Computational Fluid Dynamics
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Module No. # 01
Lecture No. # 03
Navier Stokes Equation (Contd.)

In the previous lecture we were discussing about the Navier's equation and let us continue with that.

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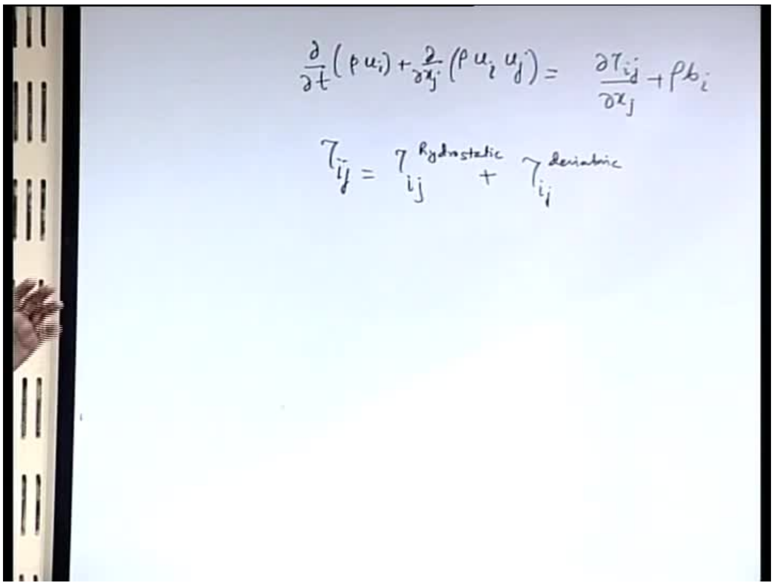

Let us say that we have the Navier's equation. This is basically representing the linear momentum equation along the i th direction. Now, this equation is quite general because it is not specific to any type of fluid. So, you can use it for all types of fluids. And how you demarcate one fluid from the other, it all depends on how do you specify this tau ij. So, what is this tau ij? Let us look into it more carefully. Now, if you see, if you consider stress at a point in a fluid, now it can be because of several things. Now, one kind of situation we can consider when the fluid has stress at a point even if it is at rest, and then the stress is totally of normal type because the fluid under rest cannot sustain shear, if there is some shear, fluid will immediately start getting deformed.

So, when the fluid is at rest, there is a normal component of the stress, negative of that which we call as pressure. So, basically when the fluid is at a static condition, the state of stress of that fluid that is called as the hydrostatic state of stress. It does not mean that this state of stress vanishes all together when the fluid is under motion. When the fluid is under motion, the state of stress is this one plus something which is different from the hydrostatic state of stress. And that plus something is called as a deviatoric stress tensor component, which depends on the deformation of the fluid.

So, we can write tau ij as tau ij hydrostatic plus tau ij deviatoric. Now, we will consider the tau ij deviatoric more intensely because that is related to the deformation of the fluid, and for that kinematic quantities need to be considered, which consider the deformation of the fluid. So, when we write the deviatoric stress tensor component, it should be related to what? It should be related to the rate of deformation. We know that for solids stress you can, for example, prescribe stress for a linearly elastic solid as proportional to strain. So, you relate stress with strain. For fluids, you relate stress with rate of deformation or rate of strain.

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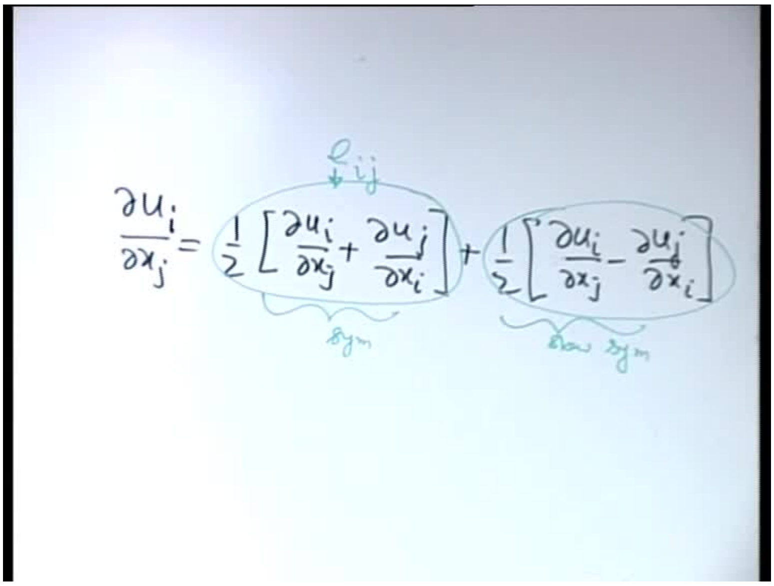

So, in general if you write a rate of deformation, now you can write a velocity radiant tensor, just like you can write a stress tensor, a general velocity gradient tensor you can write, and this you can write, you can be compose into two parts. So, you can clearly see that you have a symmetric part; that is, if you exchange i and j, this part remains the same whereas, this is a skew-symmetric or anti-symmetric part because if you exchange i and j, this part becomes minus of this one. So, this is symmetric, this is skew-symmetric. This is nothing very critical or odd. It is just very common because just like in metrics algebra any matrix can be written as a sum of a symmetric and a skew-symmetric matrix. Just like that it is a second order tensor, written as a sum of a symmetric and a skew-symmetric second order tensor.

Now, it is important for us to physically appreciate what are these. You can see that this particular term is related to the rate of deformation because the other term which is there it is sort of, it is related to the rotation of the fluid element. So, rotation of the fluid element will not directly give rise to a deviatoric stress tensor component, but the deformation will do. So, out of these two, if you give this a name e ij, then this e ij is responsible for the deformation in the fluid element; that is, this e ij rather represents the deformation of the fluid element. So, tau ij should be related to this type of like e kl, like that. So, tau should be related to e, so to say, tau deviatoric should be related to e.

Now, how they are related? Is it a linear function, it is a non-linear function, what type of function? It depends on the nature of the fluid. So, there are several fluids for which this relationship is linear; that is, tau deviatoric versus e, the relationship is a linear relationship; that is the mapping between these two is a linear mapping or a linear transformation. There are several fluids for which that does not hold true and those fluids are called as Newtonian fluids. So, for fluids for which the deviatoric stress tensor component maps linearly with the rate of deformation, that kind of fluid is called as a Newtonian fluid, as a general definition of Newtonian fluids.

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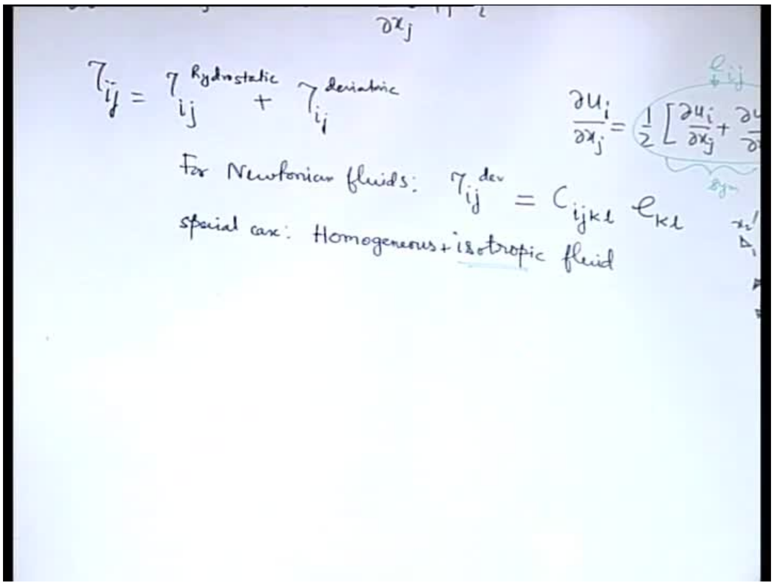

So, you can write tau ij deviatoric. So, for Newtonian fluids, you have tau ij deviatoric is somehow related to some e. Now, what maps tau to e? We had seen earlier that there is a second order tensor which maps a vector onto a vector, that we saw an example in terms of the Cauchy's theorem. Similarly, here you have a second order tensor, which needs to be mapped to another second order tensor. So, one second order tensor is the rate of deformation tensor that will be mapped to a second order tensor which is the deviatoric stress tensor. So, that is mapped by something which is a fourth order tensor, just like a second order tensor maps a vector onto a vector, a fourth order tensor maps a second order tensor to a second order tensor.

So, if you have a fourth order tensor, it should be specified by how many indices? It should be specified by four indices. So, we call that as C ijkl, where these are four indices of the fourth order tensor and C is representative of the fourth order tensor times e kl. Remember that you have ijl in the left hand side, so these are free indices. So, here ij you have already used, so you cannot repeat those anymore because then there will be a summation over those indices. So, you are using two different indices k and l over which you have summation, so another k and l appears with e, that is how these indices are designed. So, this is for Newtonian fluids.

Now, if you look at the structure of C ijkl, Cijkl will have how many components? Each of these ijkl can vary from 1 to 3. So, 3 into 3 into 3 into 3. So, it could be total 81 components. So, to specify the behavior of a fluid, you could require total 81 types of independent constants in general. But of course, we know that we do not require so many constants.

