Computational Fluid Dynamics
Prof. Dr. Suman Chakraborty
Department of Mechanical Engineering
Indian Institute of Technology, Kharagpur

Lecture No. # 21

Solution of System of Linear Algebraic Equation

We have discussed in this course so far about how to discretize several types of differential equation. Those types include the diffusions type of equations, both steady and unsteady versions. We have seen that the base of the discretize equations in many cases; we could come up with a system of Linear Algebraic equations.

So, the whole objective of the discretization method was in those cases to convert the governing differential equation into a system of Linear Algebraic equations. The objective is of this type because numerically, it is much more convenient or it is possible rather to solve a system of Linear Algebraic equations rather than directly solving a differential equation because standard methods of numerical solution of system of Linear Algebraic equation, those are well known.

So, now what we will do is, we will try to look into the standard methods for solution of systems of Linear Algebraic equations keeping in mind that we have our interest in applying these methods for solving the problems, which are for solving the equations. Linear Algebraic equations which evolve due to discretization of the CFD problems that we have discussed so far or we will be discussing subsequent to this consideration.

Now, to begin with, we will begin with level of high school or junior high school type. We will just consider very simple equations to get first some insight on the nature of the equations and the nature of the solutions and so on.

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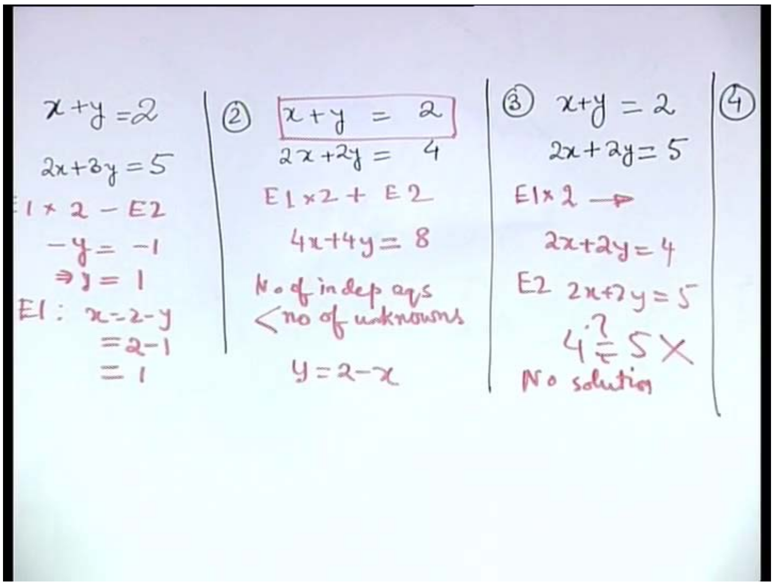

So, let us consider these following equations, say equation number 1 or system of equation 1. X plus y is equal to 1 or let us make it 2. This is one. Second problem, third problem, fourth problem and the fifth problem, these are all very simple types of equations and these equations people get familiar with these days in very junior high school classes. Now, our objective will be to look a bit deep into these equations and find out that what the special characteristics of these equations are or how the characteristics of these equations differ from one another.

Let us consider the first one. So, x plus y equal to 2, 2x plus 3y equal to 5. So, the standard method of solution, if you apply here, you are readily successfully doing that. For example, you can multiply the first equation, 1 by 2 and subtract that from equation 2 or subtract equation 2 from that. So, if you do that, so 2x plus 2y equal to 4 and 2x plus 3y equal to 5. So, you have minus y is equal to minus 1. That means, y equal to 1. This one you can now utilize in equation 1.

So, equation 1 will give you x is equal to 2 minus y that is 2 minus 1 is equal to 1. So, it can be the simplest possible type of equation and there is hardly anything to discuss on this. It is straight forward in a sense; you have two equations, two unknowns. What you are doing here is, essentially an elimination that is you are eliminating one of the variables in preference to the other and solving that particular variable first. Then, using that, you are using the other variables. You are solving the other variables.

Now, let us go to equation number, the problem number 2. So, here you have x plus y equal to 2. 2x plus 2y equal to 4. So, if you see that, if you readily give this problem to a junior high school student, there will appear to be some anomaly. Of course, at this level you know that there is no anomaly associated with it, but the apparent anomaly is there because of the fact that you see that there are two equations with two unknowns. So, this should give some solution, else it is giving some solution, but not the unique solution.

So, it is not possible for a school student to find out a unique solution to this system of equation. Not just for a school student, for anybody it is not possible to find out a unique solution. The reason is that these are two equations, but not two independent equations. So, these equations are considered to be linearly dependent equations because the second equation is just a constant multiplier of the first equation. It could also be that we can form a third equation which is a linear combination. So, some additive effect and some multiplying effect, for example, of the first two equations, just like as an example, if you multiply equation 1 with 2 and add that with equation 2, so you will get 2x plus 2 y plus 2x plus 2y. So, 4x plus 4y is equal to 8.

