. So, then r is less than equal to a very large number. So, it is not conservative when sine square theta by 2 equal to 1, r is less than equal to half. So, you have the whole requirement in one limit in in one side r is r may be large less than a large number; that means, it is upper limit may be large in another case it is upper limit is not. So, large, but half. So, out of these two which is the conservative design that you you would take you take half, because if it is less than half then automatically less than the upper limit is also satisfied, but less than the upper limit does not satisfy less than half. So, you have to satisfy both.

So, that means, we can say that r is less than equal to half. Now, concentrate the other limit that is the right hand limit so, what this will imply 4 r sine square theta by 2 is greater than equal to 0, this is trivially satisfied this is always true, because r is alpha delta t by delta x square that is always positive and sine square theta by 2 is positive. So, we do not have to bother anything about this. So, our stability criteria becomes r less than equal to half or Fourier number less than equal to half, because r is the Fourier number alpha delta t y delta x square. So, you can see that this is the same stability criteria here that we could derive from the equality of signs of all the terms all the coefficient in the discretized equation that all coefficient must be of the same sign.

So, using that we could easily derive this one even without going through such rigorous routes, because that physical consistency has a one to one correspondence with the stability, that we are talking about and you can easily make out here, that if you look out this discretized equation, I mean one step back if you go and see the discretized equation you have a coefficient 1 minus 2 r which is tied up with T i n. So, 1 minus 2 r has to be positive, because the other coefficient like r or 1 these are all positive. So, to make sure that it happens you must have 1 minus 2 r that is positive and that will make sure that r is less than equal to half.

So, just by looking at the discretized equation that is the power that is more powerful than even this analysis to my understanding, because that is very physical and just by looking at the discretized equation without going through this rigorous exercises, you can come up with the same condition that is required for stability of the corresponding discretization method, but anyway this rigorous exercise checks that whatever we could infer from the sign of the different coefficients that inference holds true even with more complex analysis. So, that is about the F T C S scheme. So, we will consider one or two more examples to illustrate these stability analysis.

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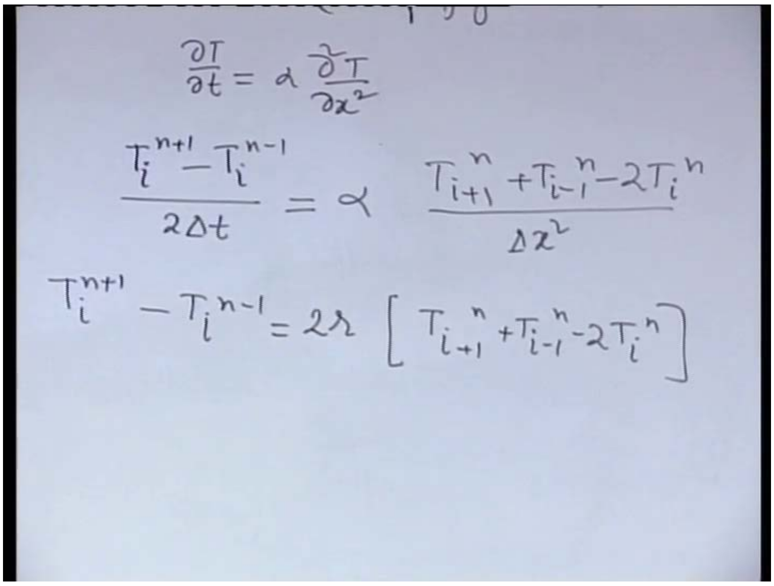

So, next we will consider a different scheme all this are for unsteady one dimensional unsteady state diffusion problems, C T C S scheme this is also called as leap frog scheme. C T C S means central time central space. So, time discretization by central difference and spacial discretization also by central difference. So, let us see let us write the governing equation so, let us write the corresponding discretization, central difference in time so, T i n plus 1 minus T i n minus 1 by 2 delta t.

So, how do you derive this central difference we have done it once, but just to recapitulate you write the Taylor series for T i n plus 1, write it in terms of T i n then write T i n minus 1 in terms of T i n and subtract the two. Now this is the left hand side the right hand side is the same as that of the previous case sorry. So, you can write here T i n plus 1 minus T i n minus 1 is equal to r by 2 T plus 1 n plus T i minus 1 n minus 2 T I n 2 r sorry not r by 2 r.