So, what helps us in simplifying the situation, let us see. To understand that, we will consider the special case of homogeneous and isotropic fluid. So, when we consider homogeneous and isotropic fluid, what do we mean by that? By homogeneous, we mean that the constitutive behavior is position independent; that is, if you have a particular constitutive property or material property at one particular point, then if you change the position you do not have a change in the same property. Isotropy means direction independence.

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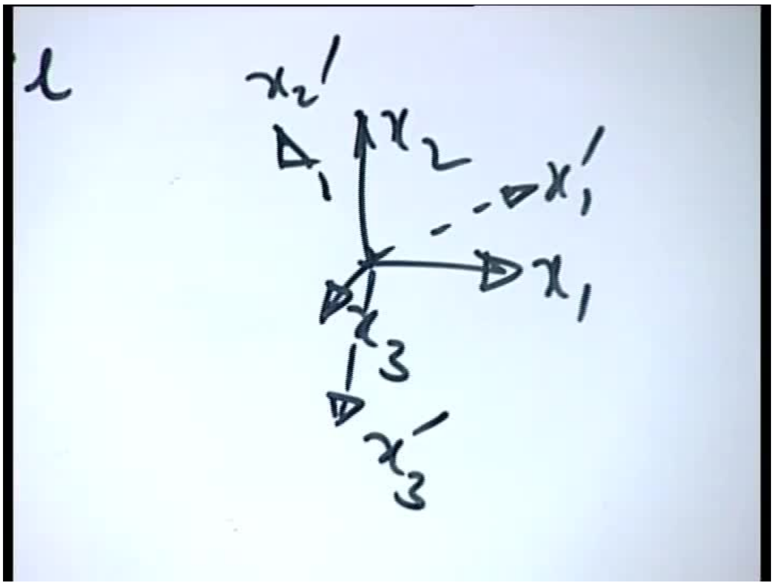

So, that means, if you measure a property with some coordinate axis x 1, x 2, x 3 and you measure the same property with respect to, for example, a rotated coordinate axis x 1 prime, x 2 prime, x 3 prime, the measured properties will not change, so it will be invariant to rotation, as an example.

So, we consider a homogeneous and isotropic fluid. So, if you have a homogeneous and isotropic fluid, first let us consider the isotropy aspect of it. So, we consider that we try to form a scalar which is isotropic. So, our objective is to form an isotropic scalar. How do we form an isotropic scalar? We will form an isotropic scalar by using C ijkl and some vectors.

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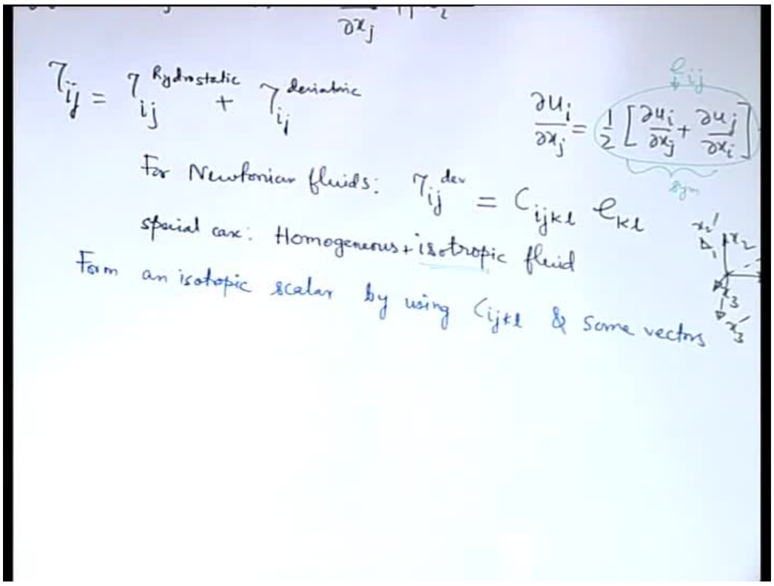

First question is, how many such vectors will you require? So, you have an isotropic scalar, your objective is since the liquid is isotropic, you want to formulate or structure a scalar which is isotropic using C ijkl and some vectors. How many such vectors will be required? Isotropic means it will be direction independent. So, no index will remain in that. So, when no index will remain in that, then each of these ijkl, these indices have to be somehow combined with another sets of these indices, so that you have a summation over these indices, so no free index will remain. So, if you consider four vectors, each vector can contribute one index. So, if you consider four vectors, then for example, A I, B j, C k, D l, by that you can represent four vectors A, B, C and D, then each of these indices may combine with this i j k and l of C ijkl and you will have an invisible summation, if you consider the product of that, so that you will come up with eventually some quantity a scalar, which does not contain any index.

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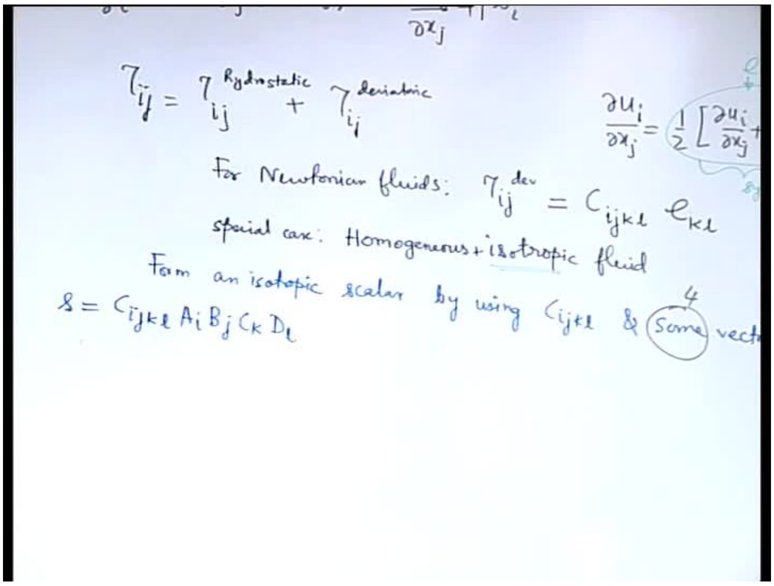

So, you can write that scalar say S as C ijkl A i B j C k D l. So, these some vectors, we have concluded that this will be four vectors.

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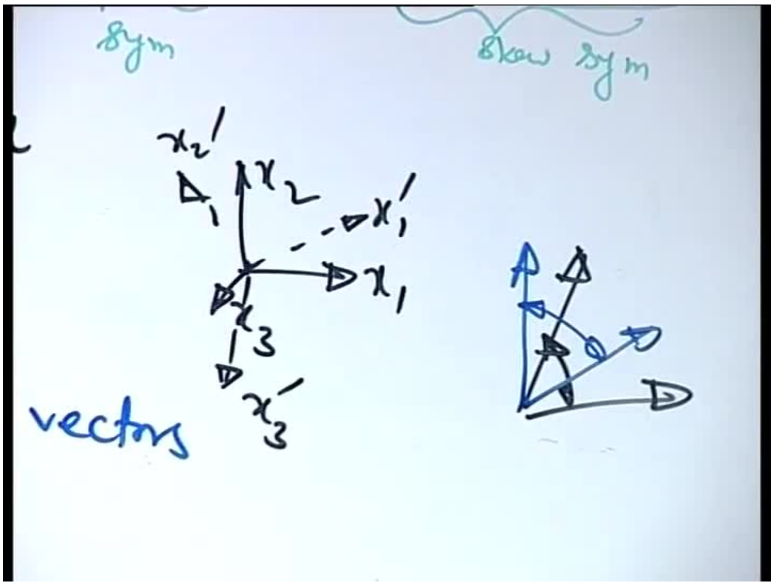

Now, if you have four vectors, remember that our objective is to formulate some directionally invariant; that means, if you have two vectors taken at a time, what you consider a directionally invariant quantity, the angle between the two vectors. Because if you rotate it, just think of this example of isotropy. Now, if you rotate these both, these two vectors by a fixed amount, then the angle between them does not change. So, if you consider such rotational invariance, you can ensure that by taking two vectors at a time and taking the dot product. Because if you take the dot product, the dot product depends on cosine of the angle between the two vectors.

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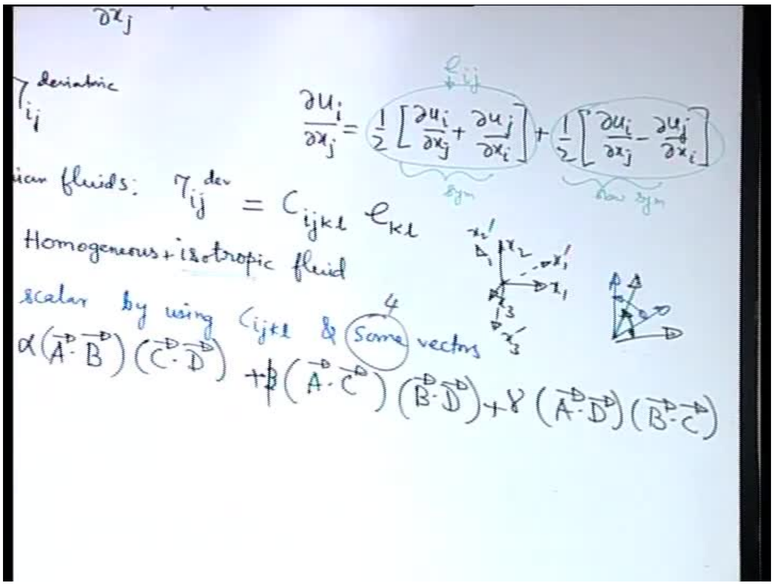

So, keeping that in mind, what you can do is, you can write this as some combination of, so how many such dot products are possible, you can take A with B, you can take A with C and, you can take A with D. And A with B and B with A are the same because A dot B and B dot A are the same. So, you can take A dot B, then automatically you have C coupled with D, then A dot C B dot D and A dot D B dot C. So, it is a linear combination of these three, so we just give these some names with some multiplying scalar coefficients. So, this is alpha, let us say this is beta and this is gamma, where alpha, beta and gamma are some scalars. So, we can make a simplification in the next step.

