Computational Fluid Dynamics

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Lecture No. # 07

Approximate Solutions of Differential Equations: Error Minimization Principles

So far, we have discussed various types of partial differential equations. Now, our next objective will be to go step by step towards solution of the different conservation equations. To do that today, we will learn one particular principle which we can use, and that principle is nothing but error minimization principle; so, its turns from the fact, that if you have an approximate solution of a differential equation, remember that we are not really looking for analytical solutions through computational techniques. So, anyway we are looking for some approximate solutions. So, when we have an approximate solution - the approximate solution will involve incur certain errors. And very good approximate solution will be that one, which in some way tries to minimize the error. And that is why error minimization happens to be one of the key principles, based on which many of the numerical methods are founded.

Before going into that details, we will first look into one important consideration of error minimization principle, and that comes through the consideration of variations. And we will do that with a brief introduction to the calculus of variations. So, we will not formally try to learn the in depth understandings of calculus of variations, but we will do that, what to whatever extent it is necessary in in the context of this numerical solutions that we are going to learn in in our subsequent lectures.

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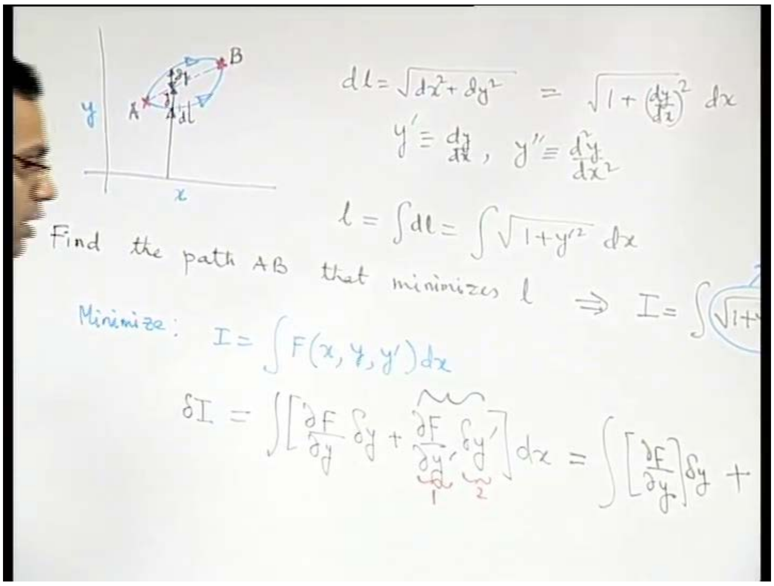

Let us say, let us start with an example: I mean, we will not formally start with with the rigor, but we will start with an example, and see that how we can come up with rigor through that example. Let us say that you have two points A and B. These are two points. The objective is to find out a path, that is the least distance between A and B. So, you can go from A to B in various ways, this may be one path, this may be another path, and of course, all of us know that the straight line path is the shortest path. But how we can formally pose this problem. So, let us say that this is a x, y plain. So, you have this as x axis, and this as y axis. And let us say that we are interested about the total length from A to B.

So, if you take a small length let us say, let us let us take any any of the curves, let us say that we take this d l. So, once we take this d l, let us say that we are interested to find out the integral of the d l from A to B, which gives the total length. So, as we can write d l is square root of dx square plus dy square, because for a very short length the curve is almost like a straight line. So, it is some of by the Pythagoras theorem, you can write this as the sum of d l square, as sum of dx square and dy square. So, you can write this as square root of 1 plus dy dx.

So, we use the short notation like y dash equivalent to dy dx, y double dash is equivalent to d square y dx square like that. These are that sorts of notation that we will be using, which are anyway common notation in calculus. So, our objective is to first find out what is the total length? That is integral of d l which is integral of this one. And the problem statement therefore becomes, find the path A, B that minimizes l. That is the formal problem statement or that means, here equivalently it minimizes this integral.