Again, you have designed an equation which is illusively new, but it is not a new equation because it is not a linearly independent form of an equation disregarding the existence of the previous two. A linearly dependent form of the previous two existing ones and even the second one is the linearly dependent version of the first one. So, effectively, no matter you can go on writing so many equations based on this, but effectively, you have only one independent equation with two unknowns.

So, here you have number of independent equations less than the number of unknowns that is what you have to remember. We will formulize this statement subsequently, but we will keep in mind that if that is the case, it can be easily observed that you have only one independent equation and two unknowns. So, you can have infinitely large number of possible solutions. So, you can substitute any value of x, the corresponding y is 2 minus x. So, its solution is y is 2 minus x. So, you independently input whatever value of x you like and you will get a corresponding value of y. So, we should be aware of the fact that, it is not a unique solution that you can expect if you have number of equations same as numbers of unknowns.

It means absolutely nothing. When we say that number of equations same as number of unknowns, many times when we are mathematically modeling a problem, we express our satisfaction in seeing that we are having number of equations same as number of unknowns, but many times we do not check that those are independent equations or not. If those are linearly independent equations, then we can make such statement. We will see that what the conditions which further lead towards that are, but if those are not linearly independent equations, then obviously, that consideration cannot be applied.

Then, let us consider the third problem, so x plus y equal to 2x plus 2y equal to 5. So, here you can see that you have equation 1 and equation 2. They are not linearly dependent that you can see because if you multiply equation 1 by 2, you will not land up in getting equation 2. So, if you multiply equation 1 with 2, what you get? If you multiply equation 1 with 2, you get 2x plus 2y equal to 4. From equation 2 you get, 2x plus 2y equal to 5. So, if you want to equate this 2, you will land up with 4 equal to 5. So, obviously, this does not have a solution.

So, the situation is, you have number of independent equations 2, number of unknowns 2, but you do not have any solutions. If you make a sketch of these two equations in the xy plain, you will find these two are parallel straight lines. They have the same slope because the y by x co-efficient is same for both. So, they are parallel straight lines and since, parallel straight lines do not intersect, you do not have their common solutions. So, these are typical cases that we encounter, but these cases are somewhat special in a way that these cases involved the right hand side which is not equal to 0. You could also have equations where the right hand side is 0 which are called as homogeneous systems of Linear Algebraic equations. So, if you have the right hand side equal to 0, then such examples are given in these problems 4 and 5.

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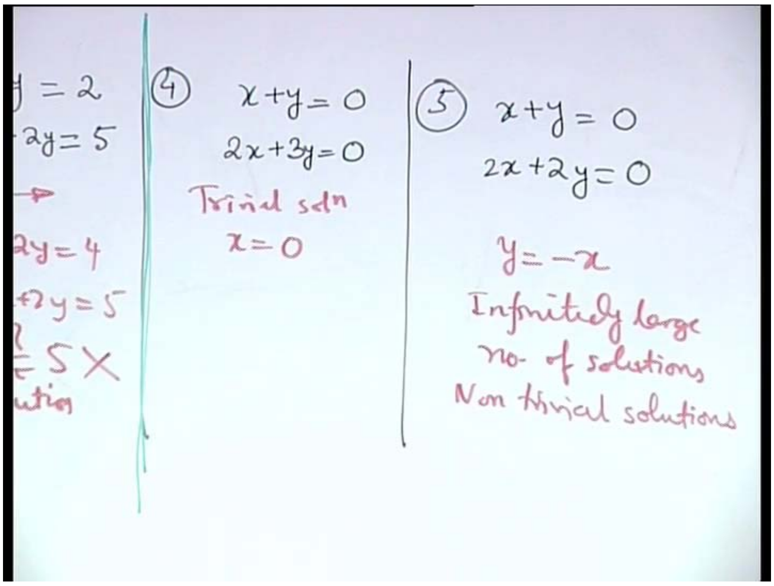

So, for discussion's sake, we can demarcate between the cases 1, 2, 3 and cases 4 and 5. So, for case 4, what we can see that here by observing the equations, you can come up with a solution which is valid always that is x equal to 0 and y equal to 0. So, x equal to 0 and y equal to 0 trivially satisfies these two equations. That is why; it is called as a trivial solution. So, you have trivial solution, but are there non-trivial solutions. You do not have any non-trivial solution, whereas if you go to the example 5, you have again the issue of linear dependence coming into the picture x plus y equal to 0. 2x plus 2y equal to 0 is a multiplier of equation 1 with 2. It is not a linearly independent version. It is just a dependent version of equation 1.