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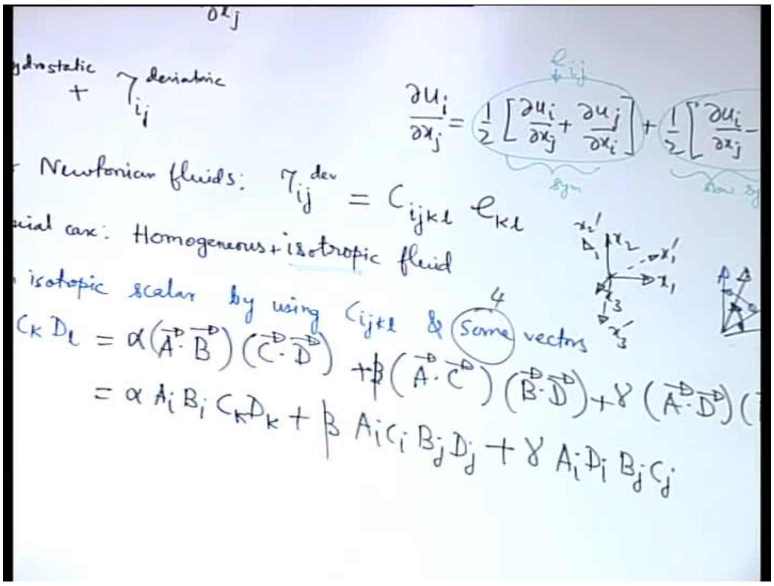

We can write this as A i B I, dot product of two vectors means what, their corresponding components are multiplied with each other. So, i th component of A is multiplied with i th component of B. And because it is a repeated index, it is summed up over i equal to 1 2 3. So, A i B i then C k D k plus A i C i B j D j, remember this j or i these are dummy. So, in place of j, you could have written k, l whatever.

Now, next to compare the left hand side with the right hand side, we must try to bring in terms like A i B j C k D l. Here it is A i, but B is not j B is i. So, we are interested to convert B i to B j.

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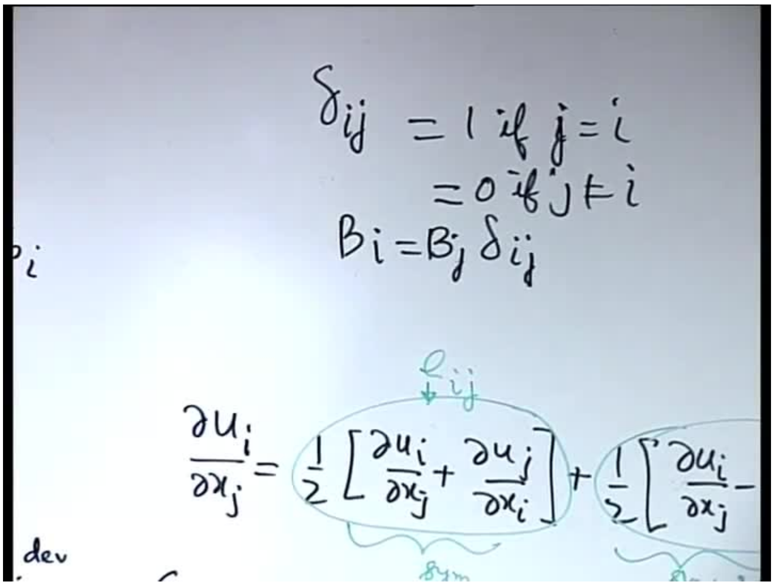

For that, what we do is, we commonly use the Kronecker delta, alternating the Kronecker delta tensor; that is, delta ij. So, what it basically does? It is equal to 1, if i is equal to j or if j is equal to i rather, is equal to 0, if j is not equal to i. So, you can write for example, B i as B j into delta ij because only when j is equal to I, you will have this as 1, and then it will be B i equal to i, for all j not equal to i, this will be 0. So, we consider such type of transformation and let us write accordingly these different terms.

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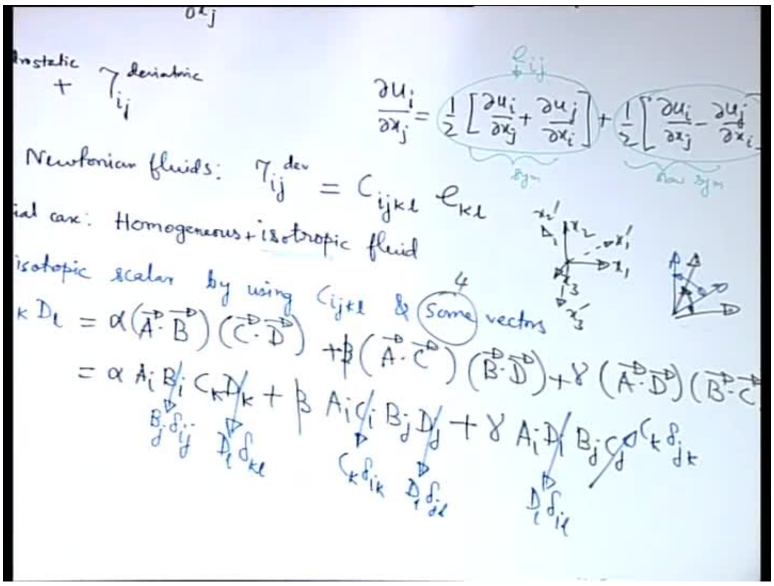

So, this will be B j into delta ij. Let us keep C k because it is also there in the left hand side. To convert D k to D l what you have to do? D l into delta kl. So, you can see it switches index, it switches from k to l. You can also write delta l k, because it is a symmetric tensor, delta ij and delta ji are the same. Similarly, for the next term, you want to change C i to C k. So, it is C k delta ik, and you want to convert D j to D l, so D l delta jl. Then in the last term you want to convert D i to D l, so D l delta i l and C j to what? C j to C k. So, C k into delta jk or kj whatever. So, if you compare the left hand side and the right hand side, you can see left hand side is C ijkl A i B j C k D l, right hand side alpha into delta ij delta kl into A i B j C k D l, similarly the other terms.

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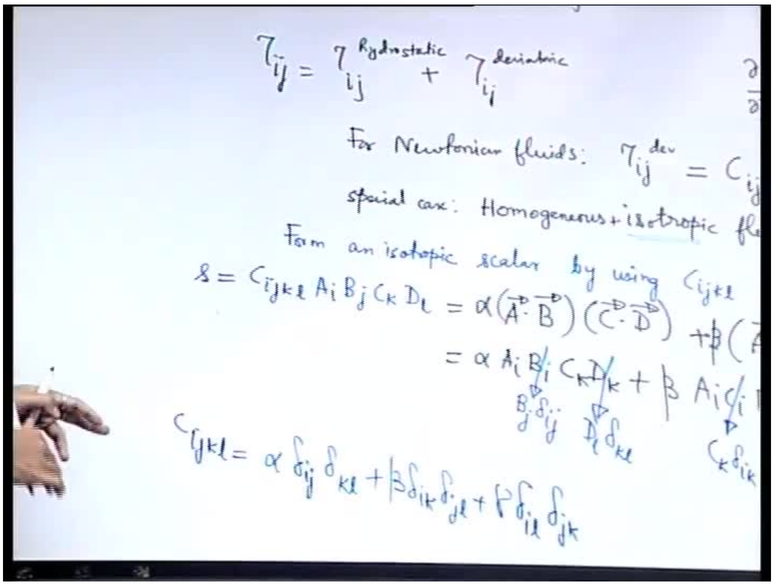

So, you can sort of as if cancel A i B j C k D l from both the sides and you come up with C ijkl is equal to alpha delta ij delta kl plus beta delta ik delta jl plus gamma delta il delta jk, where alpha, beta, gamma are position independent constants. So, you can see that we have utilized the concept of homogeneous by considering alpha, beta, gamma to be position independent and isotropic, and magically 81 constants have now boiled down to three independent constants, alpha, beta and gamma. Now, we can reduce them further. How we can do that, let us see.