This integral involves some function, which in general could be a function of x, here x is not there, but in in in in some other example x could appear, it could involve y, it could involve y dash, and it could involve even higher order derivatives. But let us just consider the case where this function itself could be a function of x, y, y dash, out of which x is the independent variable. We can clearly make out that this y, and y dash - these are themselves functions of x. This is the independent variable.

Thus capital F is a special type of a function, which involves other function. So, it is a function of some function, because it is a function of y or y prime which are themselves functions of x. So, these types of functions which are themselves functions of other functions, these are called as functional. So, this is a functional. In a simple way, it is equivalent to function of a function. Of course, one can give more formal definition to functional, but let us try to just understand it in a simple way. So, what is our objective - our objective is to minimize this integral I. So, let us now try to formalize the problem through this example, and say that minimize.

Now, how do we minimize it. There are different ways in which one can do. So, here what we will do is, we will take a first order variation which we will symbolize as delta on I. So, what is this delta? This is like an arbitrarily small virtual change in I. Just like, you have a similar concept in in terms of virtual work in mechanics, that you have virtual displacement - that is a displacement which is not really taking place, but it could have been allowed, consistent with the constrains of the physical system. So, similarly this is a virtual change in I, which we symbolize as delta I. So, how can we write this. Remember F is a function of these variables out of which, we can have virtual changes in only dependent variables, we cannot have virtual changes in independent variables. Because let us try to understand this carefully.

You are changing x, and that is that is really a differential change by which you are going from A to B. So, dx, dx, dx in this way; so far each and every change in x, you could have different possibilities with changes in y. So, your y could be this - it could even be this, it could even be this for a given x. So, if your actual path is the straight line, any other path is the actual path plus a variation in the actual path. So, if this is your correct y, then you can call this as an equivalent delta y, if this is small. So, you are having a small y possible, and if you mean if you if you, now account for the fact, that these are variation over the actual solution, you can minimize this one and come to the actual solution, which is the straight line path.

So, you can have variations over y; similarly, you can have variations over y prime, but you cannot have this variation over x, because x is the independent variable. You are changing x as per your wish, and then seeing what is the corresponding variations in y and y prime. So keeping that in mind, you can write this as…

Remember that this operators del, and delta these are mathematically behaving in a similar way. So in in in fact, d del delta all those mathematically behave in a similar way. They are conceptually different. Just like d is that exact differential - this del is the partial derivative, and this delta is a virtual change. So, these have different physical meanings, but mathematically these may be these may be treated in a in a similar way. So, that is how, it is just like the rule of partial derivatives that what we have used. Instead of dy it is delta y, because these are virtual changes.

Now, what we will do is remember this integration is from A to B from that, for the two end points. So, what are the boundary conditions which are there, the boundary conditions given are that at A you are having a specified y, at B you are having specified y. So, your problem is tied down with this constrains, that at these two points your y is specify. That means, the variation in y is 0; there is no variation in y, that we have to keep in mind. So, delta y is 0 at A and, B that is in this example. Of course, in other examples there could be other types of boundary conditions, that we will see subsequently, but this is one of the simple types of cases. Now, what we will do is we will try to simplify the second term by doing integration by parts.

This is the first term, and then for the second one we will consider one something as the first function, and something as the second function. So, first function should be such for which you will calculate the derivative. And second function which you will integrate. So, this is the first function, this is second function, because from here we want to bring down from y prime to y. So, we want reduce one order of derivative, and that is why this is the second function.