So, we can write this equation system solution as y equal to minus x. So, this also has a trivial solution x equal to 0, y equal to 0, but beyond that it has infinite number of non-trivial solutions. So, whatever is the value of x, you can have the corresponding value of y as minus of that x. So, infinitely large number of solutions will exist that includes the trivial solutions, but who also have non-trivial solutions. Now, these are certain observations that we can get from very simple systems of equations. The big challenge is that how can we extend these equations to a general case of a system of n number of equations with n number of unknowns. How can we extend or extrapolate these concepts? To do that what we will consider is, we will consider a matrix form of these equations. So, let us again come back to the equations 1 after the other.

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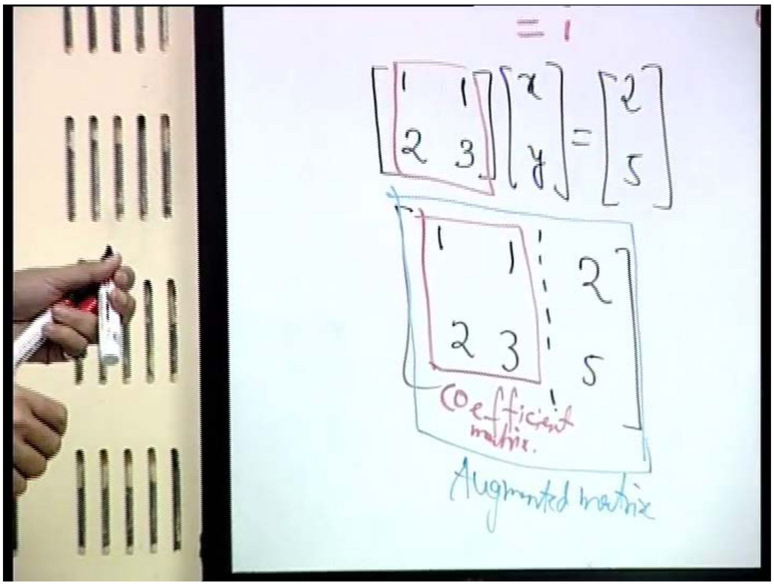

So, equation 1 has a matrix form 1 into x plus 1 into y2 into x plus 3 into y. So, this one, you can clearly see that the distinction between case 1, 2 and 3 that originated out of the difference in the co-efficient, not only that, also the differences in the right hand side. In some cases, the co-efficient are equal like cases 2 and 3, but what mattered in the differences in the nature of solution is their right hand sides are different. So, it is important to characterize the co-efficient also. It is important to characterize the differences in the right hand side.

So, we can write a unified representation where we write the co-efficient in one side and also, which make a partition and write the right hand side together with it. So, this part, where you have just the co-efficient which is this same as this one, this is called as the co-efficient matrix. The bigger picture which is there is called as augmented matrix as if the co-efficient matrix is augmented includes the right hand side. So, for the case 1, let us identify what is the rank of the co-efficient matrix and what is the rank of the augmented matrix. Why we are interested in the rank of the matrix is that the rank of the matrix, essentially talks about the sense of linear dependence and independence between different equations.

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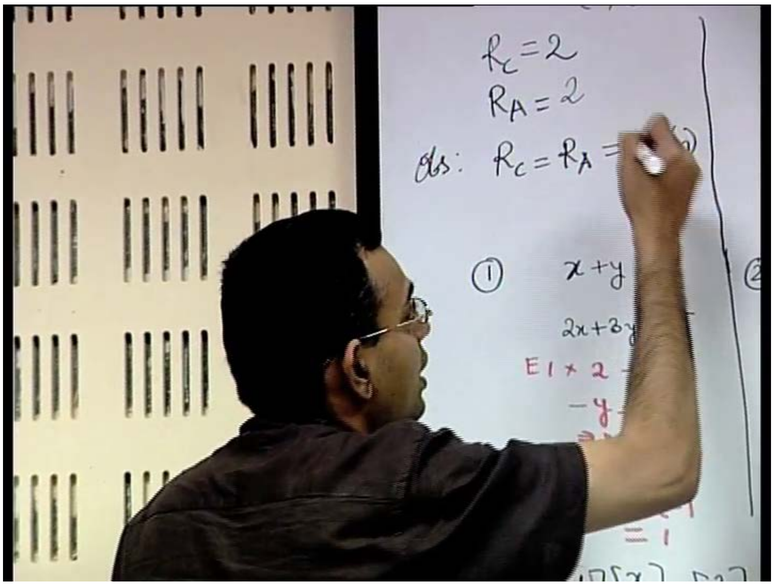

So, if you consider the rank of the co-efficient matrix, let us recall that what the rank of a matrix is. If you have a matrix of rank R, that means, there are two things that need to be satisfied. So, let us note that rank of a matrix is R. If it has at least one non 0 minor of order R, all minors of order greater than R vanish. That is become 0. What is a minor? We can go to a formal definition, but it is just a determinant that you can extract from the matrix in very simple terms. So, rank of the co-efficient matrix, if you have to find out, we have to find out that what that R is. So, that it has at least one non 0 minor that is one determinant that you can extract out of it.