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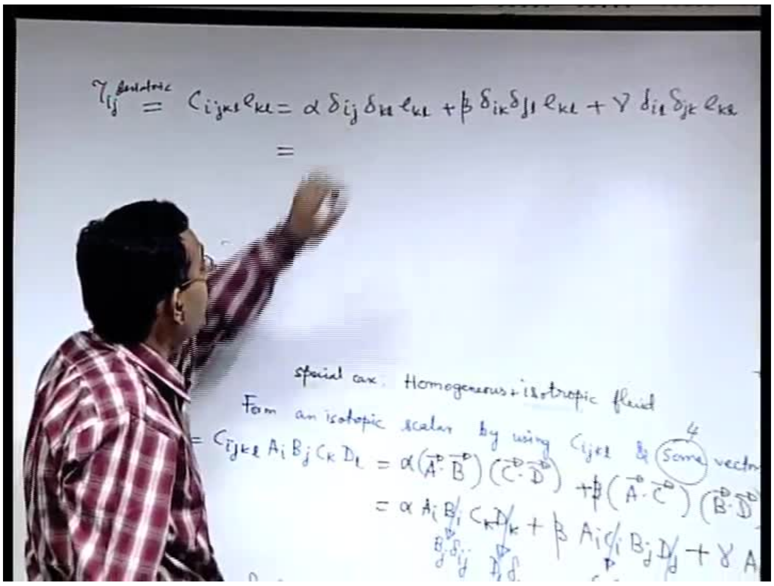

So, you have tau ij deviatoric is equal to C ijkl into e kl, that is alpha delta ij delta kl into e kl plus beta delta ik delta jl e kl plus gamma delta il delta jk into e kl. So, let us simplify this one. Delta ij, let it be as it is because i and j are free, you should not disturb i and j, summation is over the other indices, but summation is not over i and j. So, you can play with k and l. So, here you can see that you have a delta kl. So, when l equal to k, then only is it non-zero, otherwise it is 0. So, when l is equal to k this will become e kk.

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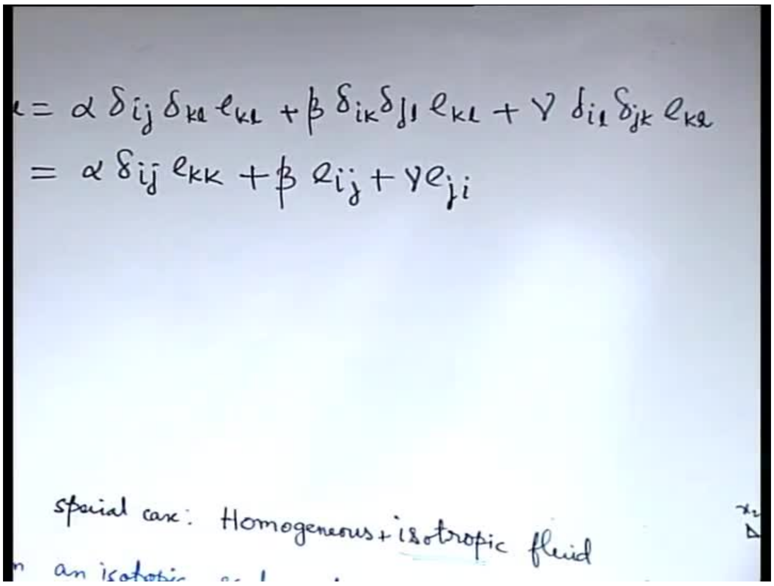

So, this becomes alpha delta ij e kk, then plus beta. Here also delta ik and delta jl, when k equal to i and l equal to j, then only these delta are non-zero. In fact, they are 1. So, k equal to i and l equal to j will make it e ij, so beta e ij. Similarly, here k equal to j and l equal to i, so it will make it e ji, but e ij and e ji are the same, so you can write this as 2 beta e ij. Sorry that will come in the next step, let us not jump steps. So, let us, it will eventually come like that, but let us write it as plus gamma e ij or e ji, if you want to write it as e ji that is also all right. Now, we can easily show that beta equal to gamma. How we can do that?

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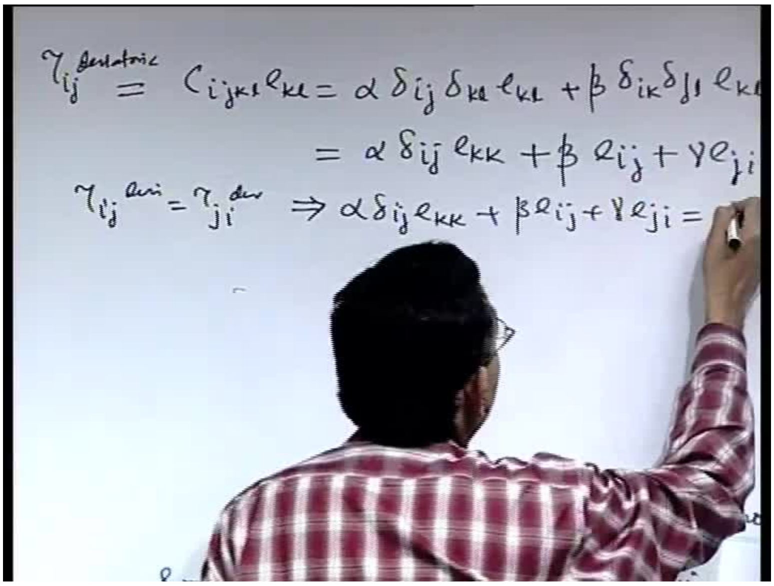

If you consider the symmetry of tau ij deviatoric, that tau ij deviatoric is equal to tau ji deviatoric. Stress tensor is symmetric and it's hydrostatic and deviatoric components themselves are individually symmetrical.

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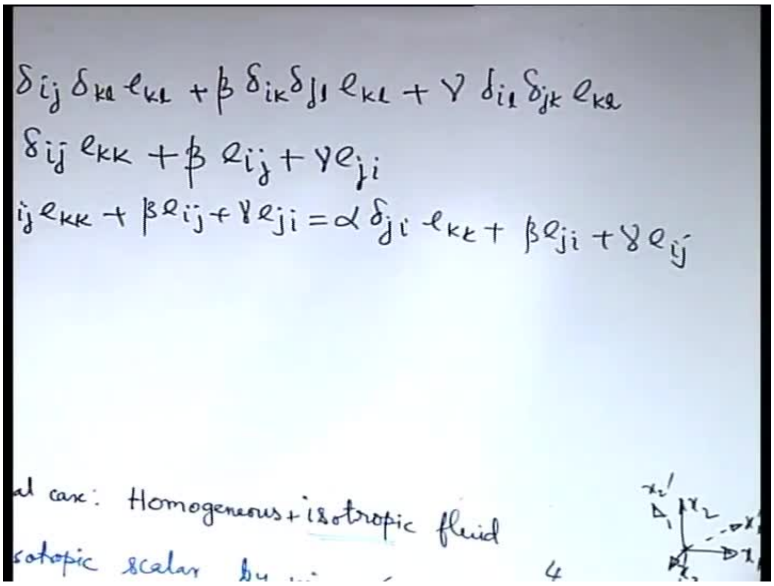

So, that means, alpha delta ij e kk plus beta e ij plus gamma e ji is equal to, just swap i and j, so alpha delta ji e kk plus beta e ji plus gamma e ij.

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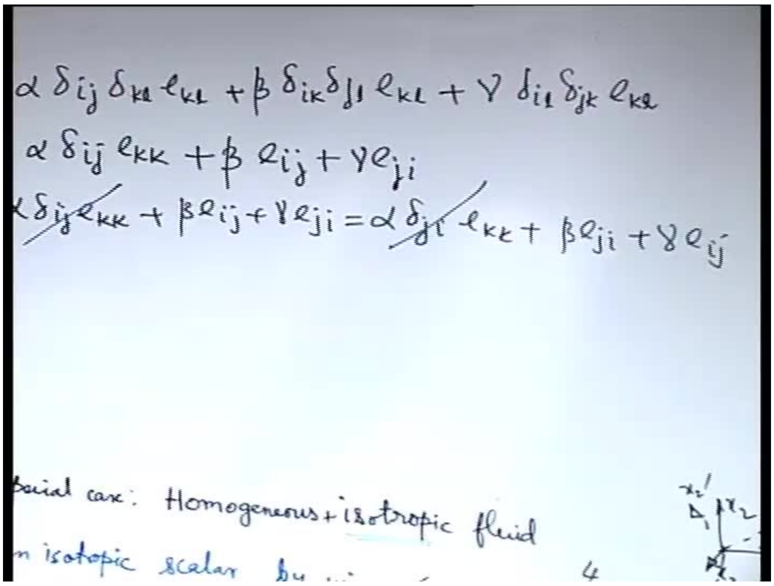

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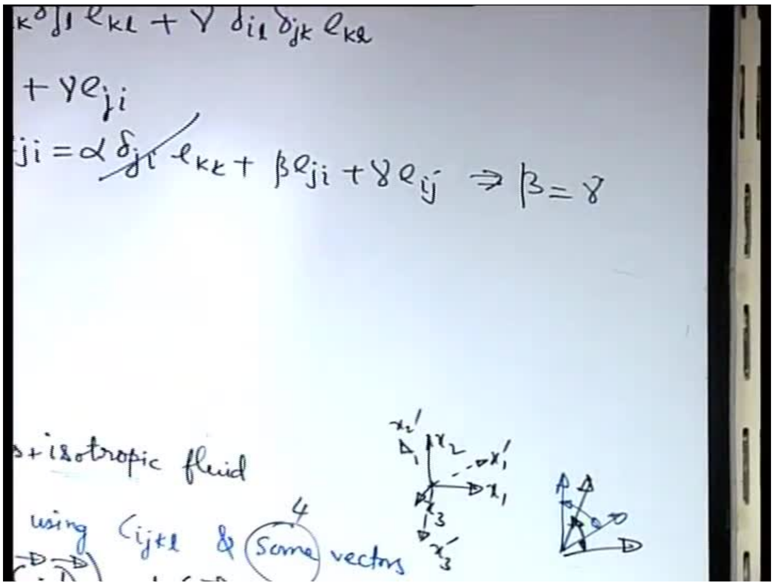

And delta ij and delta ji are the same, so these will cancel, so that from this it follows that beta is nothing but equal to gamma. So, you are now left with two independent constants, which you have come up by utilizing the symmetry of the stress tensor.