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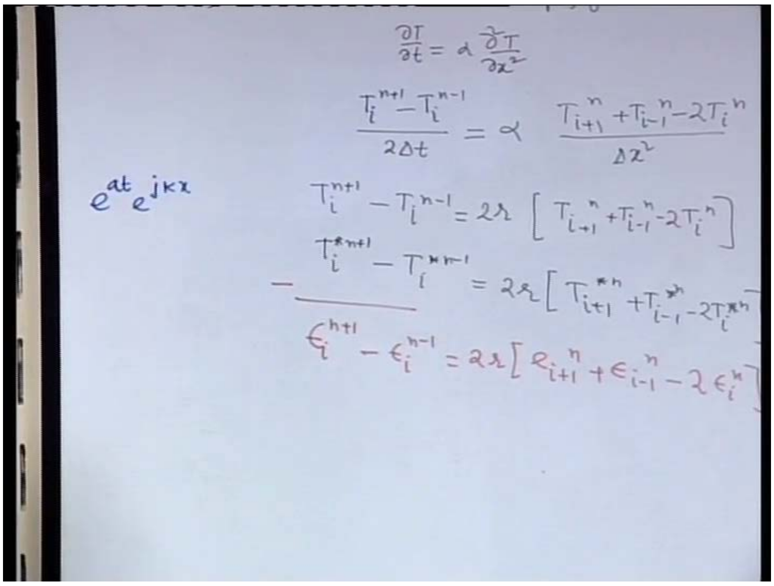

The corresponding approximate solution T I star n plus 1 minus T I star n minus 1, if you subtract this you will get the error, that is epsilon I n plus 1 minus epsilon i n minus 1 is equal to 2 r into epsilon i plus 1 n plus epsilon i minus 1 n minus 2 epsilon i n. What will be the next step? We substitute the form of the error that is e to the power a t into e to the power j k x, the multiplier A is also there it is not that it is not there, but it get cancelled from both sides that is why we are not write keeping it explicitly. So, we are just writing the form.

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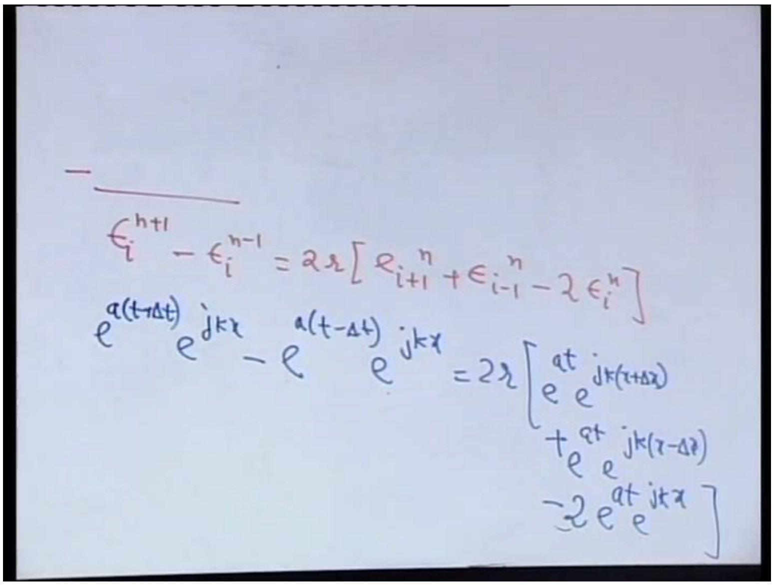

So, in place of this one what will be the term? e to the power a t plus delta t into e to the power j k x minus e to the power a t minus delta t into e to the power j k x is equal 2 r into e to the power a t e to the power j k x plus delta x plus e to the power a t e to the power j k x minus delta x minus 2 e to the power a t e to the power j k x. So, let us simplify, to simplify we divide both sides by e to the power a t into e to the power j k x see why do we have this e to the power a t e to the power j k x form, it is a separable function of time and position. So, that we can isolate the effects of the time dependence with special dependence specially we can keep it periodic still time wise we can have it either amplified or died down and these two effects are separable by virtue of taking a separable form of the expression for the error.