It has at least one non 0 such determinant and any minor of order greater than R vanish. So, to do that, we will consider the highest possible order determinant that we can extract from it. So, what is the highest possible order determinant we can extract from it? It is determinant of order 2. So, if we extract a determinant of order 2, then what we can see here? We can extract any two second order determinant, but just let us concentrate on this one, the one which is boxed with the red colour. So, the determinant value of this is 3 into 1 minus 2 into 1. So, it is non 0.

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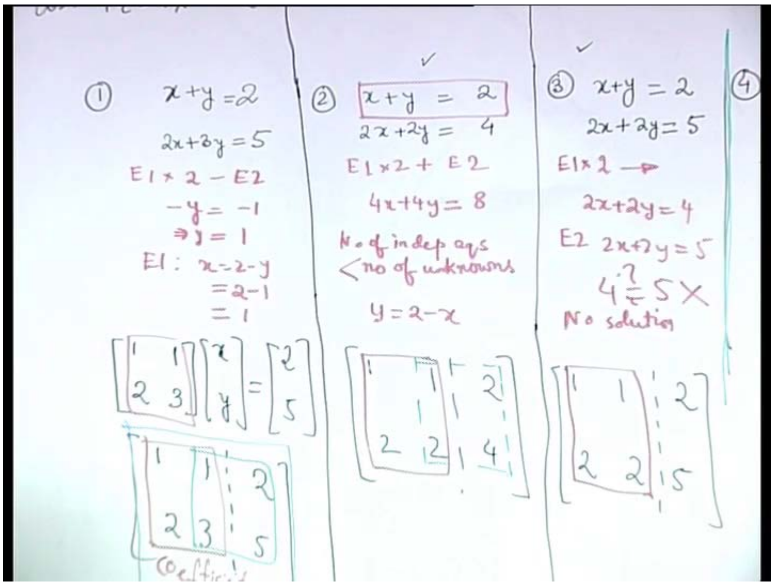

So, it is good enough that we have got at least one 0 minor of order 2 and any minor of order greater than 2 will vanish, of course, because it is not there. So, what is the rank of the co-efficient matrix 2? We are still on discussion with regard to the example 1. Rank of the augmented matrix that also is 2 because you can extract, say some 2 by 2 from the augmented matrix, say this one as an example, the green marked one, so 5 into 1 minus 3 into 2 that is non 0. So, the observation is rank of the co-efficient matrix equal to rank of the augmented matrix equal to 2. This 2 is equal to our n, where we have considered that you have n number of equations with n number of unknowns. Let us do the same exercise for the example 2.

So, let us write the co-efficient and the augmented matrix co-efficient matrix 1, 1, 2, 2. Then, the augmented will include 2 and 4. So, this is the co-efficient matrix. What is its rank? So, if you consider the determinant of order 2 that is 0, 2 into 1 minus 2 into 1 and therefore, its rank is 1 because any 1 order determinant that is individual terms that you can extract of course, this is non 0.

So, rank of co-efficient matrix equal to 1. You can see that in a way, it is talking about a linear independence. Here, it is a linear dependence rank of the augmented matrix. So, of course, in the augmented matrix, you can extract this determinant, but you can also extract other determinants which are non 0. For example, this 1, but here that is also coming out to be 0. So, that shows that you can have the second equation as the perfect multiplier of the first equation because you can see that when will the determinant be 0. You will have the determinant 0, when it may be possible that there are 2 rows and the rows are identical.

So, here instead of the rows being identical, what you have? You have one particular row as the row with a constant multiplier 2. So, if you extract the multiplier out of it, then what it becomes? Then, what it becomes is that the 2 rows are the same. So, 2 identical rows mean, basically they are the same equation. Just you have the constant multiplier as 2. So, what we get out of it? We get out of it is that the rank of the augmented matrix that is also 1 because you cannot extract a second order non 0 determinant out of the augmented matrix.

So, rank of the augmented matrix is 1. So, the observation is the rank of the co-efficient matrix equal to rank of the augmented matrix equal to 1 which is less than n. Then, let us consider the third example. In the third example, let us write the co-efficient matrix and the augmented matrix. So, let us mark the co-efficient matrix with the red box as we have done earlier. Here, what is the rank of the co-efficient matrix 1? The co-efficient matrix has not changed from the previous example.