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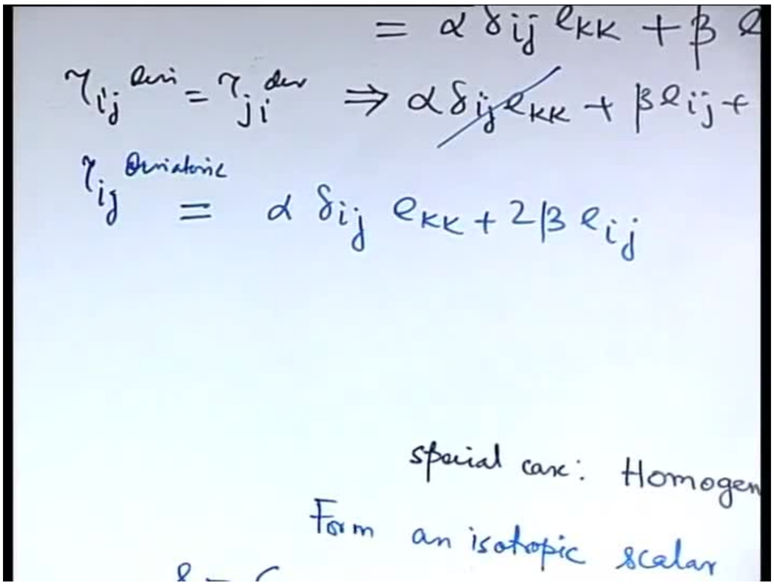

So, tau ij deviatoric is equal to alpha delta ij into e kk plus 2 beta e ij. So, this alpha and beta should have some physical implications. What are their physical implications, let us try to understand that by relating the stress with the deformation.

So, first of all let us consider beta. So for beta, we can see that this is the deviatoric stress tensor, e ij is the rate of deformation. Physically for a Newtonian fluid, you have the deviatoric stress tensor proportional to the rate of deformation or linearly related to the rate of deformation, and that if you express it by a single material property, that material property turns out to be the viscosity of the fluid, just like tau equals to mu times the rate of deformation.

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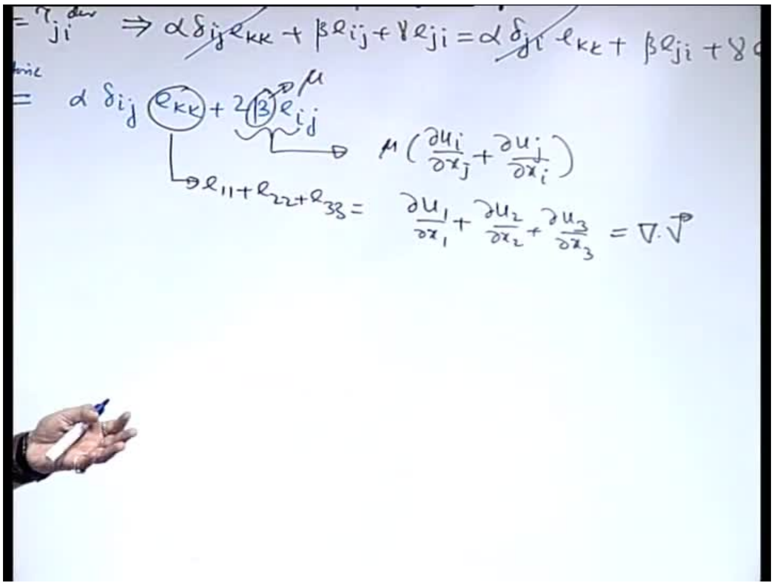

So, the same thing here you are extending with the index notation. So, then this beta becomes nothing, but the viscosity of the fluid mu. Remember 2 e ij is what? So, this term becomes mu into this one because e ij was half of this one. So, it is just like mu du dy because it is a multi-dimensional flow, so you have both u i x j and u j x i derivatives. So, this is related to the deformation of the fluid, this term physical is related to the deformation of the fluid.

This term is physically related to what? What is e kk? e kk is e 11 plus e 22 plus e 33,so it is, right, so it is nothing, but the divergence of the velocity vector which is an indicator of the volumetric deformation of the fluid. So, as you recall that, if the fluid is not deforming in terms of its change in volume; that is. It is an incompressible flow, then you have divergence of the velocity vector equal to 0. So, incompressible flow means fluid element will not change its volume and it’s volumetric strain is 0.

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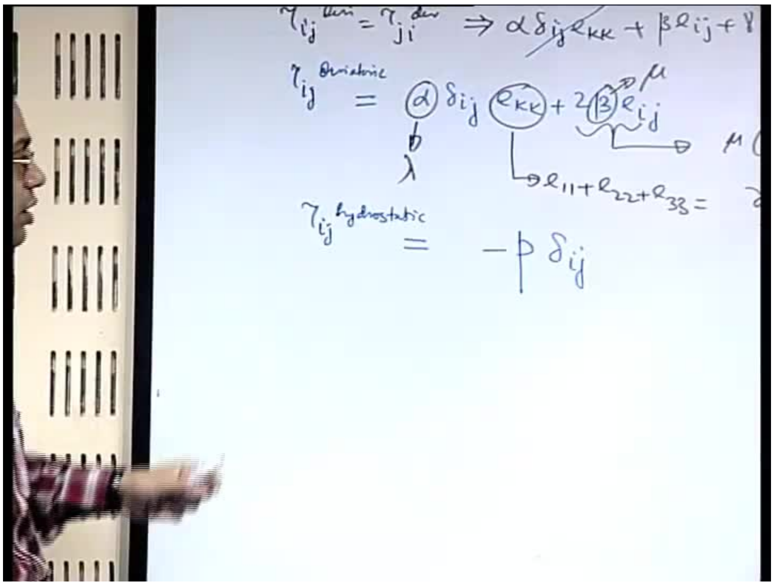

So, just like this is related to shear strain or angular strain, this is related to volumetric strain. And this is given just a different symbol in most of the textbooks as lambda, which is also called as a second coefficient of viscosity. Just like this is a viscosity or coefficient of viscosity, this is second coefficient of viscosity which is related to the volumetric dilation of fluid elements. So, we now have an expression for tau ij deviatoric. Now, what is an expression for tau ij hydrostatic? So, it has it’s magnitude as minus p which is always normal, so minus p into delta ij, so that, only when i is equal to j, that is only when you are considering a normal direction, you have the hydrostatic component, there is no shear component of the hydrostatic state.

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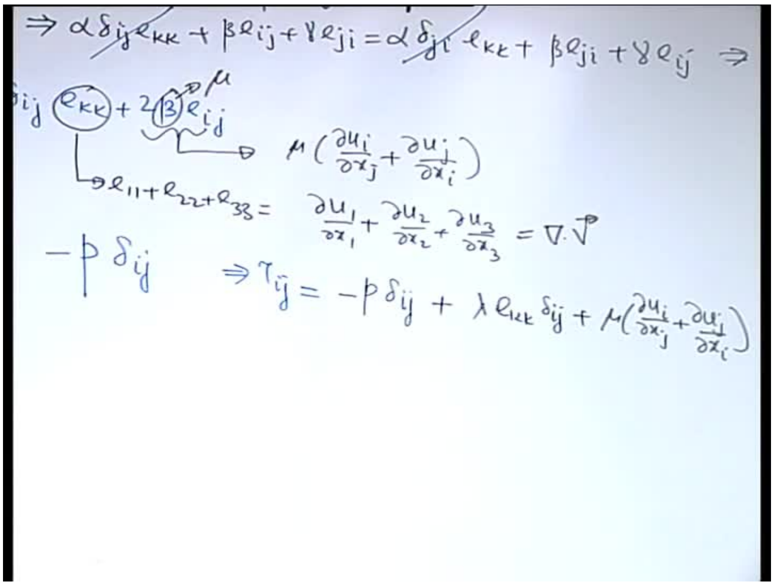

So, the total tau ij is equal to minus p delta ij plus lambda e kk delta ij plus mu into this one. See, why we have done this exercise? When we were writing the Navier's equation in terms of tau ij, six components of tau ij were unknown to us, so we needed to have additional equations on tau ij to close the number of equations and match the number of equations with the number of unknowns. So, now we are able to write the tau ij in terms of the primitive variables, the primary variables which are the velocities, of course gradients of velocities and while doing so we are coming up with an additional quantity, which is the pressure of the fluid.

So, when it was written as del tau ij in governing equation, tau ij six independent components were unknown, now we can express each of those components in terms of velocity and pressure. So, that is the final outcome from the exercise of writing a constitutive relationship for homogeneous isotropic Newtonian fluids. So, this is the expression of constitutive relationship for that type of fluid.

Now, next question is that, well what happens for the normal and the shear stresses separately, or if you just consider the normal stress, how do you relate the quantities lambda and mu by considering the normal stresses. Let us try to do that exercise.