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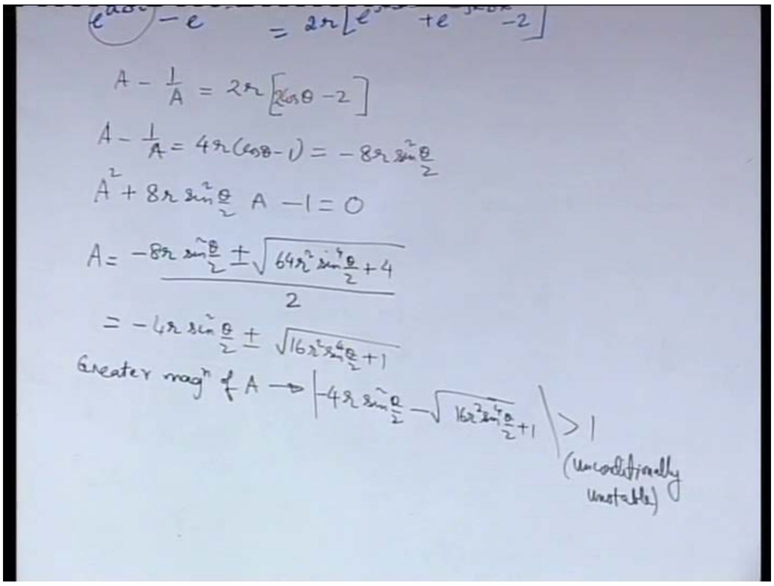

Now, if we divide both sides by e to the power a t into e to the power j k x, then we have e to the power a delta t minus e to the power minus a delta t is equal to 2 r into e to the power j k delta x plus e to the power minus j k delta x minus 2. Remember that e to the power a t, is a delta t is your amplification factor A that is e to the power a t plus delta t divided by e to the power a t. So, if this is A, this is 1 by A. So, A minus 1 by A is equal 2 r, k into delta x let us call it theta. So, 2 r Cos theta 2 Cos theta minus 2. So, A minus 1 by A is equal to 4 r into Cos theta minus 1, that is minus 8 r sine square theta by 2. So, if we write it is A square plus 8 r sine square theta by 2 into A minus 1 is equal 0, it is a quadratic equation in A, there is an important and interesting observation that you can make from this quadratic equation there are two roots of a right and you can see what is the product of the roots.

So, what is the magnitude of the product of the roots that is 1. So, magnitude of the product of the roots is 1 means, if 1 of the roots has magnitude less than 1 the other root will have magnitude greater than 1 like that. So, it has an inherent problem of the scheme has inherent problem associated with it, that is you may not be able to keep magnitude of a bounded as less than 1, less than or equal to 1, because the magnitude the product of the roots is having a magnitude of 1. So, let us try to see what are the roots and go into bit more details. So, A is minus 8 r sine square theta by 2 plus minus root over 64 r square sine 4theta by 2 . So, minus 4 r sine square theta by 2 plus minus square root of 16 r square sine 4 theta by 2 plus 1. Now, we want to make an assessment of mod A and again we have to be conservative in design. So, to do that which one we will consider here out of plus and minus plus or minus if you consider the minus 1 you can see if you consider the magnitude, the magnitude is higher, because here is a minus term which when added with another minus term will add to the magnitude, whereas the plus with A minus combination will actually reduce the magnitude.See it all it all depends on the interpretation of A, it all I mean it depends on it is it is it is not true that you can just interpret it as e to the power a delta t in this way, it is basically an amplification factor. Now the amplification factor you could write in different ways in whatever way you write, the thing is that you you are you are considering about a factor which is talking about the amplification of a particular perturbation.

So, when you consider only the only only the positive root of course, you you make sure that a is less than 1, but you have to also keep in mind that is the mod A that is important. So, essentially when we talk about A here essentially, when we talk about this one we talk about the mod of A, it is it is not just the A that is important it is the mod of A which has certain interpretation. So, here it is not that A that itself is important it is the mod of A that is important, it may not have the exponential sense that we are talking about this is just an example. So, you could write in many ways where you could write it at the new time step divided by the old time step, will come as the factor A exponential way is just one way of writing it, but it that is that is not all. So, it is just the amplification just the time dependence in the next step divided by the time dependence in the previous step that is what is A. So, exponential way of writing it is just one way of deriving it, but that is not the just just the complete way of doing, it is just one way of doing it. So, if you consider the greater magnitude of A. So, the greater magnitude of A is given by the magnitude of 4 r minus 4 r sine square theta by 2 minus 16 r square sine 4 theta by 2 plus 1 .