So, let us write that first function into integral of the second…

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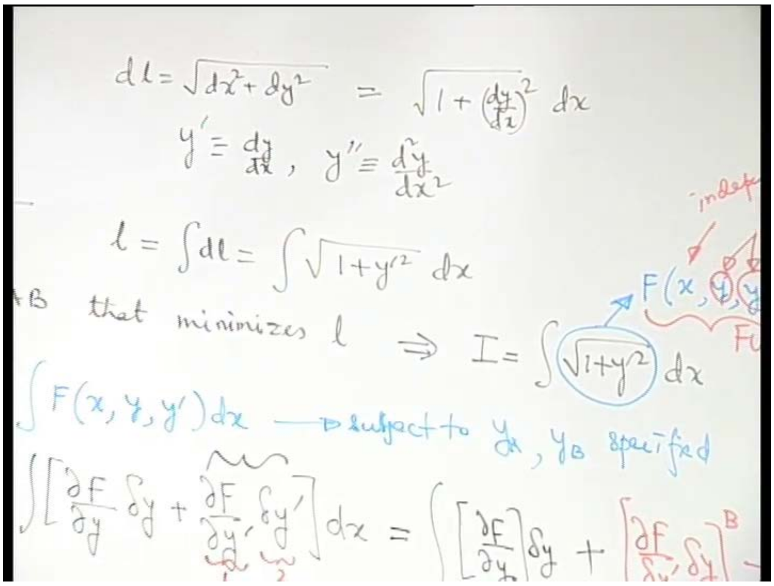

Minus integral of derivative of first into integral of this again, sorry into delta y times d x. So, you can see from this example, that these d and delta they are sort of commutative. So, it is delta of dy dx. So, that you can write as, as if d of delta y, delta x like that. So, you can interchange d and delta. That is that is very much possible. Now next, what we will do is we will use the boundary condition. So, we are minimizing this subject to what constraint or boundary condition y at A, and y at B specified. Since, y at A and B are specified, you have delta y at A, and delta y at B equal to 0. So, this term will be zero. So, you are left with what - you are left with…

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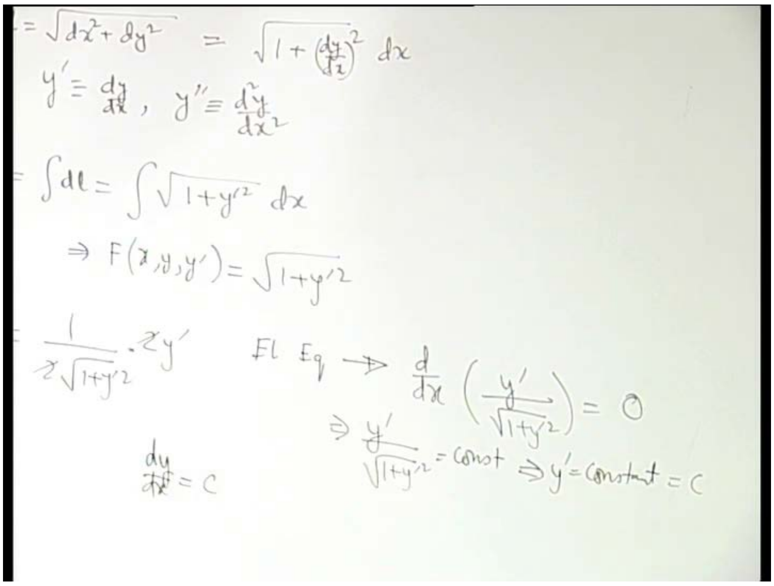

Delta I equal to this one. And for minimum I, it could also be for maximum I, whatever we generally can say that for extermination of I. It could be either maximization or minimization. You must had delta I equal to 0. Now, delta I equal to 0, and that has to be true for any arbitrary, delta I equal to 0 has to be true for any arbitrary delta y. When is it possible? It is possible, only when the integrand itself is 0. Otherwise, it is not true for any any arbitrary choice of this delta y. So, for minimization of I, you must have delta I equal to zero. That means, essentially…

Right. So, this equation is known as one of the constraining conditions for extermination of this functional, and integral integral of this functional, and this is known as Euler-Lagrange equation. Let us try to apply this Euler-Lagrange equation. For the minimization of the distance between the two points this example.

So in this example, you have I is equal to, that means F is equal to. So, partial derivative of F with respect to y is 0, partial derivative of F with respect to y prime. And essentially the Euler-Lagrange equation for this case will imply, that you have d dx of y prime by square root of one plus y square is equal to 0. Because the second term is only there, the first term is trivially zero. So, the second term is 0 that will mean, at y prime by one plus y prime square is a constant, which implies that y prime is another constant. So, let us say that is C; that means, dy dx equal to C. That shows that the path is a straight line, because straight line is a path for which dy dx is a constant. So, with this background, let us try to may be workout one or two additional problems to illustrate the use of similar principles.