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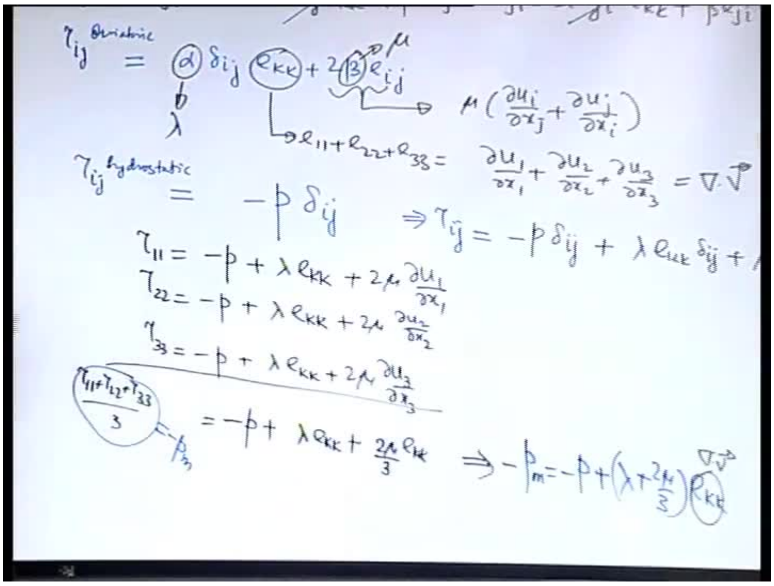

So, let us say that we are interested in tau 11. Tau 11 is equal to minus p plus lambda, see we are considering normal stress for assessing lambda because for lambda is not relevant for shear, because when j equal to I, then only this term is there, otherwise this term is not there, plus lambda e kk plus 2 mu. Similarly, let us write tau 22… and tau 33… Let us find the arithmetic mean of tau 11, tau 22 and tau 33. So, we add these 3 and divide by 3. So, tau 11 plus tau 22 plus tau 33 by 3 this equal to, right hand side it will become minus p plus lambda e kk plus 2 mu into e kk, because e kk is del u 1 del x 1 plus del u 2 del x 2 plus del u 3 del x 3 by 3, because you are dividing the equation by 3, dividing the sum by 3.

So, you can simplify this. Before simplifying, we just give a name to this one, this we call as minus of mechanical pressure. We can see in the right hand side there is another pressure tau which appears this, we call as thermo-dynamic pressure, that is a pressure which satisfies the equation of state of the fluid. So, here you have the thermo-dynamic, pressure, here you have the mechanical pressure, and they are related in this way. So, minus mechanical pressure is equal to minus thermo-dynamic pressure plus lambda plus 2 mu by 3 into e kk.

In vector notation, e kk is nothing but divergence of the velocity vector. Now, let us try to discuss a bit on the mechanical pressure and thermo-dynamic pressure, what are these all about. So, what we can very easily assess that, no matter whether it is mechanical pressure or thermo-dynamic pressure, pressure of a fluid inherently is a outcome of intermolecular interactions. So, the molecules themselves have some energy which you can broadly classify as translational energy, vibrational energy, rotational energy like that. Now, there are fluids for which you have all these modes of energy, there are certain special fluids for which you may have only some restricted modes of energy. Irrespective of that, when you consider the mechanical pressure, it considers only the translational mode of energy of the molecules. Whereas, when you consider the thermo-dynamic pressure, it considers all sorts of modes of energy, so translational, rotational vibrational like that.

Now, there are situations when these two are the same. When these two are the same? So, let us consider a process, when you are changing the thermo-dynamic state of the system. How you can change the thermo-dynamic state of the system? Let us say you heat it to change it's temperature. Now, it's pressure will try to adjust, question is how fast or how slow. How fast the pressure will try to adjust or how slow the pressure will try to adjust? It depends on the characteristic time scale of response of the system as compared to the characteristic time scale of the disturbance that is imposed on the system.

So for example, let us say that the temperature of the system is changed at a very rapid rate. So, what will happen is that, the system at each and every instance will not be able to achieve local thermodynamic equilibrium by responding to that quick change.

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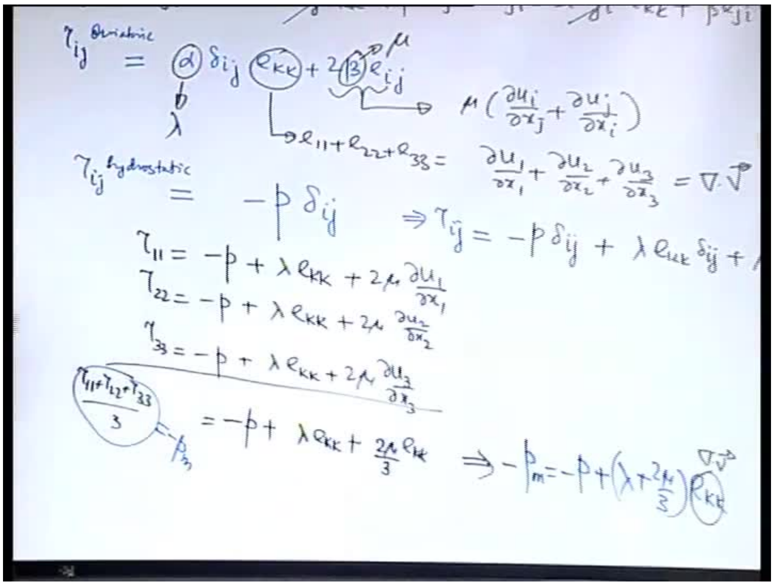

So, it is system's response time, it is a threshold amount of time, and if the disturbance time scale is faster than that, then the system cannot adjust to that immediately. And then it may not be able to equilibriate in terms of mechanical pressure and thermodynamic pressure. If you allow it sufficient time, you have all modes of energy eventually manifested in terms of the corresponding translational mode, so that you have the corresponding mechanical pressure, which sort of can be measured by a pressure measuring device.

Now, if you do not allow that sufficient time or if the process time scale is very fast. As an example, let us say that you have a bubble which is expanding and contracting alternately at a very rapid rate, suddenly it is expanding, very fast it is contracting, again it is expanding like that. So, if such a process occurs, then the equilibrium between mechanical pressure and thermo-dynamic pressure cannot take place and then mechanical and thermodynamic pressure will not be the same.

But for most of the processes that we talk about, we have mechanical pressure and thermo-dynamic pressure are the same when the system responses very quickly as compared to the time scale of imposition of the disturbance. Usually the disturbance is not imposed at a very rapid rate, at least not at a rate which sort of goes beyond the characteristic response time scale of the system. So, for most of the practical cases, we have mechanical pressure equal to thermo-dynamic pressure.

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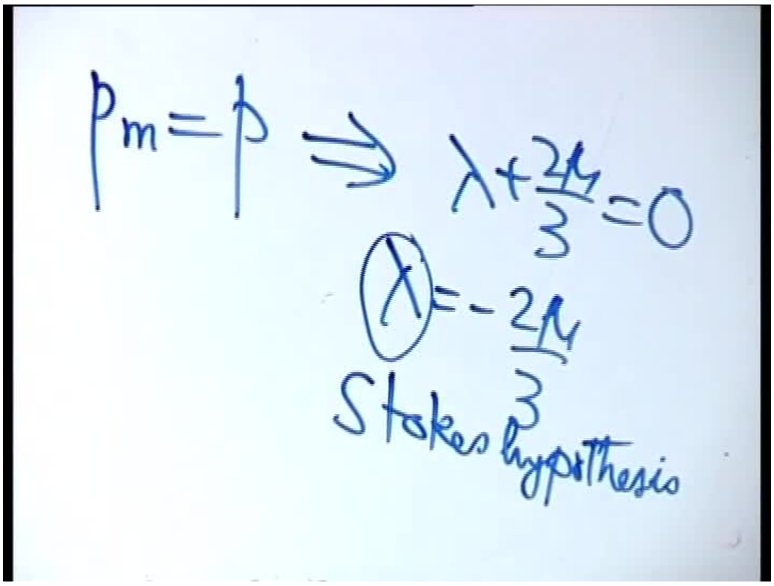

So, if we have mechanical pressure equal to thermo-dynamic pressure. Remember it is based on most of the practicalities of most of the processes, it is a reality for most of the cases, but you cannot just prove it by saying that it is valid for most cases or all cases, it is just from the physical understanding of the time scales of events and processes we can make an argument that mechanical pressure and thermo-dynamic pressure they are the same for most other processes. So, if they are the same, that means that you must have lambda plus two-third mu equal to 0. Because for a general case, divergence of the velocity vector is not equal to 0, it is 0 only for incompressible fluids. So that means, you have lambda is equal to minus two-third mu, which is called as Stokes hypothesis, and any fluid which is obeying that is called as a Stokesian fluid, just like a fluid which obeys Newton's law viscosity is called Newtonian fluid, any fluid which obeys this particular behavior or particular relationship is called as a Stokesian fluid.

Of course, you can see that mechanical pressure and thermo-dynamic pressure are identical under certain trivial conditions. What are those trivial conditions? For example, if it is incompressible flow. If it is incompressible flow, it does not matter whether lambda plus two-third mu is 0 or non-zero, because divergence of the velocity vector is 0. So for incompressible fluid, you have the mechanical pressure and thermo-dynamic pressure are identically equal. For dilute mono-atomic gas, you have only one mode of energy. So, then you have mechanical and thermo-dynamic pressure to be identically the same. And then you have the Stokes hypothesis, not hypothesis, but exactly provable theory. But otherwise, it is a hypothesis rather than an exact sort of provable theory, but this hypothesis works for almost all practical cases that we consider for solving problems. So, we can say that, like for Stokesian and Newtonian fluids you can use this stokes hypothesis. Now, let us try to write or complete the description of the governing equation, but before doing that let us focus a bit of attention on this parameter lambda.