And this is always greater than 1 right, because 1 plus some 1 plus something positive square root of that, with that again an a positive term is added, because the minus sign you can absorb from both. So, this is greater than 1 that means, it is until and always it may be equal to 1 if r is equal to 0, but r equal to 0 is not the case which corresponds to a practical numerical solution, because delta t is equal to 0 you cannot take. So, this means it is unconditionally unstable, see from the F T C S scheme we have to first understand that what motivated us to introduce the C T C S scheme, because after introducing it we have now fallen in trap that we are finding that it is unconditionally unstable, but what motivated us to introduce that scheme, what motivated us is very simple it is a greed to go for higher order accuracy. So, if you consider the time discretization in the F T C S scheme it is a forward difference and in the C T C S is the central difference the central difference discretization is expected to be more accurate in terms of it is order of accuracy as compared to the forward difference and that motivated us to go from F T C S to C T C S, but what we are saying here is that even you go for more accurate scheme that is not all, because that can give rise to undesirable conditions of stability that is here we are getting it is unconditionally unstable. So, there is no way you can ensure that it has a particular condition at least which you could maintain for it is stability. So, we need to modify this method. So, two very famous mathematicians came up with a modification of this particular scheme to give rise to some desirable features of stability.

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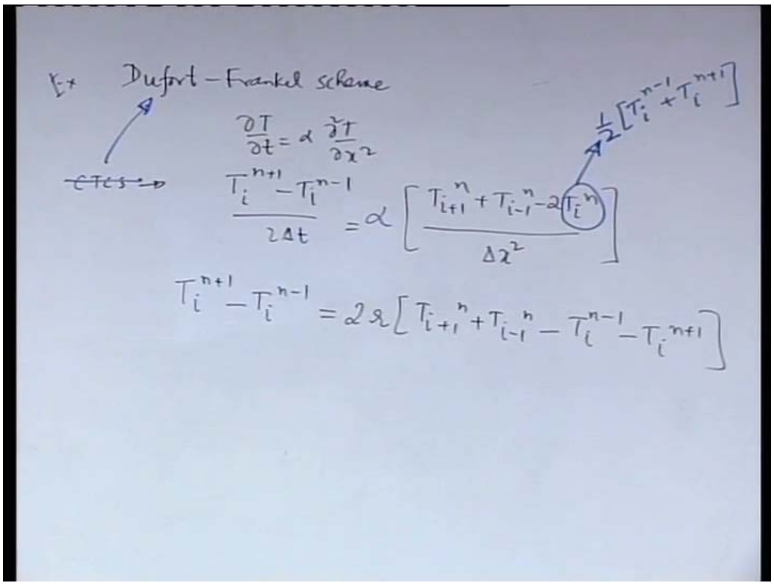

And that particular scheme is called as Dufort Frankel scheme. So, let us write first the C T C S scheme and then see that how it is modified. So, what these two mathematicians they did was as follows. So, they made a change of this term and represent represented or replaced T i n as half of T i n minus 1 and T i n plus 1 just an adhoc change that is replaced T i n as an average of T i n plus 1 and T i n minus 1, which is sort of physically intuitive that the temperature at a particular grid point at a particular time, you are replacing it with an average of that as the previous time and the next time and that makes the scheme change from C T C S to Dufort Frankel scheme. So, the presentation of the scheme is very simple, now let us try to make an assessment of the stability of this particular scheme. So, let us simplify it T I n plus 1.