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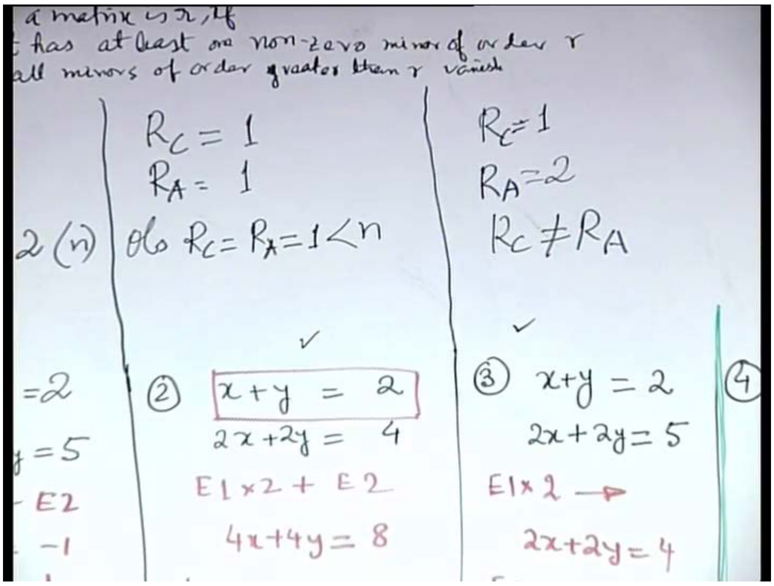

So, the rank of the co-efficient matrix still remains to be 1. What is the rank of the augmented matrix? Here, the rank of the augmented matrix is 2. Of course, you can extract a determinant with 0 determinant value like this 1, but you can also extract a non

0 determinant of order 2 like the determinant of this is 5 into 1 minus 2 into 2 and that is non 0. So, that means, rank of the augmented matrix is 2. So, rank of the co-efficient matrix is not equal to rank of the augmented matrix. These 3 cases may now be summed up.

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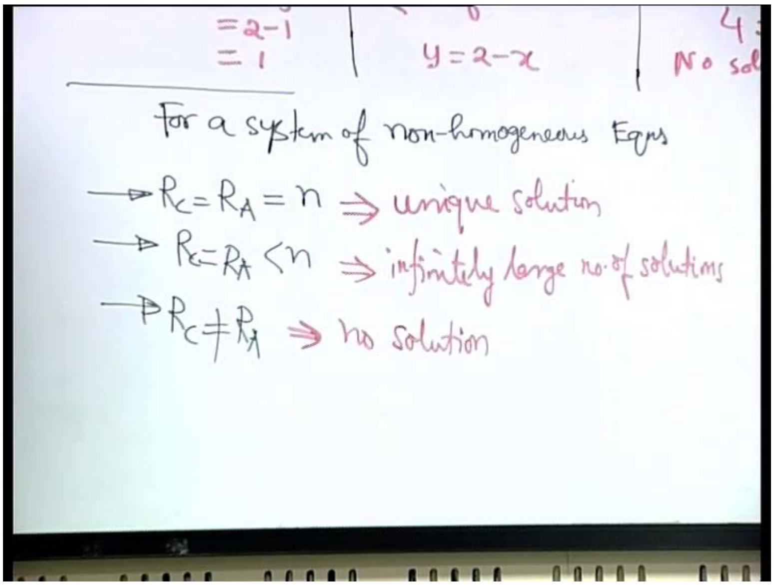

So, how can we sum up these cases for a system of non homogeneous equations which are examples 1, 2, 3. There are 3 possible cases rank of the co-efficient matrix equal to rank of the augmented matrix equal to m. What is its consequence? If that is the case, then, that will give you a unique solution. Then, rank of the co-efficient matrix equal to rank of the augmented matrix less than n, this will imply infinitely large number of solutions and rank of co-efficient matrix. Not say augmented matrix which implies no solution. So, this is for non-homogeneous equations.

Now, let us try to get the corresponding inferences for homogeneous equations. Now, when you consider homogeneous equations, you have to keep certain important things in mind. The first thing is that there is no requirement of putting any augmented matrix here because right hand is 0. Right hand side will have no contribution. So, we can just infer on the nature of the solution based on the co-efficient and determinant of the co-efficient matrix.