You can clearly see that lambda is a negative quantity, because viscosity you have viscosity as a positive quantity. So, minus of that is a negative quantity. So, what does it mean? See, this quantity lambda is related to the volumetric deformation. So, if e kk is positive, you can see that because of negative lambda the corresponding contribution to tau 11 is negative. That means for a fluid element which is already expanding, the proportional enhancement of stress to expand further is actually not an enhancement, but a reduction. So, if it is already expanding, you require less stress to expand it further, that is what in a simplified form it reflects. So, with these considerations let us now write the final governing equation step by step.

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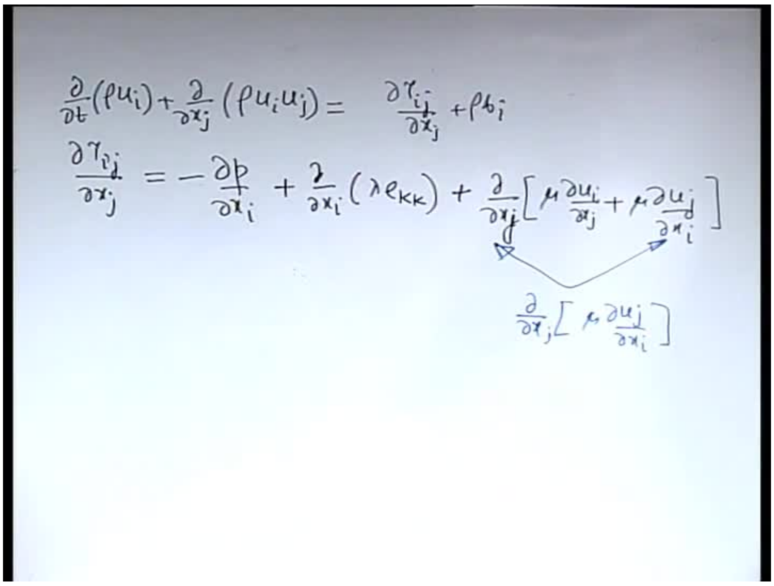

So, let us start with the Navier's equation that is… So, del tau ij del xj, tau ij is minus p delta ij. So, if you partially differentiate this with respect to xj, what will happen? Delta ij and xj both are there. So, delta ij will become non-zero only when j equal to i. So, this will become minus del p del xi. Then next term, there also only partial derivative with respect to x i will remain plus… sorry this is xj. So, we can simplify the last term further. Let us combine these two, so this is…

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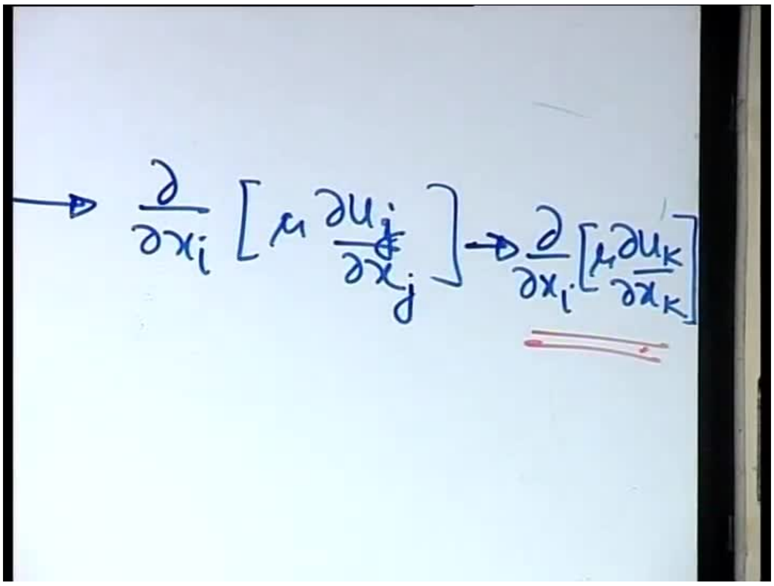

If we assume the partial derivatives to be continuous, then we can first differentiate with respect to x i and then with respect to x j without altering the result. So, this we can also write… So, we have swapped this x i and x j, this is as good as this one. Because j is a repeated index, it is a dummy index, in place of j you could write k, l, m, n whatever. Why we are writing it in this way is because we want to combine this particular term with this particular term, this is nothing but equal to e kk. So, keeping that in mind we can write the governing equation as…

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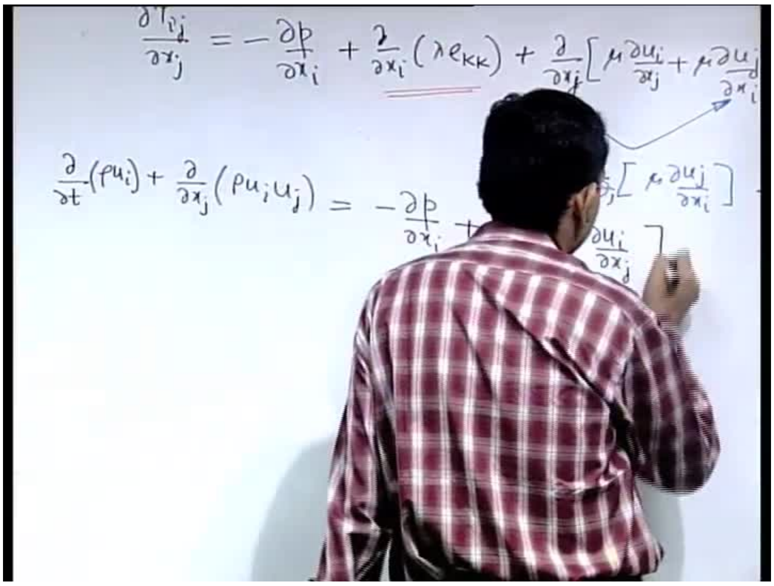

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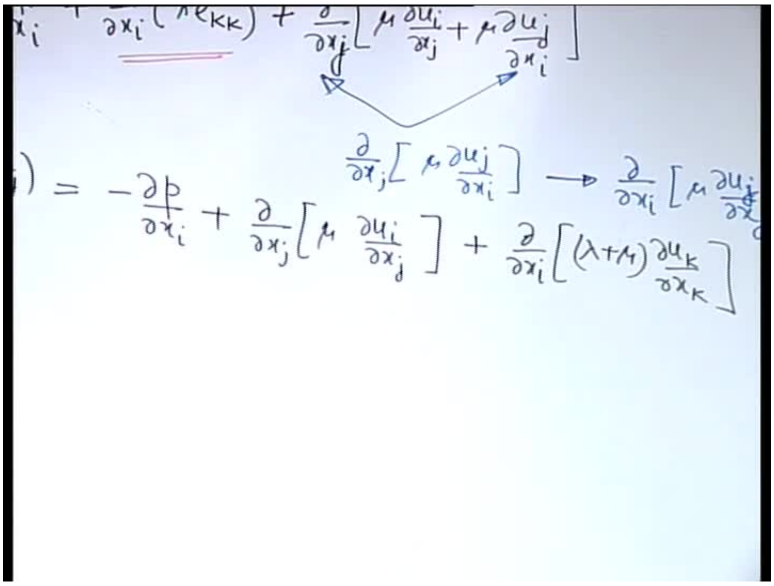

Where for Newtonian and Stokesian fluid, lambda is equal to minus two-third mu, but this is general this is independent of how lambda is related with mu. So, you can clearly see that the last term becomes identically 0, if it is an incompressible flow, only for compressible flows, this extra term in the governing equation will appear. And it is directly related to the rate of change of volume per unit volume or volumetric strain rate.

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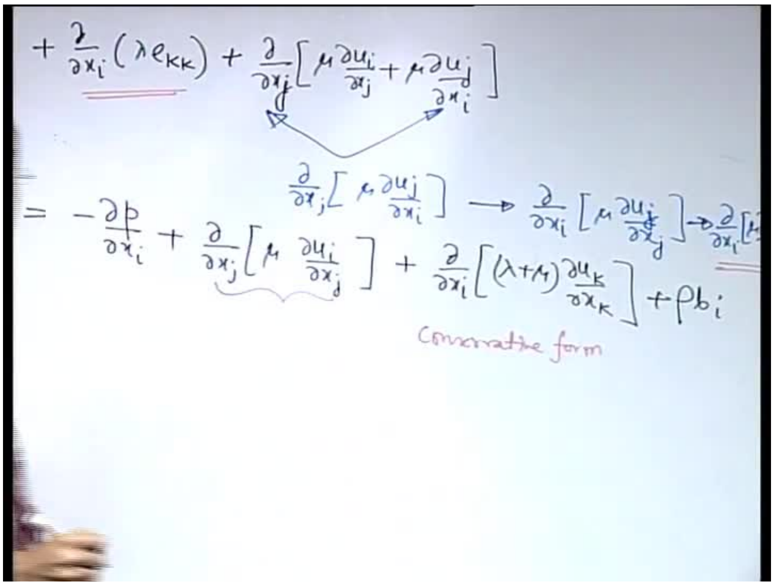

Now, you can see first of all certain things like this is not a single term, it is a summation because you have the index j repeated. So, you have an invisible summation over this with j equal to 1 2 3. So, you can clearly make out that this is the pressure gradient term, this is the viscous term, this is the volumetric dilation term and this is the body force term. And this form of the equation, this equation of course is known as the Navier-Stokes equation and this form of the equation is called as conservative form of Navier-Stokes equation.