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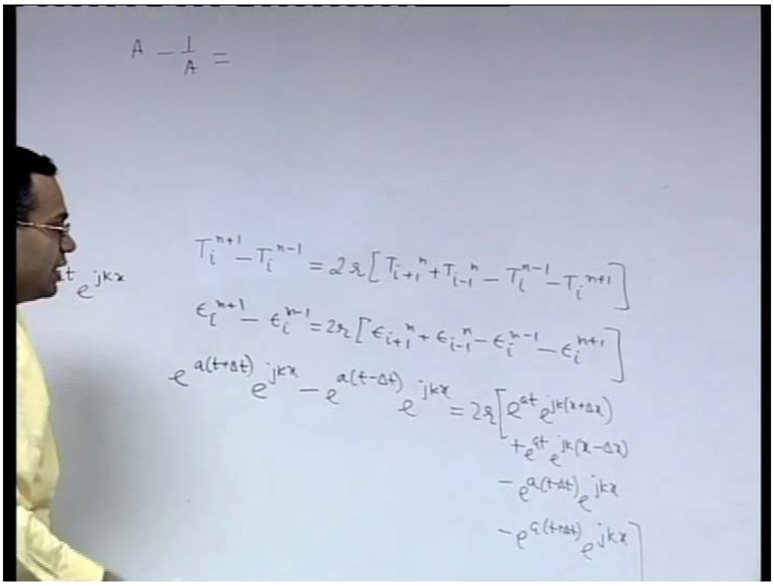

So, we substitute the form e to the power a t e to the power j k x just to remind you again this is just a form. So, something which is an amplification which can get amplified with time or decayed with time exponential is just a representative of that, but it need not be the only representative there could be several ways in which it could be written, and similarly this is written as the specially periodic function of space.

Now, let us write the corresponding error terms. So, by the time we know that the errors will follow the same as that of this equation of the form of the discretized equation, because of the linearity of the equation . So, we have epsilon i n plus 1 minus epsilon i n minus 1 is equal to 2 r epsilon i plus 1 n plus epsilon I minus 1 n minus epsilon i n minus 1 minus epsilon i n plus 1, we substitute here e to the power a t plus delta t into e to the power j k x minus e to the power a t minus delta t into e to the power j k x is equal to 2 r into e to the power a t e to the power j k x plus delta x plus e to the power a t e to the power j k x minus delta x minus e to the power a t minus delta t into e to the power j k x and minus e to the power a t plus delta t into e to the power j k x, the next step is divide both sides by e to power a t into e to the power j k x.

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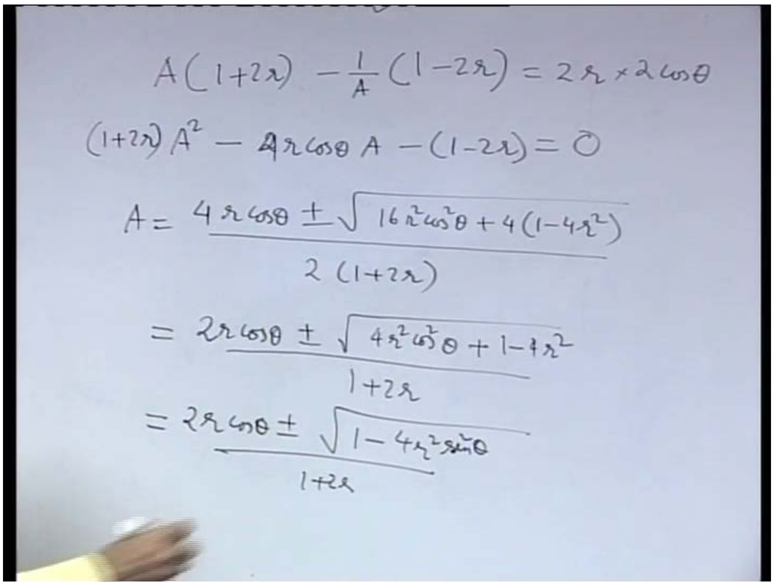

So, we have A minus 1 by A, the change is there in the right hand side it is 2 r then e to the power a t into e to the power j k x is gone, e to the power j k delta x plus e to the power minus j k delta x then yes minus minus 1 by A right minus A. So, A minus 1 by A if we write then this into sorry you cannot write A minus 1 by A in this way. So, A into 1 plus 2 r minus 1 by A into 1 minus 2 r is equal to 2 r into 2 Cos theta. So, if we multiply by A then A square 1 plus 2 r into A square minus 2 minus 4 r Cos theta into A minus 1 minus 2 r equal to 0 right, if there is any error in algebra please point it out. So, A becomes 4 r Cos theta plus minus root over 16 r square Cos square theta plus 4 into 1 minus 4 r square by 2 into 1 plus 2 r. So, this is 4 r Cos theta plus minus you can take the 2 inside. So, square root of 4 r square Cos square theta sorry we can make it 2 r Cos theta 2 r Cos theta plus minus square root of 1 minus 4 r square into 4 r square into 1 minus Cos square theta that is 4 r square sine square theta by 1 plus 2 r.