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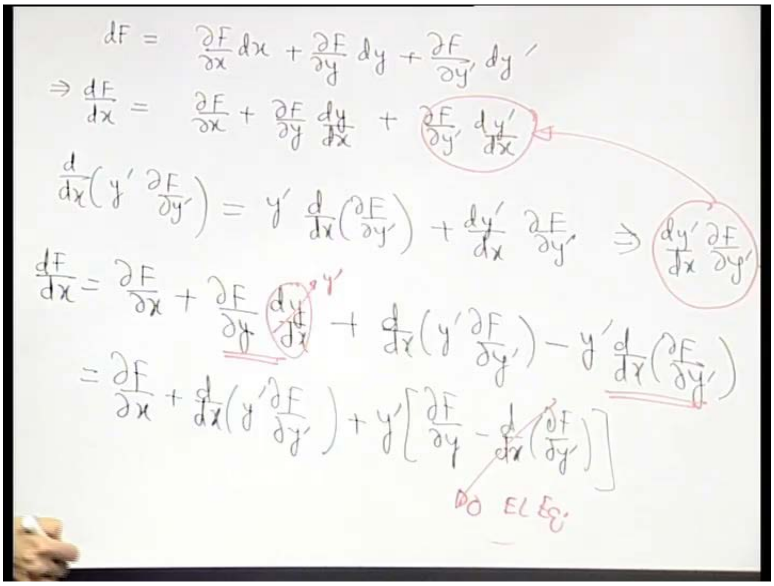

So, the problem statement is like this. Show that an alternative form of the Euler-Lagrange equation is given by dF dx minus d dx of y prime d del F del y prime minus del F del x equal to 0. We have another part of the problem, but that we will state after we complete this part, where F is a function of x, y, y prime. So, since F is a function of x, y, y prime - what is dF? We can use the rule of partial derivatives remember that these are not virtual changes, these are partial derivatives. We have use the rule of the partial derivatives. So, what is dF d x?

We can simplify the last term by noting that d dx of y dash del F del F y dash which is one of the terms, that we have to incorporate in the derived equation. This is nothing but y dash into d dx of del F del y dash plus dy dash dx into del F del y dash, which implies that dy dash dx into del F del y dash is equal to d dx of y dash del F del y dash minus y dash d dx of del F del y dash. We will incorporate this expression for dy dash dx into del F del y dash, here or substitute that expression here. If we substitute that expression here, then what we will get - we will get dF dx is equal to del F del x plus del F del y into dy dx plus in place of the last term plus d dx of y prime del F del y prime minus y prime d dx of this one.

We also have to keep in mind, that this dy dx is nothing but y prime. So, we can club up these two terms, and write that this is nothing but this one minus or rather plus, the combination of these two terms which are marked here in red. So, plus y prime into del F del y minus d dx of del F del y dash. The last term in the square bracket is 0, because this is the Euler-Lagrange equation. So, this is the Euler-Lagrange equation that is equal to zero. So, then we are left with the equation that we have to show if we bring, all terms in one side, that completes the proof. So, that is the first part of the problem.

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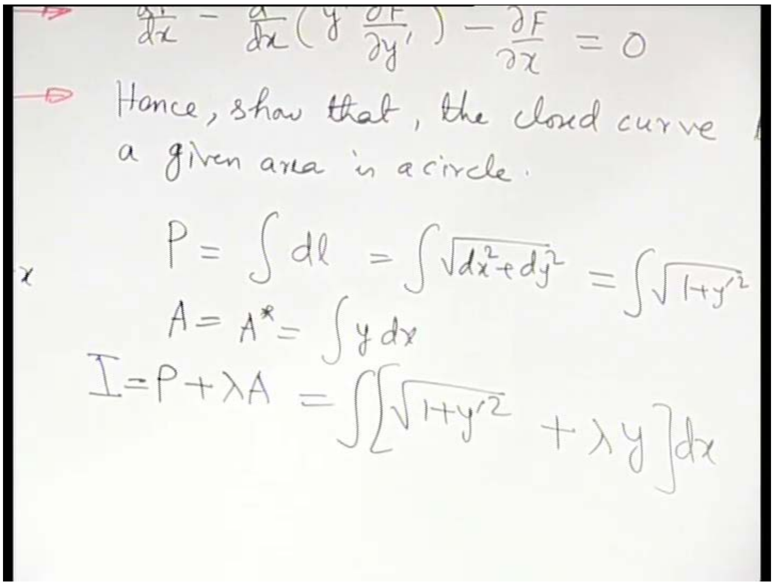