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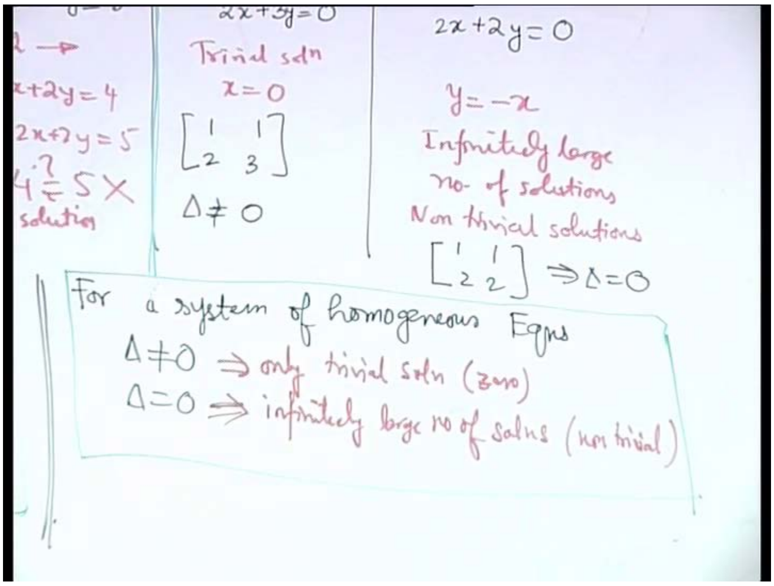

So, in example 4, the co-efficient matrix is 1, 1, 2, 3. What is its corresponding determinant? It is not equal to 0, whereas in case 5, that corresponding determinant 1, 1, 2, 2. So, determinant is equal to 0. So, what we can conclude for a system of homogeneous equations? Remember, both homogeneous and non-homogeneous, we are considering only Linear Algebraic equations, not non-linear.

For a system of homogeneous equations determinant not equal to 0 will imply only a trivial solution 0 and determinant equal to 0, this implies infinitely large number of solutions non-trivial. So, what we have got at the end? These are certain rules basis that you have learnt through your elementary courses in linear algebra and this is a brief recapitulation of the act.

The whole objective of going to this very simple route or well, understandable route of coming up with these kind of inferences is that let us not consider these as certain rule basis which cannot be retrieved, if you have forgotten it. So, you can see that through very simple examples, you can retrieve very easily that what can be the nature of the solution given. You have found out, what is the rank of co-efficient and augmented matrix for non-homogeneous equation and the determinant of the co-efficient matrix for homogeneous equation.

So, you need not forcefully remember these. You can see that very easily. You can recover from simple examples of systems of Algebraic equations with just 2 equations and 2 unknown as a matter of example. Now, while we are going through this exercise, we are going through this exercise because we have, say obtained some discretized equations. Now, the discretized equations have given rise to certain system of Linear Algebraic equations. We need to first access that what is the nature of the solution.

For example, if you have a system of non-homogeneous equations, then you are accessing the solution and you are getting that rank of co-efficient matrix equal to rank of augmented matrix which is less than n. This case we have obtained that means, it may not be a well posed boundary value problem. So, somehow, if you had a differential equation, you can always discretize it into a system of algebraic equations, but where the guarantee is that it is well posed.

So, we have seen that one of the hallmarks of a well posed boundary value problem is that it will have a unique solution, when you solve it, so that uniqueness of the solution will be disturbed. If you could have infinitely large number of possible solutions, that means, you can suspect that perhaps there is something wrong with the definition of the problem itself, rather than like your mechanical exercise of discretization.

So, the nature of solutions of a system of equation assessment of that to me, it is a very critical thing before solving the equations itself because that will give you a clue and an insight on what you can expect out of the solutions of the equations. Now, the next issue is that, once you have solve or once you have obtained the nature of the equations and say, you are satisfied with that, the next objective will be to solve the system of Linear Algebraic equations. So, what could be the method of solutions?

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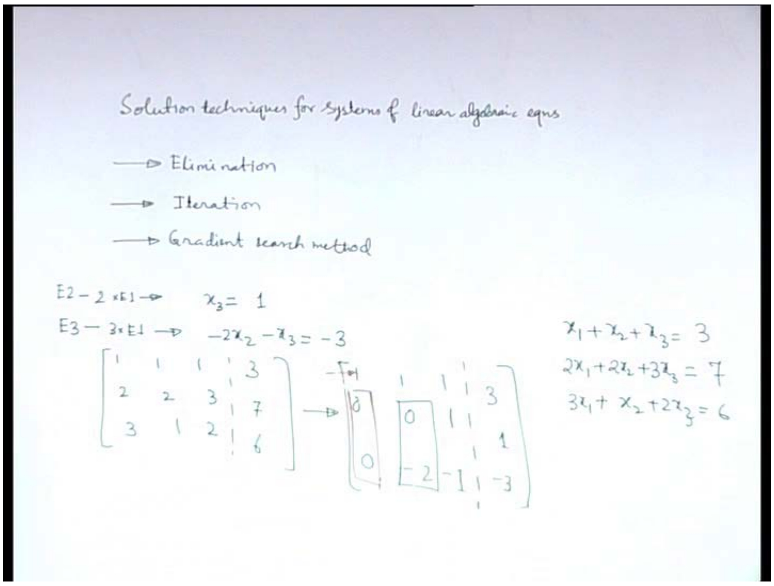

So, now we will discuss on the Solution techniques for systems of Linear Algebraic equations. For systems of Linear Algebraic equations, broadly the solution techniques are of 3 types. The first type is called the Elimination method, then Iteration method and third is Gradient search method. We will cover some aspects of all these methods to whatever extent and to whatever details possible within the scope of this particular course.