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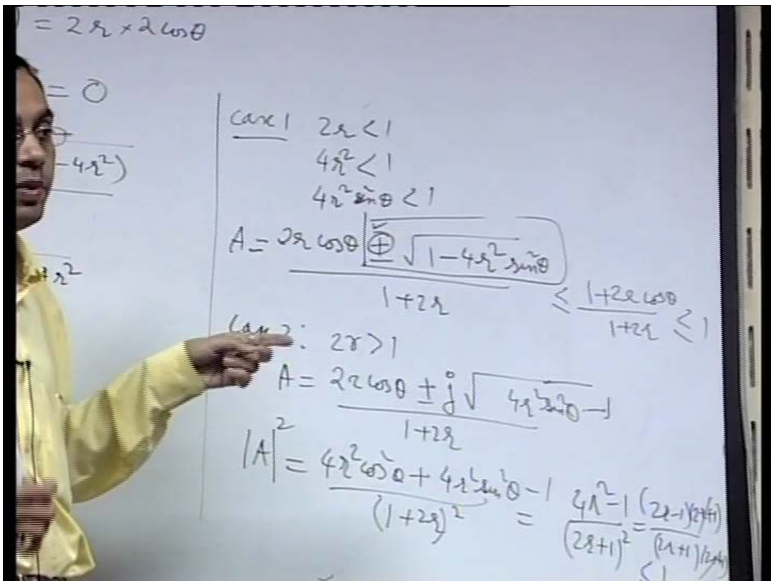

Now we can divide it into two cases, case 1: 2 r is less than 1, if 2 r is less than 1 then 4 r square is less than 1 right remember r is positive. So, it is a that means, we are considering 2 r less than 1 means r is a fraction. So, 4 r square is also that is 2 r is a fraction; that means, 4 r square less than 1. So, 4 r square sine square theta that is also less than 1; that means, this square root will give you something imaginary. So, then that a will be 2 r Cos theta plus minus j where j is square root of minus 1 into yes right right. So, very right. So, this is less than 1 means this is a positive root right. So, then we will consider the plus sign of it or plus minus, but considering that this is a real number now out of plus or minus which one we should take plus for the conservative estimate. So, you take the plus sign. So, 1 minus 4 r sine square theta plus 2 r Cos theta by 1 plus 2 r. So, this is less than or equal to this particular side of it is less than or equal to 1 right, because 1 minus 4 r square sine square theta.

Let us see, let us take it we have we have ensure it is positivity that is all. So, this is positive. So, this then this plus 2 r Cos theta. So, it is always less than or equal to 1 plus 2 r Cos theta, because this is the term that reduces it. So, if you consider the plus sign then this is less than equal to 1 plus 2 r Cos theta by 1 plus 2 r right, because 1 minus 4 r square sine square theta will always reduce it from 1. So, this will have a maximum value of 1 and then 1 plus 2 r Cos theta. So, it is less than or equal to 1 plus 2 r Cos theta and this is less than or equal to 1. So, that is case 1, case 2: 2 r greater than 1. So, if 2 r is greater than 1 then that is the imaginary root that we were talking about so, then a will be

2 r Cos theta plus minus 4 r square sine square theta minus 1 right. So, smod of A is 4 r square Cos square theta plus 4 r square sine square theta minus 1 by 1 plus 2 r mod of A square basically by 1 plus 2 r whole square. So, 4 r square minus 1 by 2 r plus 1 whole square. So, this is 2 r minus 1 into 2 r plus 1 by 2 r plus 1 whole square so, if you cancel this will come out to be less than equal to 1.

So, in either way we show that this is less than or equal to 1; that means, this scheme is unconditionally stable. So, both for 2 r less than 1 and 2 r greater than 1, but ironically this scheme is not consistent. So, I will leave it on you as a homework, you try to show it that this scheme is unconditionally stable that we have proved, but it is not consistent; that means, the truncation error will not tend to 0 in the limit as delta x, and delta t are simultaneously taken to 0. So, we stop here today, thank you.