Now, let us state the second part of the problem. The second of the problem is as follows: Hence show that, the closed curve that minimizes the perimeter - the perimeter for a given area is a circle. So, let us try to make a sketch of what we are going to show that you have a curve, the perimeter of the curve. So, if this is y axis, this is x axis, the perimeter of the curve is essentially the cyclic integral of d l - the total length of the curve, the contour integral of d l. So, square root of dx square plus dy square. So, root over one plus y dash square dx. So, we have to minimize this with a constraint - what is the constraint? The constraint is that the area is given, A is A star which is a given area. So, area yon can write as integral of y dx. Because if you take a small element at a distance x, you take a small element dx, corresponding y is y, you can always deal with half of the curve if it is symmetric with respect to x axis. So, let us make such as simplification if that is appropriate that will not change the method of solution.

So, integral y dx, if we you have with symmetric with respect to x axis then of course, it is two into that. But essentially it is it is some some constant into y dx integral of y dx, that is the total area. So, we can deal with half of the curve with if it is symmetric with respect to x axis, length of half of the curve that is the perimeter, and area of the half included with the x axis. So, with that understanding our objective is to minimize P. So, that A is a constant. So, to do that we can use the Lagrange multiplier technique. That is we introduced a function if which is P plus lambda A. Well lambda is the Lagrange multiplier. Or we can call it I, because we want to extremize the integral I that is the notation that, we had used by this time.

So, this becomes our functional F. Here of course, there is no explicit x, but you have y and y prime both. Now, let us try to apply the alternative form of the Euler-Lagrange equation. So, what is dF dx? So, we have in this alternative form d dx of F minus y dash del F del y dash minus del F del x equal to 0. dF dx is not 0, why because F contains y and y prime which are themselves functions of x, but partial derivative of F with respect to x is 0, because there is no explicit x appearing here. So, that means you have this equal to 0. So that means, you have F minus y prime del F del y prime is equal to a constant. So, let us write that square root of 1 plus y prime square plus lambda y minus y prime into del F del y prime.

This allright. So, we can simplify it by writing at writing this as. So, if we multiply both the sides by square root of 1 plus y prime square. So, it will become 1 plus y prime square plus lambda y minus y prime square is equal to C into square root of 1 plus y prime square.

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Lambda y into into square root of 1 plus y prime square. So, this into right, then the remaining part is the mathematical simplification.

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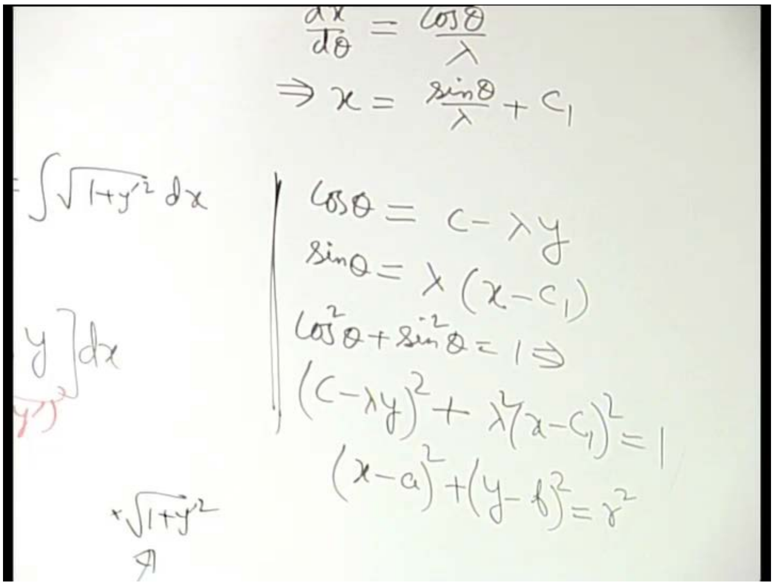