We will start with the Elimination method because that is the most intuitive one. Then, we will talk about the Iteration method. Gradient search method is based on an optimization technique and we will see that how that optimization technique leads to the solution of systems of Algebraic equations. Our initial effort will be on the Elimination method.

So, Elimination method from the name elimination, it is quite clear what we are attempting to do. In an Elimination method, we are attempting to eliminate variables. So, if there are 2 equations with 2 variables, we first try to eliminate 1 of the variables. Let us say, let us look into that example. So, if you have x1 plus x2 is equal to 2 and 2x1 plus 3 x 2 equal to 5. So, what we essentially try to do? We try to have equation 2 and subtract equation 1 into 2 from that. The objective is that, when you multiply these by 2 and subtract x1 will be eliminated. So, effort is to eliminate in this way, if there are 3 equations.

Let us let us take an example with 3 equations x1 plus x2 plus x3 equal to 3, 2 x 1 plus 2 x2 pus 3 x 3 equal to 7 3 x 1 plus x2 plus 2 x 3 equal to 6. So, if you consider these equations, now you can eliminate x1 and x2 or x2 and x3 or any 2 of the 3 variables to solve the remaining one, that elimination may not be very systematic. So, usually when we take up these problems in junior high school classes, we try to do it in the most optimal manner in which we think that process, we may not do it in a very systematic way. So, systematic way means something which has a common algorithm that can be implemented with a computer.

Whereas, when we do it in high school classes, we do it with our intuition which is many times more effective than what you could have done with a computer, but at the same time, it is not as fast as you as what you could have done with a computer. So, to have a corresponding understanding, what we will do is, we will now think that how we could systematically do it. In an algorithm that can be easily translated to a computer program.

So, to do that, let us try to do it step by step. This is equation 1, this is equation 2 and this is equation 3. So, if you now consider equation 2, minus 2 into equation 1, what it will give. So, 2x1 minus 2x1 that will be gone. Then, 2x2 minus 2x2 that will be gone. Then, 6x3 minus x3, so oh sorry, 3x3 minus 2x3, so x3 equal to 7 minus 6 that is 1. Then, equation 3 minus 3 into equation 1, so that this 3x1 is eliminated. So, what you get? So, 3x1, then 3x1 gone 3x then x2 minus 3x2 is minus 2x2. Then, 2x3 minus 3x3. So, minus x3 is equal to 6 minus 9 that is minus 3.

So, in this way, what you have achieved is as follows. The co-efficient matrix originally was 1 1one, then 2, 2, 3, 3, 1, 2. Let us also write the augmented matrix because right hand side here is there non 0. So, this is now transformed to 1 1one, then 0, 0, 1, 0 minus 2 minus 1 and the right hand side 3, 1 minus 3.

So, what has happened in this particular example is a very special case. I have very carefully chosen this example because if you now try to eliminate using the same policy as that you used in the first equation, you may be in a little bit of a trouble. Why? So, how you have eliminated the variable? The first variable here you have eliminated, the first variable by effectively equation 2 minus 2 into equation 1.

So, we will see that equation 2 minus 2 into equation 1, when you say, then equation 1 has a co-efficient which has a particular characteristic. It is bounded by some values where you do not expect that to be 0. Now, if you do it for this particular equation, then for this particular equation if you try to eliminate these variables using this equation, now whatever you multiply it with, this is 0. So, does not work.

For example, if you want to eliminate x2. By considering equations 2 and 3, then I mean how do you do that? If you consider the elimination of these particular columns here, you can see that what you have eliminated, you have eliminated all the terms in the first column below the diagonal term and these particular rows was acting like a pivot. So, you can see everywhere, you have used equation 1 some multiplier of that one to eliminate something from equation 2 and 3.

So, this is like a pivot. Using the diagonal element of the pivot, you have eliminated the corresponding terms in the column. The same thing if you want to do equation 3 minus this 1, then you see that like minus 2 minus, whatever into equation, this 1 that will remain as minus 2. So, what could be a way out? You could interchange the order of the equations. So, you could also reorder.