It is very important to appreciate that, because in computational fluid dynamics we will work very often with conservative forms of the equations, because conservative forms directly talk about the conservation in a mathematical sense. So, what we have done is, we have started with a physical principle of linear momentum conservation and this form has evolved, if you consider the left hand side, this form has evolved by considering the conservation of linear momentum and there you had an unsteady term and you had an outflow minus inflow term which was converted into the corresponding flax, and that is how corresponding divergence, and that is how you have come up with these two terms.

Now, this is called as a conservative term, a conservative form of the equation and how you can make out whether it is a conservative form or not. If you see rho inside any expression, you will understand that it is a conservative form. So, this is a conservative form of the Navier-Stokes equation.

How can you convert the conservative form to a non-conservative form? You have to then simplify the left hand side. Usually for computational purpose we use the conservative form, because it directly gives you the sense of physical conservation through mathematical expressions. Whereas, for analytical work or for hand calculations, we usually use the non-conservative form. So, how can we do that?

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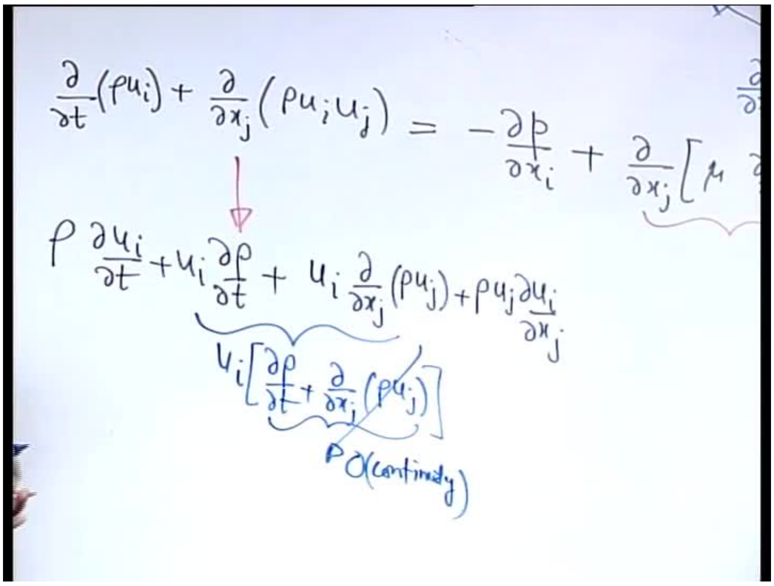

So, here if you want to simplify it, you can write it as rho now… You combine these two terms. So, what we have basically done is, we have basically considered the product rule of derivative for the second term and written it as sum of two terms Similarly, the first term also. Now, if you combine this… which is identically equal to 0 by the continuity equation. So, the left hand side can also be written as…

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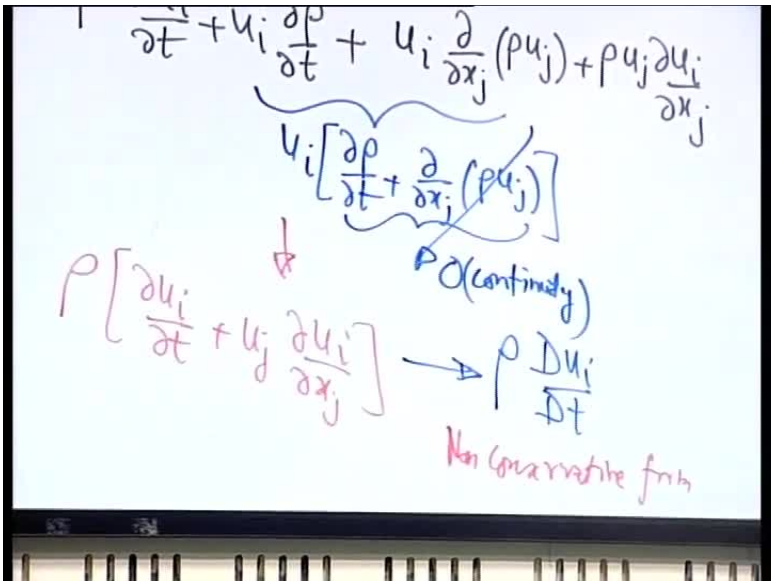

You can see the rho goes out of the expression as a simplification, not because it is a constant, here we have not used anywhere that rho is a constant, it can be a variable. Because of simplification using the continuity equation, this rho has come out of the derivative. So, this is called as a non-conservative form. So, when rho is inside the derivative, it is called as a conservative form. Conservative form simplified with the aid of the continuity equation will give something which is called as a non-conservative form. And this non-conservative form you can write this in terms of the total derivative, you can write this as rho capital DD t of u i.

The total derivative of velocity is sort of the total acceleration, it is the sum of the temporal component of acceleration and the convective form of the acceleration. So, it is as if mass into the total acceleration per unit volume. Because it is Newton's second law expressed for a control volume, the right hand side should also be force per unit volume. So, all the terms in the expression in the right hand side are force per unit volume, this is force due to pressure gradient per unit volume, it is force due to Viscous effect per unit volume, this is force due to volumetric dilation per unit volume and this is body force per unit volume.

So, it is basically a simplified version or simplified understanding to state that it is nothing, but Newton's second law of motion for a control volume expressed in a differential form. One important observation that we can make out of this equation is that, so you can write no matter whether you are writing it in a conservative or a non-conservative form, how many equations and unknowns you have. So, i is a free index, i equal to 1 will give x component, i equal to 2 y component, i equal to 3 z component. So, you have three equations. How many unknowns you have? You have four unknowns. So, three components of u; that is, u 1, u 2, u 3 and the pressure. So, to close this system you require another equation which fortunately is provided by the continuity equation, but we have to remember that continuity equation explicitly does not contain pressure.

So, that is one of the challenges in CFD, where you are given the task of solving these equations numerically, where pressure is an unknown variable, but you do not have an explicit governing equation for pressure. And we will see later on, when we see how to numerically solve the flow field, that how to get rid of this problem in a somewhat innovative way.

So, to summarize, we have seen that how to derive the governing equation starting from the Reynolds transport theorem. We have considered one example of continuity equation, we have considered an example of momentum equation and we have derived the special of momentum equation for Newtonian Stokesian fluids and we have come up with a Navier-Stokes equation as a consequence.

And in the next lecture we will see that, using this same philosophy how can you derive the energy equation also, which is another important equation in thermo-fluid sciences. And finally, to see that even if it is not a continuity equation or a momentum equation or an energy equation, but any other equation which talks about conservation of some quantity, then how can we write all these equations in a common mathematical structure or the mathematical form. Why we are interested in doing that? Because once we do that, once we figure out that we can cast all these equations which talk about conservation of something, these need not necessarily follow from fluid dynamics or thermo-dynamics or heat transfer, these equations may follow from electro-magnetic or electro-hydro-dynamics or whatever, but if we can cast these equations in a common generic form which is a signature of the conservation of that particular physical quantity, then it will be possible for us to apply a generic mathematical principle to solve these equations.

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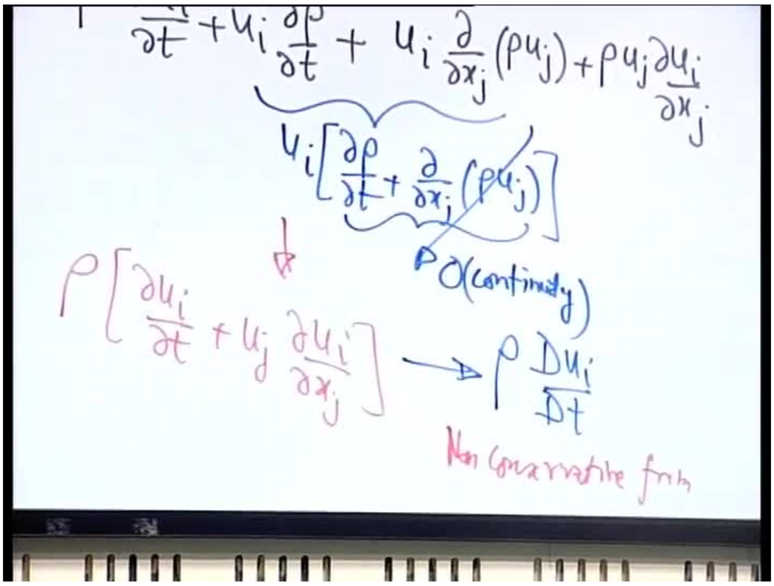

Because if we can develop a method of solving numerical solution of these types of partial differential equations, which sort of represent the conservative nature of a particular physical system, and then we can use that method for any type of equation, then the method will be just like a mathematical tool, it will not understand whether you are talking about the physics of electro-magnetism or physics of heat transfer or fluid mechanics, it will be the responsibility of the analyst to interpret and it say what is the physical situation pertinent to that particular condition. So, we stop here for this lecture and in the next lecture we will continue with the subsequent discussion on the conservative equations. Thank you.