So, we can write one is equal to C minus lambda y into 1 plus y prime square alright. So, for further simplification, you can write say y prime as tan theta. So, that is nothing but dy dx. So, we have given it name tan theta. So, that will mean that C minus lambda y, this is 1 plus tan square theta setsquare theta. So, that will go in the other side will become cos theta. So that means, you get an expression for cos theta. Similarly, you can write dy dx as dy d theta by dx d theta.

So, from here you will get an expression for cos theta. Similarly, what you get from here. So, dy d theta you can write - dy d theta is minus 1 by lambda or cos theta. So, that minus and this minus will cancel. So, dy d theta will be one by lambda sin theta. Since, you know y, you can write dy d theta, and divided by dx d theta.

So, you can get an expression for sin theta, you can get an expression for cos theta, and you can eliminate those by writing sin square theta plus cos square theta equal to one. You you can you will involve x also well, how can you involve x, how can you involve x here, which one will you integrate.

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Yes.

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dx d theta. So, you mean that you have dx d theta equal to.

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So, this will become, this is sin theta by cos theta right. So, cos theta by lambda. So, from here you will get an expression involving x. So, what will you get x equal to sin theta by lambda plus sum C 1. Now, let us see. So, you have cos theta is equal to C minus lambda y, sin theta equal to lambda into x minus C 1 right. You have cos square theta plus sin square theta equal to 1. This implies C minus lambda y whole square plus lambda into x minus C 1 whole square lambda square right into x minus C 1 whole square is equal to 1. So, you can simplify this in a form x minus a whole square plus y minus b whole square is equal to sum r square. Is it possible to write it in this way?

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So, you can take you can observe this lambda with C. Now till now, we have discussed about cases where the functional F involves function of x, y and y prime, but it can also contain functions of higher order derivatives. So, let us take an example where it involves a second order derivative also, that is y double prime.

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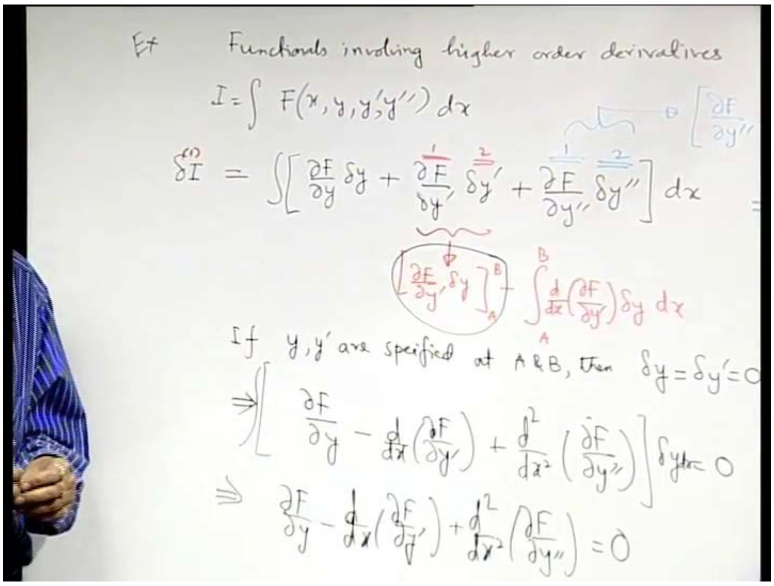

So, we will consider an extension of the Euler-Lagrange equation, where you have higher order derivatives - functional involving higher order derivatives. So, we have F I is integral of F x, y, y prime say y double prime dx. So, let us consider delta I.