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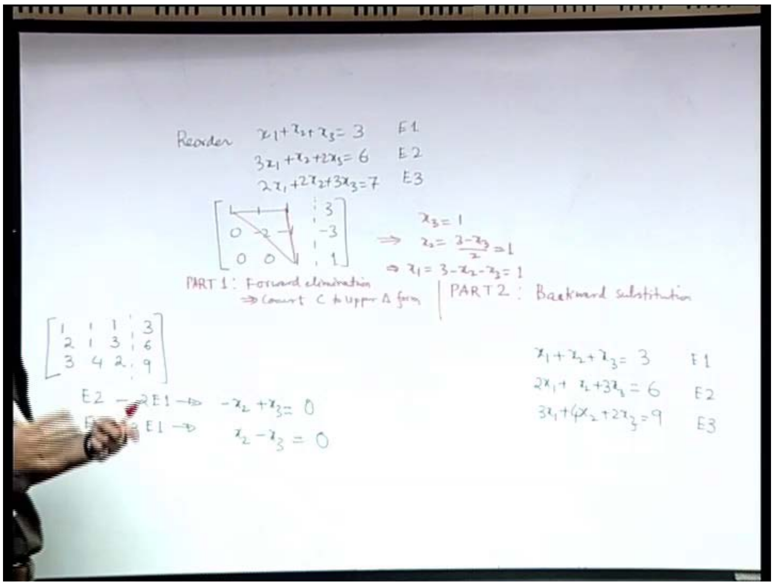

So, we come with an example where we see that reordering of the equations may be necessary. This is what we will keep in mind. So, how will you reorder? We will keep x1 plus x2 plus x3 equal to 3 3 x1 plus x2 plus 2x3 equal to 6 and 2x1 plus 2x2 plus 3x3 equal to 7. So, this is equation 1, this is equation 2 and this is equation 3. So, then what will be your corresponding equations 1 1one 3. Then, the third equation now will become the second equation, so 0 minus 2 minus 1 minus 3 and 0, 0, 1, 1.

So, in this way, what you have achieved is something very special. You have converted the system into an upper triangular matrix. All terms below the diagonal f0. So, now from here, you can easily solve for x. So, xx3 equal to 1, then what is x2 is minus 3 plus x3 by, sorry 3 minus x3 by 2 that is 1. What is x13 minus x2 minus x3? That is 1.

So, again we could make this problem or break this problem into 2 parts. First part is forward elimination convert co-efficient matrix to upper triangular form. So, in that process you have eliminated many variables. So, finally, you will get x3 equal to what in general for a system with n number of equations xn equal to what. So, that is part 1 and part 2. Once you have found out what is xn or x3, in this example using x3, you have substituted that back into the second equation to get x2 and these 2 back in equation 1 to get what is x1.

So, backward substitution in this particular example because of a particular simplicity, the forward Elimination was over in 1 step, but usually, you will not have the forward elimination over in 1 step. How many steps you will require for these 3 equations? So, what you expect is that in the first set of forward elimination, let us consider a different example where you require the full number of steps. May be, let us change one of the co-efficient. Let us try to see how many steps we can complete the elimination for this particular case for these 3 equations.

So, the first step of elimination is, so first let us write the co-efficient matrix 1 1one 2 1 3 3 4 2 equation 2 minus 2 into equation 1. So, you will have x2 minus 2x2 is minus x2 3x 3 minus 2x3 is x3 6 minus 6 is 0. Then, E3 minus 3E 1 4x2 minus 3x2 is x2 2x3 minus 3x3 is minus x3 is equal to 9 minus 9 equal to 0.

What you see here? So, this is not an improved version of the problem where I am trying to show you through repeated examples that the equation may look like to be elusive. I mean, it may just be a change of co-efficient, but you can see here that the Elimination method will fail here also. It will fail not because it does not have a solution, but it has infinitely large number of solutions. So, it shows that equation 2 and equation 3 are not linearly independent.

In fact, if you multiply equation 3 with 2 by 3, you will get equation 2. Is it or no or maybe. By some linear combination of this, if you, I think equation 1 plus equation 2 is equation 3 that is the linear combination. Not 2 by 3 that is also not. Then, what is the linear combination? You tell. No. Two third of E3, if you say, then what about this 4?

So, there are mistakes here. There is no mistake here. Then, what is the linear combination, you find it out. So, there must be some linear combination because essentially, yes 5E1 minus E2 right 5 into 3 15 minus 6. So, 5 minus 3 is 2 5 minus 1 is 4 5 minus 2 3. So, 5 E 1 minus E 2 is E 3. So, there must be a linear combination and because of this linear combination, see it is not so easy to find out what is the linear combination. If you do not, if you have not worked out yourself the linear combination, but somehow, it will be revealed that it was a linear combination because of the fact that you come up with same equations.

So, the Elimination approach will fail. So, we have seen 2 examples where the Elimination approach does not work straight away. It either goes through a twist where you require a reordering of the equations or it will fail altogether.

Now, in the next lecture, we will see that there are many more examples which are much straight forward and where you can directly use the Elimination approach through the forward elimination and backward substitution and find out the solution of the systems of equations. We will consider one particular systematic algorithm for that one that is a Gaussian Elimination method. That we will take up in the next lecture. Thank you.