Sometimes in this delta notation, if one is interested to write it more rigorously one may use superscript one, to indicate that it is a first order variation, because there could be even higher order variations that one can apply. But we will omit this superscript, and usually write delta I. So, we will now consider y, y prime, and y double prime, and write it as sum of three variations.

Now, we will do exactly the same thing what we do, what we did while deriving the Euler-Lagrange equation. The policy will be reduce the orders of the highest order derivative. So, you reduce y prime to y, y double prime not in one step it can be reduced to y. So, y double prime to y, y single prime, and then that to y. So, to do that let us let us consider this term, so this will be we will do integration by parts, as this is the first function, this is the second function.

Similarly, let us consider the third term. This as the first function, and this as the second function. So, let us write it. So, we have been successful in reducing the derivative by one order. We will reduce it by further one order. So, we will integrate the second term by parts as this as the first function, and this as the second function.

So we have simplified this one, and we have we will now pay attention to some important terms. The important terms are first the boundary terms. You see this is one boundary term, and these are the other boundary terms. This boundary terms are identically 0, when when delta y and delta y prime is zero. That means, y and y prime are both specified at the points A and B. So, let us assume that that is the case, if y, y prime are specified at A and B.

So, we are left with or since delta y is arbitrary, there is of course one dx. Since, delta y is arbitrary, we have this equal to zero. So, if you have even a higher order derivative, now you can get the parity. So, first order derivative minus sign, second order derivative plus sign, again minus sign will come again plus sign will come. In this way, depending on the number of higher order derivatives, you will get the additional number of terms. So, that generalized equation is known as Euler-Poiso equation. Just like the equation up to the first order derivative in y is an Euler-Lagrange equation, that is called as Euler-Poiso equation. So, our objective is not go into the very very general form, but at least to appreciate that if there are higher order derivates, it is possible to incorporate that in the framework by going through a very similar exercise that we do even for the first order derivative cases.

Now, we have seen that how we can make a formulation of a problem through a variational statement. But we are not interested in any arbitrary problem - we are interested in solving differential equations or in particular we are interested in approximate solutions of the differential equations through the variational formulation.

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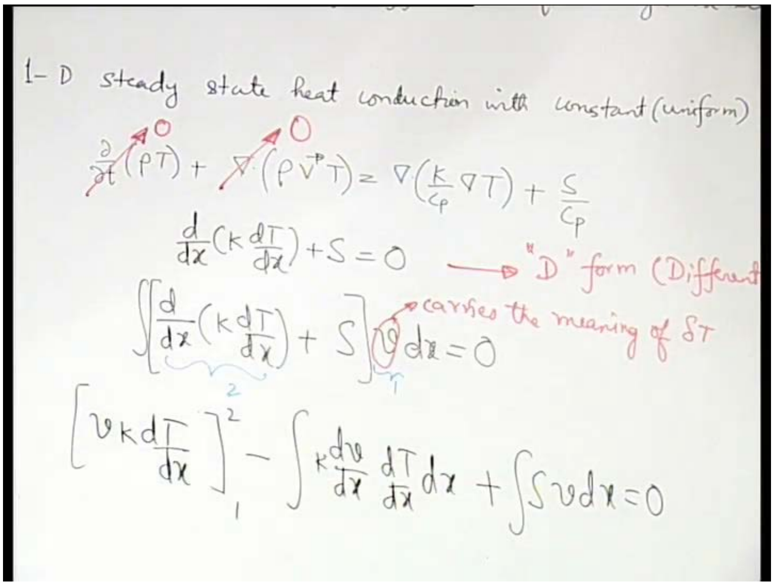

So, our next agenda will be approximate solutions of differential equation through variational formulation. To do that, let us take the example of one-dimensional steady state heat conduction with constant or uniform heat source. Let us write the general governing differential equation in a conservative form.

Let us make a further simplification that not only, it is a one-dimensional steady state heat conduction with constant heat source. But all the thermal properties are constant; like K is constant, C p is constant with constant constant properties. So, we can effectively take K by C p or C p or K in or out of the derivatives without any constraint. So, first of all it is a steady state. So, this unsteady term will be 0, then it is a conduction problem. So, there is no flow velocity involved. So, this term is 0. It is a one-dimensional problem. So, only the gradient along one particular directions say x direction becomes important. So, it will become effectively d dx of K dT dx plus x equal to 0.

For generality, what we do is we keep in mind that of course, we have taken an example with constant properties, but if even if K were variable that can be easily incorporated in this framework. So, this equation is not just valid for constant K our example, we have taken for constant K, but even for variable K we want it to be valid. So, that is why I have deliberately put K inside the derivative. So, what we will do now, we will make a variational formulation of this differential equation, this is known as the D form of the problem or differential form of the problem D form.

To make a variational form of the problem what we will do is, we will multiply this with a variational parameter. And integrate over the domain, where this V carries the meaning of delta y or here y is T. So, next what we will do. So, this is the first step in a variational formulation. So, we have multiplied it with a variational parameter which sort of carries the meaning of variation in the dependent variable, and integrate it over the domain and set it to 0. We will see, subsequently that why we do it physically, but first let us try to understand the principle.

So, next what we will do is, we will integrate it by parts our objective will be to reduce the order of derivative for the T, and eventually that will lead to increase in order of derivative for V. So, we will integrate it by parts with which as the first function, you have this as one function and V as another function. So, which is the first function?

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V is the first function. So, second function is the one for which order of derivative will be reduced. So, first function into integral of this again. So, K dT dx at the boundaries, let us say the boundaries are 1 and 2 minus integral of the derivative of the first function. Now, what we can see that at the boundary, if you want these terms to be specified. How you can specify that. One is you could specify T or that is the temperature is specified at the boundary; if T is specified at the boundary.

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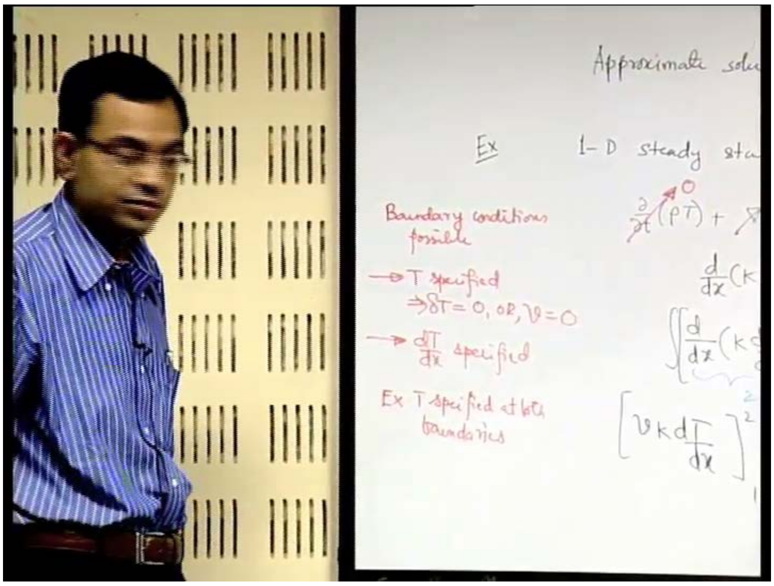

So, let us note the boundary conditions possible. So, you could have T specified - if T is specified that will imply that delta T is equal to 0 or V equal to 0. You could have dT dx specified. Of course, not both simultaneously either T specified or dT dx dx specified or T as a function of dT dx. Fundamentally either T specified or dT dx specified, third is the mixed type of boundary condition which is the combination of this. So, if dT dx is specified, then that you can substitute here.

So, let us take an example, where you have T specified at both the boundaries. We start with this simple example. So, that the boundary term vanishes goes away totally. So, if that becomes 0, then you are left with integral of K dT dx dv dx dx is equal to integral of S V dx. So, this is a particular simplified form at which we arrive, and what we will do is from the… in the next class we will take it from here, and see that how we can device approximate solution techniques by using these variational form. We stop here for this class, thank you.